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EIGENFREQUENCIES OF THE REINFORCED CONCRETE BEAMS – METHODS OF CALCULATIONS

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Abstract. The paper presents method of calculation of eigenfrequencies of the cracked reinforced concrete beams including discreet model of crack. The described method is based on the stiff finite elements method. It was modified in such a way as to take into account local discontinuities – cracks. In addition, some theoretical studies as well as experimental tests of concrete mechanics based on discrete crack model were taken into consideration. The calculations were performed using the author's own numerical algorithm. Moreover, other calculation methods of dynamic reinforced concrete beams presented in standards and guidelines are discussed. Calculations performed by using different methods are compared with the results obtained in experimental tests.

Keywords: beam, crack, eigenvibrations, reinforced concrete, stiff finite elements method.

1. Introduction

Calculation of reinforced concrete structures requires special attitude because it involves interaction of two materials, such as concrete and steel used in this type of structures. Furthermore, reinforced concrete elements are overloaded and that causes their cracking and stiffness degradation.

There are many theories regarding displacement and redistribution of internal forces in the cracked reinforced concrete beams (e. g. Branson 1977) and structures (e. g. Kobielak *et al.* 2010). The methods proposed describe performance of the reinforced concrete structures including cracks. Typically, the cracking effect and its influence on the distribution of internal forces and deformations is taken into account globally by means of introduction of effective stiffness. This kind of approach assures simplicity of calculations by analogy to homogenous structures without cracks.

The experimental tests which were performed (Eibl *et al.* 1988; Jerath and Shibani 1985; Johns and Belanger 1981; Müller *et al.* 1983; Wlazło 1987) proved that appearance of cracks has significant impact not only on steel – concrete bond (Khalfallah 2008), deflection and redistribution of the internal forces, but also on the dynamic parameters, such as: eigenfrequencies and damping. Progressive cracking causes lowering of eigenfrequencies of the reinforced concrete beams. Moreover, such cracking increases damping properties of element.

Most papers dealing with the dynamics of the cracked reinforced concrete structures describe it globally basing on the dynamic substitutional stiffness of the cracked element (Jerath and Shibani 1985; Johns and Belanger 1981; Wittig 1977).

Further, this sort of approach makes it possible to apply solutions concerning dynamics of homogenous structures (Baušys *et al.* 2008; Clough and Penzien 1993; Lewandowski and Grzymisławska 2009) and is characterized by the simplicity of calculations. Nevertheless, it limits observation of structure to the final, summary effects connected with the impact of the element overloading on the dynamic properties. In addition, there are no explicit relations connecting dynamic and static stiffness assumed to calculate deflections. Some experimental tests prove that it is less or equal to effective stiffness (Wlazło 1987) while others confirm it is bigger (Jerath and Shibani 1985).

The paper presents alternative approach based on discrete crack model. Calculations were performed using the author's own numerical programme related to Mathematica® (Wolfram 1999; Glabisz 2003). In addition, the obtained records were compared with the existing results acquired in experimental tests. Discussion and comparison of the results was conducted according to the Polish Standard (1993) requirements for calculation of support structures for machines and Eurocode 2 (2004) directives.

2. Stiff finite element method

2.1. Homogenous beams

Dynamic calculations of most of the structures with continuous mass distribution are connected with discretization. Discretization methods can be divided into two groups: mathematical (with global approximation of displacement state – for ex. the Ritz method with local approximation of displacement state – elastic finite element method) and physical which refers to mass granulation that leads to classical discrete system.

The second group includes stiff finite element method (Kruszewski *et al.* 1975). This method was first applied in naval industry. Later it was used by J. Langer (Langer 1980) for calculation of bar structures. Beam model consists of stiff mass discs which represent force of inertia of a structure. Discs are connected by elastic constraints (one rotation and two translation) responsible for elastic features of a structure. Movement of each mass discs is described by three general coordinates. In case of transverse vibrations which are considered in this paper, elastic constraints and general coordinates are reduced to two. Example scheme and calculation model of a beam divided into four elements are shown in Fig. 1.

