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EFFECTIVE FINITE-DIFFERENCE METHODS FOR THE SOLUTIONS OF FILTRATION PROBLEMS IN MULTILAYER DOMAINS

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ABSTRACT

In papers [1,2] there were consider different assumptions for averaging methods along the vertical coordinate.These methods were applied for the mathematical simulation of the mass transfer process in multilayered underground systems. A specific feature of these problems is that it is necessity to solve the 3-D initial-boundary-value problems for parabolic type partial differential equations of second order with piece-wise parameters in multilayer domain.Therefore here an effective finite-difference method for solving a problem of the above type is developed.This method may be considered as a generalization of the method of finite volumes [3] for the layered systems. In the case of constant piece-wise coefficients we obtain the exact discrete approximation of steady-state 1-D boundary-value problem.This procedure allows to reduce the 3-D problem to a system of 2-D problems and the 2-D problem to a system of 1-D problems.

1. FORMULATION OF PROBLEM

The process of filtration we will consider in 3-D domain of parallelepiped $D = \{(x,y,z) : (x,y) \in \Omega, H_0 \le z \le H_N\}$, where $\Omega = \{(x,y) : -l_x \le x \le l_x, -l_y \le y \le l_y\}$ is rectangle in the horizontal x, y directions with length of edges $2l_x, 2l_y$, $H_N - H_0$ is the height of domain in the vertical z - direction. The domain D consist of multilauer medium of N layers (cylinders)

$$D_k = \{(x,y,z) : (x,y) \in \Omega, H_{k-1} < z < H_k\} \qquad k = \overline{1,N}, \tag{1}$$

with horizontal interfaces

$$S_k = \{(x,y,H_k) : (x,y) \in \Omega\} \qquad k = \overline{1,N-1}, \tag{2}$$

where $H_k - H_{k-1}$ is the height of layer D_k. We will to find the distribution of physical magnitudes $u_k = u_k(x, y, z, t)$ in every layer D_k at the point $(x, y, z) \in D_k$ and time $t > 0$ by solving the partial differential equation in the following form [2]

$$\partial(\lambda_k \partial u_k/\partial z)/\partial z + L_k(u_k) = -F_k(x, y, z, t) \qquad k = \overline{1, N}, \tag{3}$$

where F_k is continuously differentiable function of external sources in every layer D_k.The physical parameters λ_k of heat conductivity, diffusion coefficients or coefficients of filtration and the differential operator L_k are depends only of x,y,t ($\lambda_k > 0$). We assume that the function λ_k and the coefficients of differential operator are piece-wise constants functions of the vertical coordinate z with discontinuity on the surface S_k. The operator L_k we can consider in the form

$$\begin{aligned} L_k(u_k) &= \partial(\lambda_k \partial u_k/\partial x)/\partial x + \partial(\lambda_k \partial u_k/\partial y)/\partial y \\ &\quad - d_k \partial u_k/\partial t - a_k \partial u_k/\partial x - b_k \partial u_k/\partial y - c_k u_k \quad k = \overline{1, N}, \end{aligned} \tag{4}$$

where an example $w_k = (a_k, b_k)$ is the velocity of filtering for the flow of the underground water in layer D_k, d_k, c_k mass transfer coefficients($d_k > 0$). The equations(3,4) are considered in every layer D_k of different properties of medium.The physical magnitude u_k and the flux $\lambda_k \partial u_k/\partial z$ must be continuous on the interior boundary. Therefore we have on the S_k following continuity conditions

$$u_k = u_{k+1}, \quad \lambda_k \partial u_k/\partial z = \lambda_{k+1} \partial u_{k+1}/\partial z, \qquad k = \overline{1, N-1}. \tag{5}$$

