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Robust Numerical Method for Singularly Perturbed Convection-Diffusion Type Problems with Non-local Boundary Condition

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Abstract. This paper presents the study of singularly perturbed differential equations of convection diffusion type with non-local boundary condition. The proposed numerical scheme is a combination of classical finite difference method for the initial boundary condition and nonstandard finite difference method for the differential equations at the interior points. Maximum absolute errors and rates of convergence for different values of perturbation parameter and mesh sizes are tabulated for the numerical examples considered. The method is shown to be first-order convergence independent of the perturbation parameter ε.

Keywords: singular perturbation, boundary value problem, non-standard fitted operator scheme, uniform convergence, non-local boundary condition.

AMS Subject Classification: 65L10; 65L11; 65L12; 65L20; 65L70.

1 Introduction

Singularly perturbed differential equations are typically characterized by the presence of a small positive parameter ε multiplying some or all of the highest order terms in differential equations. Such types of problems arise frequently

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in mathematical models of different areas of physics, chemistry, biology, engineering science, economics and even sociology. The well-known examples are heat transfer problem with large Peclet numbers, semiconductor theory, chemical reactor theory, reaction-diffusion process, theory of plates, optimal control, aerodynamics, seismology, oceanography, meteorology, geophysics and so on. Solutions of such equations usually possesses thin boundary or interior layers where the solutions change very rapidly, while away from the layers the solutions behaves regularly and change slowly. More details about these problems can be found in [25, 31, 32, 35] and also the literature cited there. Due to the presence of these boundary layers, the usual numerical treatment of singularly perturbed problems gives rise to computational difficulties. Standard numerical methods are not appropriate for practical applications when the perturbation parameter ε is sufficiently small. Therefore, it is necessary to develop suitable numerical methods that are uniformly convergent with respect to ε: To solve these problems, there are generally two types approaches, such as fitted operator methods that are reflect the nature of the solution in the boundary layers and fitted mesh methods which use layer-adapted meshes. In recent years, many authors have worked for solving singularly perturbed problems with one or two boundary layers using uniformly convergent numerical methods [17, 19, 24, 28, 29, 34]. Boundary value problems including nonlocal conditions which connect the values of the unknown solution at the boundary with values in the interior are known as nonlocal boundary value problems. The study of this kind of problems was initiated by Il'in and Miseev in [21, 22], motivated by the work of Bitsadze and Samarskii on nonlocal linear elliptic boundary value problems [4]. These problems have been used to represent mathematical models of a large number of phenomena, such as problems of semiconductors in electronics, the vibrations of a guy wire of a uniform cross-section, heat transfer problems, problems of hydromechanics, catalytic processes in chemistry and biology, the diffusion-drift model of semiconducting devices and some other physical phenomena [1, 20, 33]. The existence and uniqueness of the solutions of nonlocal boundary value problems have been studied by many authors [3, 23]. Some approaches for the numerical solution of singularly perturbed nonlocal boundary value problems have been proposed in [6,7,11,13,14,15,16,18] and [26]. Uniformly convergent numerical methods of order second and high for solving different singularly perturbed problems have been studied in [5,8,9,10,12,13,14,15,16,27] and [36]. To the best of our knowledge, the problem under consideration has not been done using nonstandard fitted finite difference method. Motivated by paper [8], we develop a uniformly convergent numerical method for solving singularly perturbed problem under consideration.

2 Definition of the problem

Consider the following singularly perturbed problem with non-local condition of the form

$$Ly(x) \equiv \varepsilon y''(x) + a(x)y'(x) = f(x), x \in \Omega, \tag{2.1}$$

with the given conditions

$$y'(0) = A/\varepsilon, \tag{2.2}$$

$$y(0) + \gamma y(l_1) = By(l) + d, l_1 \in \Omega, \tag{2.3}$$

where $0 < \varepsilon \ll 1$ is a small positive parameter, A, B, γ and d are given constants, l_1 and l are given real numbers, and $\Omega = (0, l)$ and $\bar{\Omega} = [0, l]$. We assume that $a(x) \geq a > 0$ and $f(x)$ are sufficiently smooth functions on $\bar{\Omega}$. Under these assumptions, singularly perterbed nonlocal Equations (2.1)–(2.3) possesses a unique solution indicating a boundary layer of exponential type at $x = 0$.