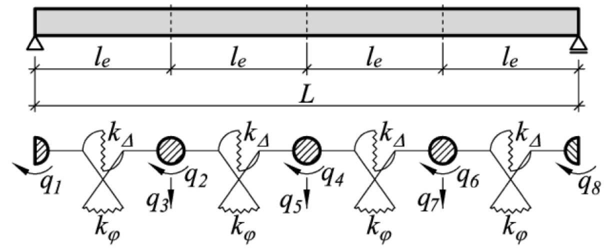

Fig. 1. Scheme and numerical model of homogenous beam

Stiffness of constraints connecting mass discs is computed on the basis of potential energy from the following Eqs. (1), (2):

$$k_\varphi = \frac{EI}{l_e}, \tag{1}$$

$$k_\Delta = 12\frac{EI}{l_e^3}, \tag{2}$$

where: EI – beam bending stiffness, l_e – length of finite element.

Stiffnesses of constraints are grouped in diagonal matrix $\{k\}$, which for the case shown in Fig. 1 is given below:

$$\{k\} = diag\{k_\varphi, k_\Delta, k_\varphi, k_\Delta, k_\varphi, k_\Delta, k_\varphi, k_\Delta\}. \tag{3}$$

Global stiffness matrix $[\mathbf{K}]$ is calculated from the following equation:

$$[\mathbf{K}] = [\mathbf{A_k}]^T \cdot \{k\} \cdot [\mathbf{A_k}], \tag{4}$$

where: $[\mathbf{A_k}]$ – transformation matrix.

Transformation matrix $[\mathbf{A_k}]$ transforms general coordinates vector $\mathbf{q}$ on relative transposition vector $\mathbf{r}$. It has repeatable character and it can be easily generated automatically for optional boundary conditions.

Inertia matrix $[\mathbf{B}]$ is a diagonal matrix. Masses of individual discs m correspond to translation coordinates while their mass inertia moments J_m correspond to rotational coordinates. For the model shown in Fig. 1 inertia matrix is as follows:

$$[\mathbf{B}] = diag\{J_{m1}, J_{m2}, m, J_{m2}, m, J_{m2}, m, J_{m1}\}. \tag{5}$$

Eigenfrequencies ω are resulted from following matrix equation:

$$Det[\mathbf{K} - \omega^2\mathbf{B}] = 0. \tag{6}$$

The second way of solution is calculation of matrix $[\mathbf{A}]$ being converse product of matrix $[\mathbf{B}]$ and matrix $[\mathbf{K}]$. Eigenvalues of matrix $[\mathbf{A}]$ are the squares of angular eigenfrequencies ω (7):

$$ev([\mathbf{A}]) = ev([\mathbf{B}]^{-1} \cdot [\mathbf{K}]) = diag\{\omega^2\}. \tag{7}$$

2.2. The reinforced concrete cracked beams

The presented approach enables to include local discontinuities (among others cracks) in a discrete way (Musiał *et al.* 2009). Adequate division into finite elements allows the introduction of cracks by means of reduction of stiff rotation constraints while calculations are performed as for the homogenous beam. In this attitude the consideration of cracks involves necessity of algorithm improvements as in case of classical finite elements method (Jovicic *et al.* 2010).

Stiffnesses of constraints k_φ, k_Δ are commuted using the element stiffness in phase I (EI_I). The stiffness of rotation constraints is reduced and has value k_φ^{cr} in the place where the cracks appear. The scheme and calculation model of the segment of beam with cracks is shown in Fig. 2.

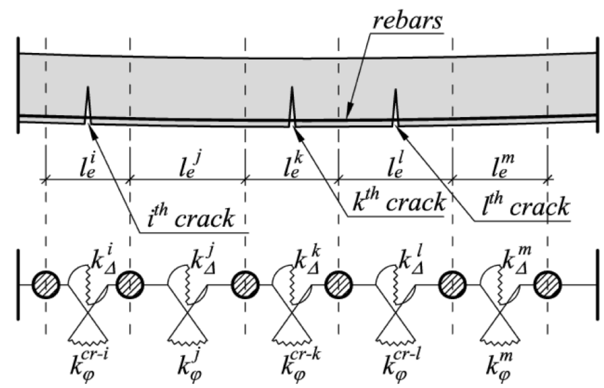

Fig. 2. Scheme and numerical model of the reinforced concrete beam with cracks

The rotational susceptibility resulted from crack was estimated on the basis of elementary relations of geometry and strength of materials. The scheme as in Fig. 3 was considered.