We assume that the whole N-layered system is bounded above and below with the plane surfaces S_0, S_N (2).The boundary conditions on the S_0, S_Nmay be written corresponding as

$$\nu_0 \lambda_1 \partial u_1/\partial z - \alpha_0 u_1 = -\alpha_0 \Phi_0(x, y, t), \tag{6}$$

$$\nu_1 \lambda_N \partial u_N/\partial z + \alpha_N u_N = \alpha_N \Phi_1(x, y, t), \tag{7}$$

where $(x, y) \in \Omega, t \geq 0$,
$\nu_0 = 0$ or $\nu_1 = 0$ for the corresponding Dirichlet boundary conditons: $u_1 = \Phi_0$ or $u_N = \Phi_1$;
$\nu_0 = 1$ or $\nu_1 = 1$ for the corresponding Neumann ($\alpha_0 = 0$ or $\alpha_N = 0$) or general form of boundary conditions;
$\alpha_0 \geq 0, \alpha_N \geq 0$, Φ_0, Φ_1 are given continuosly differentiable functions. The equations(3,4) with conditions (5-7) along the z-coordinate has been solved in a domain D with different boundary conditions in the x, y directions at $x = \pm l_x; y = \pm l_y$ and with initial condition at $t = 0$ in the case of time depending problem.The form of this conditions are not assential for obtaining

the numerical algorithm.

2. FINITE VOLUMES METHOD

The approximation of differential problem is based on the conservation law approch.Therefore it develops the monotonous difference scheme using a physical conservation law. This method is based of the application of the method of finite volumes [3]. We consider the nonuniform grid in the z-direction placed on the interval (H_0, H_N) with blocks centered at the grid points $z_j, \quad j = \overline{1, M}, \quad M \geq N \quad (z_0 = H_0, z_M = H_N)$. We shall refer to the endpoints of the interval about the point z_j as $z_{j\pm 0.5}$.This interval $(z_{j-0.5}, z_{j+0.5})$ is refered to as the control volume associated with the grid point z_j (the j-th cell). The grid contain the z-coordinates H_k of surfaces $S_k, \ k = \overline{0, N}$ and in addition some grid points in layers $D_k, k = \overline{1, N}$ when this is necessary for demonstrating the behaviour of discrete solution in this layers. To derive a difference equation associated with the j-th grid point z_j we integrate differential equation (3) to the j-th cell.We define the functions $G_k = -(F_k + L_k(u_k)), \quad W = \lambda \partial u/\partial z, \quad W_{j\pm 0.5}$
and the integrals $B_j = (\lambda_j)^{-1} \int_{z_{j-1}}^{z_j} dz \int_{z_{j-0.5}}^{z} G_j d\xi$,
where $W_{j\pm 0.5} = W\mid_{z=z_{j\pm 0.5}}, \quad h_j = z_j - z_{j-1}, \quad z_{j\pm 0.5} = (z_j + z_{j\pm 1})/2$.
We shall consider corresponding from central grid point z_j 4 cases for the applying the finite volumes method: $z_j \in S_k, k = \overline{1, N-1}$; $\quad z_j \in D_k, k = \overline{1, N}$; $\quad z_j \in S_0$ if $\nu_0 = 1$ and $z_j \in S_N$ if $\nu_1 = 1$.
2.1. Let $z_j = H_k \in S_k$ is the central grid point and z_{j-1}, z_{j+1} are the others grid points. We integrate the equation (3) from $z_{j-0.5}$ to z_j and z_j to $z_{j+0.5}$. We get

$$W_{j+0.5} - W_{j-0.5} = \int_{z_{j-0.5}}^{z_j} G_j dz + \int_{z_j}^{z_{j+0.5}} G_{j+1} dz, \tag{8}$$

where $W\mid_{z_j-0} = W\mid_{z_j+0}$.
This is the integral form of the conservation law to the interval $(z_{j-0.5}, z_{j+0.5})$. In the classical formulation for finite volumes method [3] it is assumed that the flux terms $W_{j\pm 0.5}$ in (8) are approximated with the difference expressions. Then the corresponding difference scheme is not exact for given functions G_j. Here we have the possibility to make the exact difference scheme. Therefore we integrate equation (3) from $z_{j-0.5}$ to $z \in (z_{j-1}, z_j)$, divid this expression by λ_k and integrating from z_{j-1} to z_j. We obtain $u_k(z_j) - u_k(z_{j-1}) = (A_j)^{-1} W_{j-0.5} + B_j$, where $A_j = \lambda_k / h_j$ and $u_k(z_j), u_k(z_{j-1})$ represents the value of function u_k at z_j, z_{j-1}. Hence