3 Properties of continuous solution

The following lemmas are necessary for the existence and uniqueness of the solution and for the problem to be well-posed [22].

Lemma 1. *(Continuous minimum principle)*
Assume that $v(x) \in C^2(\bar{\Omega})$ be any function satisfying $v(0) \geq 0$, $v(l) \geq 0$ and $Lv(x) \leq 0, \quad \forall x \in \Omega = (0, l)$. Then $v(x) > 0, \quad \forall x \in \bar{\Omega} = [0, l]$.

Proof. Let x^* be such that $v(x^*) = \min_{x \in [0,l]} v(x)$ and assume that $v(x^*) < 0$. Clearly $x^* \notin \{0, l\}$, therefore $v'(x^*) = 0$ and $v''(x^*) \geq 0$. Moreover, $Lv(x^*) = \varepsilon v''(x^*) + a(x^*)v'(x^*) \geq 0$, which is a contradiction. It follows that $v(x^*) > 0$ and thus $v(x) \geq 0, \quad \forall x \in [0, l]$. □

The uniqueness of the solution is implied by this minimum principle. Its existence follows trivially (as for linear problems, the uniqueness of the solution implies its existence). This principle is now applied to prove that the solution of Equations (2.1)–(2.3) is bounded. The following lemma shows the bound for the derivatives of the solution.

Lemma 2. *Let $a, f \in C[0, l]$ and $1 + \gamma - B \neq 0$. Then, the solution $y(x)$ of the Equations (2.1)–(2.3) and its derivative satisfy the following bounds:*

$$\|y\|_\infty \leq M, \tag{3.1}$$

where

$$M = m^{-1}[|d| + a^{-1}(|B| + |\gamma|)(|A| + \|f\|_1)] + a^{-1}(|A| + \|f\|_1), m = |1 + \gamma - B|,$$

and

$$|y^k(x)| \leq C(1 + \varepsilon^{-k} e^{\frac{-ax}{\varepsilon}}), \quad x \in \bar{\Omega}. \tag{3.2}$$

Proof. We first prove Equation (3.1). We can write Equation(2.1) in the form

$$\begin{aligned} y'(x) =& y'(0) e^{\frac{-1}{\varepsilon} \int_0^x a(\eta) d\eta} + \frac{1}{\varepsilon} \int_0^x f(\xi) e^{\frac{-1}{\varepsilon} \int_\xi^x a(\eta) d\eta} d\xi \\ =& \frac{A}{\varepsilon} e^{\frac{-1}{\varepsilon} \int_0^x a(\eta) d\eta} + \frac{1}{\varepsilon} \int_0^x f(\xi) e^{\frac{-1}{\varepsilon} \int_\xi^x a(\eta) d\eta} d\xi. \end{aligned} \tag{3.3}$$

Integrating Equation (3.3) from 0 to x, we get

$$\begin{aligned} y(x) =& y(0) + \frac{A}{\varepsilon}\int_0^{\tau} e^{\frac{-1}{\varepsilon}\int_0^{\tau} a(\eta)d\eta} d\tau + \frac{1}{\varepsilon}\int_0^x d\tau \int_0^{\tau} f(\xi) e^{\frac{-1}{\varepsilon}\int_{\xi}^{\tau} a(\eta)d\eta} d\tau \\ =& y(0) + \frac{A}{\varepsilon}\int_0^{x} e^{\frac{-1}{\varepsilon}\int_0^{\tau} a(\eta)d\eta} d\tau + \frac{1}{\varepsilon}\int_0^x d\xi f(\xi) \int_{\xi}^{x} e^{\frac{-1}{\varepsilon}\int_{\xi}^{\tau} a(\eta)d\eta} d\tau. \end{aligned} \tag{3.4}$$