Forces acting in the cross-section (A–A) in the place of crack occurrence are shown in Fig. 4. Triangular stress distribution in compressed concrete was assumed.

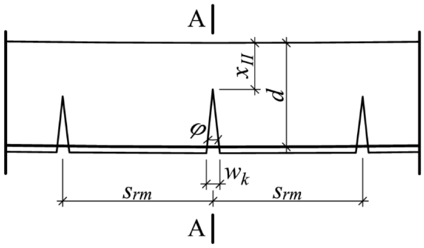

Fig. 3. Considered model of beam with cracks

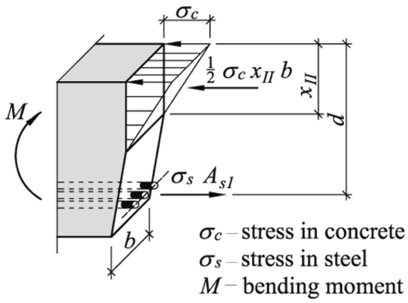

Fig. 4. Forces acting in cross-section

Figs 3 and 4 enable to formulate following expression allowing calculating the rotational susceptibility which is consequence of crack occurrence in static solution:

$$d_{\varphi,static}^{cr-i} = \frac{\psi_z\, s_{rm}}{E_s\, A_{s1}\,(d - \frac{x_{II}}{3})(d - x_{II})}, \tag{8}$$

where: ψ_z – coefficient describing violation of interaction between steel and concrete calculated according to (9), s_{rm} – average crack spacing, E_s – Young's modulus of steel, A_{s1} – reinforcement cross-sectional area, d – useful beam height, x_{II} – height of the compressed zone in phase II.

$$\psi_z = 1.3 - s\frac{M_{cr}}{M}, \tag{9}$$

where: s – 1.1 in case of immediate loading, 0.8 in case of long-term loading, M_{cr} – cracking moment, M – maximum moment up to which the cross-section was overloaded.

It was proved with own experimental investigations that dynamic susceptibility is not equal to the static one. The dynamic susceptibility could be calculated with following expression:

$$d_{\varphi,dynamic}^{cr-i} = \alpha_d \cdot \frac{\psi_z\, s_{rm}}{E_s\, A_{s1}\,(d - \frac{x_{II}}{3})(d - x_{II})}. \tag{10}$$

The used coefficient α_d is connected with static to dynamic transition. It is calculated with following formula:

$$\alpha_d = \left(\frac{M_{cr}}{M_s}\right)^2, \tag{11}$$

where: M_s – bending moment in considering place (crack) induced with static load.

Assuming that the susceptibility of finite elements connections is a sum of susceptibilities resulting from the beam deformation (for phase I) and susceptibility resulting from the crack appearance the following relationship can be written:

$$d_{\varphi}^{II-i} = (k_{\varphi}^{I})^{-1} + d_{\varphi}^{cr-i}, \tag{12}$$

where: k_{φ}^{I} – stiffness of rotation constraints calculated according (1) for the phase I (EI_I).

Knowing susceptibility (12) stiffness of rotation constraints for the cracked cross-section can be computed:

$$k_{\varphi}^{cr-i} = (d_{\varphi}^{II-i})^{-1}. \tag{13}$$

3. Calculation methods based on bending stiffness

The approach proposed in Polish Standards (Foundations and support structures for machines) referring to the support structures for machines recommends calculation of global stiffness of the bended element according to the Young's modulus of concrete and inertia moment for the gross concrete cross-section but does not take into consideration reinforcement. While this sort of approach seems to be quite correct in calculations performed for phase I, it is less convincing in case of the cracked beam. Assuming constant stiffness for the total scope of element work may cause errors.