$$W_{j-0.5} = A_j(u_k(z_j) - u_k(z_{j-1})) - A_j B_j. \tag{9}$$

Similarly the flux term

$$W_{j+0.5} = A_{j+1}(u_{k+1}(z_{j+1}) - u_{k+1}(z_j)) - A_{j+1}B_{j+1}, \qquad (10)$$

where $A_{j+1} = \lambda_{k+1}/h_{j+1}$ and $u_{k+1}(z_j), u_{k+1}(z_{j+1})$ are the value of function u_{k+1} at z_j, z_{j+1}. To derive a 3-point difference equation associated with the central grid point $z_j = H_k$ we want to apply equation (8) in the form

$$A_{j+1}(u_{k+1}(z_{j+1}) - u_{k+1}(z_j)) - A_j(u_k(z_j) - u_k(z_{j-1})) = R_j, \qquad (11)$$

where $R_j = R_j^- + R_j^+$, $R_j^- = (h_j)^{-1}\int_{z_{j-1}}^{z_j}(z - z_{j-1})G_j dz$, $R_j^+ = (h_{j+1})^{-1}\int_{z_j}^{z_{j+1}}(z_{j+1} - z)G_{j+1}dz$.

2.2. If $z_j \in D_k, h_j = h_{j+1}$, then difference equation (11) associated with point z_j has the form

$$\lambda\delta_z^2(u_k)_j = h_j^{-1}R_j, \qquad (12)$$

where $\delta_z^2(v)_j = (v_{j+1} - 2v_j + v_{j-1})/h_j^2$ denotes a central difference expression of second order for approximation the derivative $\partial^2 v/\partial z^2$ at the central grid point z_j.

2.3. Let $z_j = z_0 = H_0 \in S_0$ and $\nu_0 = 1$. We apply the integral form of the conservation law to the half interval $(z_0, z_{0.5})$ marked off to the right of the boundary point z_0. We get

$$A_1(u_1(z_1) - u_1(z_0)) - \alpha_0(u_1(z_0) - \Phi_0) = R_0, \qquad (13)$$

where $A_1 = \lambda_1/h_1$ and $u_1(z_0), u_1(z_1)$ represents the value of function u_1 at z_0, z_1, $R_0 = (h_1)^{-1}\int_{z_0}^{z_1}(z_1 - z)G_1 dz$.

2.4. If $z_j = H_N \in S_N, \nu_1 = 1$, then similarly in advance we obtain

$$-\alpha_N(u_N(H_N) - \Phi_1) - A_N(u_N(H_N) - u_N(z_{j-1})) = R_N, \qquad (14)$$

where $A_N = \lambda_N/h_N$ and $u_N(z_{j-1})$ represent the value of function u_N at z_{j-1}, $R_N = (h_N)^{-1}\int_{z_{j-1}}^{H_N}(z - z_{j-1})G_N dz$.

We see that the difference equations (11-14) are exact approximations for solving steady-state 1-D boundary - value problem (3),(5 - 7) depending only of $z, (L_k(u_k) = 0, l_x = l_y = \infty, c_k = 0)$.