It is more reasonable to calculate frequency using the relationship given below which is recommended in Eurocode 2 (2004) (14):

$$\alpha = \zeta\alpha_{II} + (1 - \zeta)\alpha_I. \tag{14}$$

Parameter α is the one which is considered (for example cross-section deformation, curvature, rotation or deflection) and α_I and α_{II} are the values of this parameter calculated under the assumption that cracks do not occur and for the completely cracked objects respectively, while ζ is coefficient of distribution. It is assumed in the paper that the parameter to be considered is eigenfrequency. It should be noted that element overloading is accompanied by the decrease of bending stiffness and resulting from this eigenfrequency.

The approach based on calculation of dynamic stiffness different from the static one is presented in ACI Journal (Jerath *et al.* 1985). Dynamic stiffness is expressed by the following formula (15):

$$EI_D = E_C\left[\left(\frac{\alpha M_{cr}}{M}\right)I_I + \left(1 - \frac{\alpha M_{cr}}{M}\right)I_{II}\right], \tag{15}$$

where: E_C – Young's modulus of concrete, α – constant parameter (α = 0.6 – 0.8), I_I – inertia moment in phase I, I_{II} – inertia moment in phase II.

Similarly as in case of the dependency (15) overloading is accompanied by the decrease of element stiffness.

As literature studies proved, the final approach presented gained the most popularity. Empirical dependencies are drawn in order to include estimation of substitutional dynamic stiffness of element. Thus, application of closed solutions of the structure dynamics can be considered in calculations of the cracked reinforced concrete structures.

4. Numerical example – verification and comparison

4.1. Experimental studies

In order to verify numerical analysis some experimental results were applied (Jerath and Shibani 1985). All units were presented in the mentioned paper in Anglo – Saxon system. They were transformed to SI system. Experimental tests were performed on the series of the beam elements such as shown in Fig. 5. The dynamic computational scheme is included in Fig. 6.

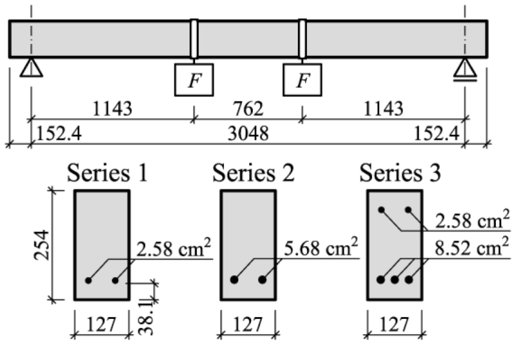

Fig. 5. Analysed beam (dimensions in mm)

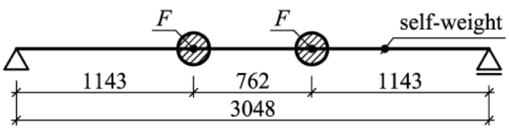

Fig. 6. Computational dynamic scheme (dimensions in mm)

The beam was struck at the center of span by a light hammer. Then the natural period was measured and eigenfrequency was calculated. Static load F was added slowly on each hanger. The beam was vibrated and the natural period was noted again. After each increment of static load the testing procedure was repeated. Increasing loading and degradation of the beam stiffness (progressive crack propagation) caused the decrease of eigenfrequency. The rest of data (concrete and reinforcing steel properties mainly) taken to further analyses are included in the Table 1.

Table 1. Input data

Properties	Series		
	1	2	3
Compressive strength of concrete [MPa]	41.37	48.26	42.10
Tensile strength of concrete [MPa]	4.00	4.33	4.04
Modulus of elasticity of concrete [GPa]	30.48	32.89	30.68
Yield strength of steel [MPa]	276		
Modulus of elasticity of steel [GPa]	200	201	200
Self-weight of beam [kg]	251.3	271.8	279.4
Spacing of cracks [mm]	175	160	150

The spacing of cracks was not measured in the experiment. It was calculated according to Eurocode 2 (2004). Moreover, own experimental investigations proved that cracks' spacing does not influence significantly on eigenfrequencies of reinforced concrete beams.