3. ONE-DIMENSIONAL EXACT DIFFERENCE SCHEME

Suppose that $L_k(u_k) = 0, u_k = u_k(z), F_k = F_k(z), \quad \lambda_k; \Phi_0; \Phi_1$ are constants and the grid points are $z_k = H_k, k = \overline{0, N}$. If $v_j = u_j(z_j)$ is the value of function u_j at the grid point $z_j, j = \overline{0, N}$, then,evaluating the integral R_j in the right side of (11-14), one obtains exact 1-D steady-state difference scheme ($\nu_0 = \nu_1 = 1$)

$$A_{j+1}(v_{j+1} - v_j) - A_j(v_j - v_{j-1}) = R_j, \quad j = \overline{0, N}, \tag{15}$$

where $A_0 = \alpha_0 \geq 0, A_{N+1} = \alpha_N \geq 0, v_{-1} = \Phi_0, v_{N+1} = \Phi_1, A_j = \lambda_j/h_j > 0, j = \overline{1, N}$. Therefore, the difference scheme (15) is monotone and has an unique solution [4]. We can to consider in addition the new grid points or interpolating points for approximationof functions u_k in layers D_k.In the case of uniform grid we use the difference equations (12) .The finite-difference scheme (15) can be solving by factorisation method for tri-diagonal matrix (Thomas algorithm [3]).

4. SOLUTION OF ONE-DIMENSIONAL PROBLEM

We can solve the difference scheme (15) also in a more simplest form. For this purpose from the first equation (15) we conclude that $A_1(v_1 - v_0) - \alpha_1^+(v_1 - \Phi_0) = \alpha_1^+(\alpha_0)^{-1}R_0$, where $\alpha_1^+)^{-1} = (\alpha_0)^{-1} + A_1^{-1}$ is the inverse value of the interaction coefficient of the two layers in directe direction. Further more, from the second equations (15) follows $A_2(v_2 - v_1) - A_1(v_1 - v_0) = R_1$. Therefore $A_2(v_2 - v_1) - \alpha_1^+(v_1 - \Phi_0) = \alpha_1^+ R_1^+$, where $R_1^+ = R_1/\alpha_1^+ + R_0/\alpha_0$. Hence

$$A_{m+1}(v_{m+1} - v_m) - \alpha_m^+(v_m - \Phi_0) = \alpha_m^+ R_m^+, \tag{16}$$

where

$$(\alpha_m^+)^{-1} = (\alpha_{m-1}^+)^{-1} + A_m^{-1} = (\alpha_0)^{-1} + A_1^{-1} + \cdots + A_m^{-1},$$
$$R_m^+ = R_m/\alpha_m^+ + R_{m-1}^+/\alpha_{m-1}^+ = R_0/\alpha_0 + R_1/\alpha_1^+ + \cdots + R_m/\alpha_m^+$$
$$m = \overline{1, N-1}.$$

From the last equation (15) and from (16) for $m = N-1$ follows $\alpha_{N-1}^-(\Phi_1 - v_{N-1}) - \alpha_{N-1}^+(v_{N-1} - \Phi_0) = R_{N-1}^{\pm}$ and

$$v_{N-1} = \frac{\alpha_{N-1}^-\Phi_1 + \alpha_{N-1}^+\Phi_0 - R_{N-1}^{\pm}}{\alpha_{N-1}^- + \alpha_{N-1}^+}, \tag{17}$$

where $R_{N-1}^{\pm} = R_{N-1}^+\alpha_{N-1}^+ + \alpha_{N-1}^-(\alpha_N)^{-1}R_N$. Similarly can be obtained $v_N = (\alpha_N\Phi_1 + \alpha_N^+\Phi_0 - R_N^{\pm})/(\alpha_N + \alpha_N^+)$, where $R_N^{\pm} = R_N + \alpha_N^+ R_{N-1}^+$. For Dirichlet boundary condition ($\nu_0 = 0$ or $\nu_1 = 0$) we can take $\alpha_0 = \infty$ or $\alpha_N = \infty$ ($v_0 = \Phi_0$ or $v_N = \Phi_1$).