4.2. Results of numerical analysis

Numerical analysis were carried out for the data as in 4.1. Calculations were performed using three methods:

- according to Eurocode 2 (2004),
- according to ACI,
- according to Polish Standard,
- using author's own algorithm based on stiff finite elements method (SFEM) related to Mathematica®.

The obtained results are shown in diagrams of eigenfrequencies (Figs 7–9). Vertical axes illustrate load applied on one hanger. Horizontal axes correspond to eigenfrequency. The results of numerical analyses are presented with the results of the experimental studies (Jerath and Shibani 1985) for comparison.

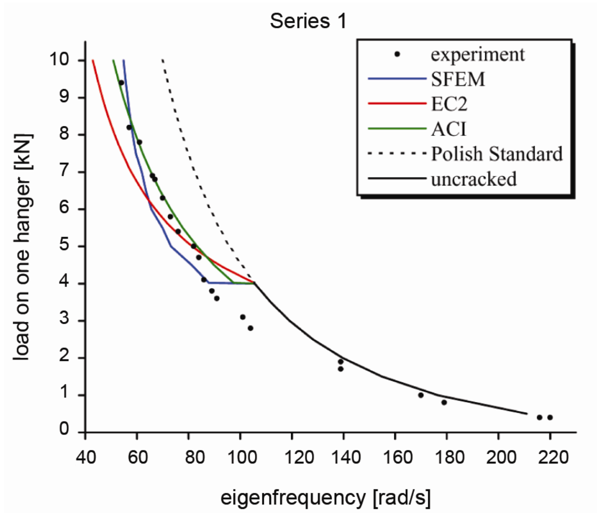

Fig. 7. Load vs eigenfrequency for beams of series 1

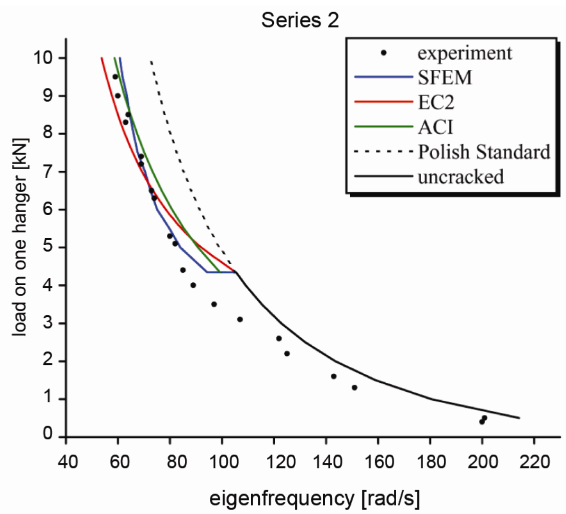

Fig. 8. Load vs eigenfrequency for beams of series 2

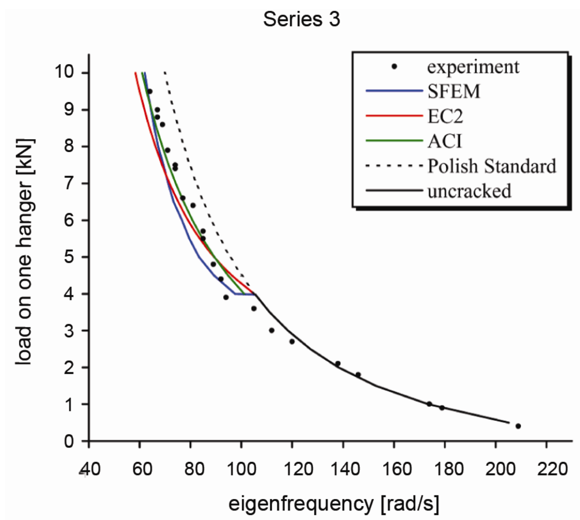

Fig. 9. Load vs eigenfrequency for beams of series 3

Diagrams (Figs 7–9) show quite noticeable decrease of eigenfrequency depending on the level of beam overloading. The results of the experiments and some of the theoretical charts (SFEM, ACI) demonstrate significant decrease of eigenfrequency when $M \approx M_{cr}$. In case of the theoretical values it occurs at lower loading level. In authors' opinion it is consequence of internal cracks (Pędziwiatr 2008) which influence on interaction between steel and concrete was not taken into account in calculations. In studies following rule was observed: reinforcement ratio is higher the decrease connected with first cracks occurrence is lower.