We can also consider the opposite direction. Then similary (16) follows

$$\alpha_{N-n}^-(\Phi_1 - v_{N-n}) - A_{N-n}(v_{N-n} - v_{N-n-1}) = R_{N-n}^-\alpha_{N-n}^-, \tag{18}$$

where $(\alpha^-_{N-n})^{-1} = (\alpha_N)^{-1} + (A_N)^{-1} + \cdots + (A_{N-n+1})^{-1}$,
$R^-_{N-n} = R_{N-n}/\alpha^-_{N-n} + R_{N-n+1}/\alpha^-_{N-n+1} + \cdots + R_N/\alpha_N$.
From the first equation (15) and from (18) by $n = N-1$ follows
$\alpha^+_1(\Phi_0 - v_1) - A_1(v_1 - v_0) = \alpha^+_1(\alpha_0)^{-1}R_0$ and

$$v_1 = \frac{\alpha^-_1\Phi_1 + \alpha^+_1\Phi_0 - R^{\pm}_1}{\alpha^-_1 + \alpha^+_1}, \tag{19}$$

where $R^{\pm}_1 = R^-_1\alpha^-_1 + \alpha^+_1(\alpha_0)^{-1}R_0$. The value of v_0 can be obtained in the form $v_0 = (\alpha^-_0\Phi_1 + \alpha_0\Phi_0 - R^{\pm}_0)/(\alpha^-_0 + \alpha_0)$, where $R^{\pm}_0 = R_0 + R^-_1\alpha^-_0$. ¿From the expression (16),(18) by $m = k-1$ and $n = N-k$ follows $\alpha^+_k(\Phi_0 - v_k) + A_k(v_k - v_{k-1}) = \alpha^+_k R^+_{k-1}$ and

$$v_k = \frac{\alpha^-_k\Phi_1 + \alpha^+_k\Phi_0 - R^{\pm}_k}{\alpha^-_k + \alpha^+_k}, \tag{20}$$

where $R^{\pm}_k = R^-_k\alpha^-_k + \alpha^+_k R^+_{k-1}, \quad k = \overline{1, N-1}$.
We can consider in addition for example the grid point z_i with steps h^+_i, h^-_i between the nearests grid points $z_i \pm h^{\pm}_i$. Then we have also the terms in the corresponding summ
$(\alpha^+_k)^{-1} = (\alpha_0)^{-1} + A^{-1}_1 + \cdots + h^-_i/\lambda^-_i + h^+_i/\lambda^+_i \cdots + A^{-1}_k$,
$(\alpha^-_k)^{-1} = (\alpha_N)^{-1} + A^{-1}_N + \cdots + h^-_i/\lambda^-_i + h^+_i/\lambda^+_i \cdots + A^{-1}_{k+1}$,
$R^+_k = R_0/\alpha_0 + R_1/\alpha^+_1 + \cdots + R_i/\alpha^+_i + \cdots + R_k/\alpha^+_k$,
$R^-_k = R_N/\alpha_N + \cdots + R_i/\alpha^-_i + \cdots + R_k/\alpha^-_k$,
where λ^-_i, λ^+_i are the corresponding parameters of layers.

5. DISCRETE APPROXIMATION OF FIRST AND SECOND ORDER

If $L_k(u_k) \neq 0$ and the functions $\lambda, F_k, \Phi_0, \Phi_1$ are depends of others variable, then the difference scheme (15) is not exact (this is the case of 2-D or 3-D problems with $l_x \neq \infty, l_y \neq \infty$) . In this cases is the accuracy of order $O(h_x + h_y + h_z)$ or $O(h^2_x + h^2_y + h^2_z)$,where h_x, h_y, h_z are the steps of uniform grid in the corresponding directions. We consider different approximations for right side function R_j in equations (11-14).
5.1. To approximate R_j of (11) on the nonuniform we consider the following Taylor series expansions of function G_k :
$G_k(z) = G_k(z_j) + (z - z_j)G'_k(z_j) + O(z - z_j)^2, z \in (z_{j-1}, z_j)$
$G_{k+1}(z) = G_{k+1}(z_j) + (z - z_j)G'_{k+1}(z_j) + O(z - z_j)^2, z \in (z_j, z_{j+1})$
where $G'_k = \partial G_k/\partial z, z_j = H_k$.
Then $R_j = \frac{1}{2}(G_{k+1}(z_j)h_{j+1} + G_k(z_j)h_j) + \frac{1}{6}(G'_{k+1}(z_j)h^2_{j+1} - G'_k(z_j)h^2_j) + O(h^3)$, where $h = max(h_j, h_{j+1}), 0 < z_j < H_N$. Since $G'_{k+1} = (G_{k+1}(z_{j+1}) - G_{k+1}(z_j))/h_{j+1} + O(h_{j+1})$,

$G'_k = (G_k(z_j) - G_k(z_{j-1}))/h_j + O(h_j)$ the expression can to obtain in the form

$$R_j = \frac{h_{j+1}}{6}(G_{k+1}(z_{j+1}) + 2G_{k+1}(z_j)) + \frac{h_j}{6}(G_k(z_{j-1}) + 2G_k(z_j)) + O(h^3). \quad (21)$$

With $h_j = h_{j+1}$ in the case of uniform grid the expression for R_j can be rewritten in the form

$$R_j = ((G))_j h_j + \frac{1}{6}[G']_j h_j^2 + O(h_j^3), \quad (22)$$

where $((G))_j = (G_{k+1}(z_j) + G_k(z_j))/2$ is the averaged value of G;
$[G']_j = G'_{k+1}(z_j) - G'_k(z_j))$ is the jump of G' at the point z_j.
In this case $(z_j = H_k)$:

$$\begin{aligned} ((G))_j = -[((F))_j h_j + \partial(((\lambda))\partial u_k/\partial x)/\partial x + \partial(((\lambda))\partial u_k/\partial y)/\partial y \\ - ((d))\partial u_k/\partial t - ((a))\partial u_k/\partial x - ((b))\partial u_k/\partial y - ((c))u_k] \end{aligned} \quad (23)$$

where
$((F))_j = (h_j)^{-1} \int_{z_{j-1}}^{z_j} (z - z_{j-1}) F_j dz + (h_{j+1})^{-1} \int_{z_j}^{z_{j+1}} (z_{j+1} - z) F_{j+1} dz.$
The expressions (21,22) approximates R_j to the second order in h_j.
5.2. From (13) evaluating R_0 we se , using a Taylor series expansion, that $G_1(z) = G_1(z_0) + (z - z_0)G'_1(z_0) + O(z - z_0)^2, z \in (z_0, z_1)$.
So $R_0 = 0.5(G_1(z_0)h_1 + (G'_1(z_0)h_1^2/6 + O(h_1^3)$ or

$$R_0 = \frac{h_1}{6}(G_1(z_1) + 2G_1(z_0)) + O(h_1^3). \quad (24)$$

The expressions (24) approximate R_0 to the second order in h_1.
5.3. Similarly from (14), evaluating R_N we can show that
$R_N = 0.5(G_N(H_N)h_N - (G'_N(H_N)h_N^2/6 + O(h_N^3)$, or

$$R_N = \frac{h_N}{6}(G_N(z_{j-1}) + 2G_N(H_N)) + O(h_N^3). \quad (25)$$

The second order of accuracy in x, y directions can be obtain by the central difference approximation for derivatives in the expressions (21 - 25). If $a_k \neq 0, b_k \neq 0$ then the monotonous difference schemes can be consider [4].

6. SOME EXAMPLES

In the following examples we discuss the applications the finite- difference scheme (15).