In case of curves obtained on the basis of Eurocode 2 (2004) the decrease of eigenfrequency at the cracking moment is not as clear as in case of the methods described above. When $M = M_{cr}$ Eurocode 2 (2004) curve changes inclination angle abruptly (first derivative is not continuous).

Polish standard (1993) allows assuming constant stiffness in whole range of beam work. Therefore the decrease of eigenfrequency is resulted only from the increase of load on hangers (not from the stiffness degradation).

5. Final conclusions and remarks

The experimental tests carried out so far prove that element overloading causes changes of dynamic characteristics (eigenfrequencies, damping parameters). Thus including this factor in calculations seems to be quite reasonable.

The paper presents different methods of calculation of eigenfrequencies of the cracked reinforced concrete beams. It can be noticed according to some literature studies that the most popular approach is based on the global description of the effect (substitutional element stiffness). This sort of attitude makes it possible to use the closed solutions of the structure dynamics for simple static schemes.

The author's own method is the alternative approach which considers the crack morphology in a detailed way. It allows following processes connected with the influence of overloading on the eigenfrequencies of the cracked reinforced concrete beams. Moreover, in case of dynamic loads the influence of fatigue should be taken into account. The modified SFEM presented in the paper allows to consider this not only for concrete and steel as materials but for every single crack as well. It is known that dynamic load causes fatigue grow of crack width (Szata and Lesiuk 2009).

Conducted comparative analyses shown that each presented method gave similar results. The obtained curves have similar character and in more or less precise way they resemble actual element work. The selection of method depends on intended accuracy and specifity level in calculations, conditions in which structure exists or calculation's tool. The highest discrepancies were observed in case of method proposed by Polish Standard. However, in authors' opinion it could not be rejected in particular with regard to beams with high reinforcement ratio, where the differences are insignificant. Furthermore, the method recommended by Polish Standard (1993) is intended for common engineering calculations and quite simple in use. If higher precision is demanded or structure is more sophisticated the other methods based on the dynamic stiffness or stiff finite elements method can be used.

Nowadays the described method based on SFEM is being developed. Theoretical studies coupled with experiments are in progress. The own method will be verified in case of beams subjected to immediate and long – term vibratory loads. Authors are going to applied own method in random vibrations issues (Kamiński and Szafran 2009).

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GELŽBETONINIŲ SIJŲ TIKRINIAI DAŽNIAI – SKAIČIAVIMO METODAI

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Santrauka

Straipsnyje pristatomas įtrūkusių gelžbetonio sijų tikrinių dažnių skaičiavimo metodas, įskaitant diskretinį įtrūkio modelį. Metodas pagrįstas standžių baigtinių elementų metodu. Jis buvo patobulintas siekiant įtraukti vietinius nevienalytiškumus – įtrūkius. Taip pat buvo įvertintos kai kurios betono mechanikos teorinės studijos ir eksperimentiniai bandymai, pagrįsti diskrečiu įtrūkių modeliu. Skaičiavimai buvo atlikti taikant autorių sukurtą skaitmeninį algoritmą. Be to, pristatomi ir aptariami kiti dinaminiai gelžbetoninių sijų skaičiavimo metodai, pateikti standartuose ir gairėse. Skaičiavimai, atlikti taikant skirtingus metodus, palyginami su rezultatais, gautais eksperimentinių bandymų metu.

Reikšminiai žodžiai: sija, įtrūkis, tikriniai dažniai, gelžbetonis, standžių baigtinių elementų metodas

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