6.1. We assume that the boundary-value problem of mathematical physics (3) - (7) for the two-layered system ($N = 2$) is steady-state ($c_k = 0$) with the boundary conditions at the side $x = \pm l_k, y = \pm l_y$:
$\partial u_k/\partial x = \partial u_k/\partial y = 0$. Let $H_0 = 0, H_1 = \epsilon, H_2 = 1, \nu_0 = 1, \nu_1 = 0$, $\alpha_0 = \alpha_2 = 1, F_1 = -\epsilon^{-1}, F_2 = 0, \Phi_0 = \Phi_1 = 0$. Then from (15) follows the system of two equations
$\lambda_1\epsilon^{-1}(v_1 - v_0) - \alpha_0 v_0 = R_0, \quad -\lambda_2(1-\epsilon)^{-1}v_1 - \lambda_1\epsilon^{-1}(v_1 - v_0) = R_1,$
where $R_0 = \epsilon^{-1}\int_0^\epsilon(1 - z/\epsilon)dz = \epsilon^{-1}\int_0^\epsilon z/\epsilon dz = 0.5$.
We obtain the exact values of solution at the 3 grid points $z_0 = 0, z_1 = \epsilon, z_2 = 1$ in the form
$v_0 = ((1-\epsilon)\lambda_1 + 0.5\epsilon\lambda_2)/p, \quad v_1 = 0.5(\epsilon - 1)(\alpha_0\epsilon + 2\lambda_1)/p, v_2 = 0,$
where $p = \alpha_0\lambda_1(1-\epsilon) + \lambda_2(\lambda_1 + \alpha_0\epsilon)$.
6.2 We consider the 2-dimensional steady -state process with conditions $\partial u_k/\partial x \mid_{x=\pm l_x} = 0$, $u_k = u_k(y,z), b_k = c_k = 0, F_k = F_k(y,z)$, $\lambda_k = const$ and of uniform grid in the y direction with points $y_i = -l_y + ih_y, i = \overline{0, 2N_y}(h_y N_y = l_y)$. Since the functions $\lambda_k, F_k, \Phi_0, \Phi_1$ are continously differentiable in the x, y, t directions the continuity condition (5) can to derivate with respect to x, y, t one or more time.Then from (15), (23-25) follows the finite-difference scheme

$$\nu_0\lambda_1 h_1^{-2}(v_{i,1} - v_{i,0}) - \alpha_0 h_1^{-1}(v_{i,0} - (\Phi_0)_i) + \nu_0(\lambda_1/2 + h_1\alpha_0/6)\delta_y^2(v_0)_i = (F_0^*)_i \qquad i = \overline{1, 2N_y - 1};$$
$$\Lambda_z v_{i,j} + \Lambda_y v_{i,j} = -(F_j^*)_i, \qquad j = \overline{1, N-1}, i = \overline{1, 2N_y - 1};$$
$$\alpha_N h_j^{-1}((\Phi_1)_i - v_{i,N}) - \nu_1\lambda_N h_N^{-2}(v_{i,N} - v_{i,N-1}) + \nu_1(\lambda_N/2 + h_N\alpha_1/6)\delta_y^2(v_N)_i = (F_N^*)_i, \qquad i = \overline{1, 2N_y - 1},$$

where
$v_{i,j} = u_j(y_i, z_j), (p)_i = p \mid_{y=y_i}, \delta_y^2(p)_i = ((p)_{i+1} - 2(p)_i + (p)_{i-1})/h_y^2$, $p = F_j; v_j; \Phi_0; \Phi_1$, $(F_0^*)_i = \nu_0(\alpha_0 h_1\delta_y^2(\Phi_0)_i/6 - (F_1)_i/2 - h_1(F_1')_i/6)$,

$$(F_N^*)_i = \nu_1(\alpha_N h_N\delta_y^2(\Phi_1)_i/6 - (F_N)_i/2 + h_N(F_N')_i/6),$$
$$(F_j^*)_i = ((F_j)_i h_j + (F_{j+1})_i h_{j+1})/(2\hbar_j),$$
$$\Lambda_z v_{i,j} = (\hbar_j)^{-1}(\lambda_{j+1}/h_{j+1}(v_{i,j+1} - v_{i,j}) - \lambda_j/h_j(v_{i,j} - v_{i,j-1})),$$
$$\Lambda_y v_{i,j} = (\lambda_j h_j + \lambda_{j+1}h_{j+1})/(2\hbar_j)\delta_y^2(v_j)_i,$$
$$\hbar_j = 0.5(h_j + h_{j+1}), \quad F_j' = \partial F_j/\partial z \mid_{z=z_j}.$$

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