Taking into account the boundary condition (2.3), we obtain

$$\begin{aligned} y(0) =& \frac{1}{1+\gamma-B}\{d + \frac{AB}{\varepsilon}\int_0^{l} e^{\frac{-1}{\varepsilon}\int_0^{\tau} a(\eta)d\eta} d\tau \\ &+ \frac{B}{\varepsilon}\int_0^{l} d\xi f(\xi) \int_{\xi}^{l} e^{\frac{-1}{\varepsilon}\int_{\xi}^{\tau} a(\eta)d\eta} d\tau - \frac{A\gamma}{\varepsilon}\int_0^{l_1} e^{\frac{-1}{\varepsilon}\int_0^{\tau} a(\eta)d\eta} d\tau \\ &- \frac{\gamma}{\varepsilon}\int_0^{l_1} d\xi f(\xi) \int_{\xi}^{l_1} e^{\frac{-1}{\varepsilon}\int_{\xi}^{\tau} a(\eta)d\eta} d\tau\}. \end{aligned} \tag{3.5}$$

From Equation (3.5) it follows that

$$\begin{aligned} |y(0)| \leq & m^{-1}\{|d| + \frac{|A||B|}{\varepsilon}\int_0^{l} e^{\frac{-a\tau}{\varepsilon}} d\tau + \frac{|B|}{\varepsilon}\int_0^{l} d\xi |f(\xi)| \int_{\xi}^{l} e^{\frac{-a(\tau-\xi)}{\varepsilon}} d\tau \\ &+ \frac{|A||\gamma|}{\varepsilon}\int_0^{l_1} e^{\frac{-a\tau}{\varepsilon}} d\tau + \frac{|\gamma|}{\varepsilon}\int_0^{l_1} d\xi |f(\xi)| \int_{\xi}^{l_1} e^{\frac{-a(\tau-\xi)}{\varepsilon}} d\tau\} \\ \leq & m^{-1}\{|d| + a^{-1}|A||B|(1-e^{\frac{-al}{\varepsilon}}) + a^{-1}|B|\int_0^{l} |f(\xi)|(1-e^{\frac{-a(l-\xi)}{\varepsilon}})d\xi \\ &+ a^{-1}|A||\gamma|(1-e^{\frac{-al_1}{\varepsilon}}) + a^{-1}|\gamma|\int_0^{l_1} |f(\xi)|(1-e^{\frac{-a(l_1-\xi)}{\varepsilon}})d\xi\} \\ \leq & m^{-1}\{|d| + a^{-1}|A||B| + a^{-1}|B|\int_0^{l} |f(\xi)d\xi + a^{-1}|A||\gamma| + a^{-1}|\gamma|\int_0^{l} |f(\xi)d\xi\} \\ \leq & m^{-1}\{|d| + a^{-1}|A||B| + a^{-1}|B|\|f\|_1 + a^{-1}|A||\gamma| + a^{-1}|\gamma|\|f\|_1\}. \end{aligned}$$

So, we obtain

$$|y(0)| \leq m^{-1}\{|d| + a^{-1}(|B|+|\gamma|)(|A|+\|f\|_1)\}. \tag{3.6}$$

From (3.4) we see that

$$\begin{aligned} |y(x)| \leq & |y(0)| + \frac{A}{\varepsilon}\int_0^{x} e^{-(\frac{1}{\varepsilon})\int_0^{\tau} a(\eta)d\eta} d\tau + \frac{1}{\varepsilon}\int_0^{x} d\xi |f(\xi)| \int_{\xi}^{x} e^{-(\frac{1}{\varepsilon})\int_{\xi}^{\tau} a(\eta)d\eta} d\tau \\ \leq & |y(0)| + |A|a^{-1}(1-e^{\frac{-al}{\varepsilon}}) + a^{-1}\int_0^{l} |f(\xi)|(1-e^{\frac{-a(l-\xi)}{\varepsilon}})d\xi \\ \leq & |y(0)| + |A|a^{-1} + a^{-1}\int_0^{l} |f(\xi)|d\xi, \end{aligned}$$

which, together with (3.6), leads to (3.1).

Next, from (3.3) it follows that

$$|y'(x)| \leq \frac{|A|}{\varepsilon} e^{-\frac{1}{\varepsilon}\int_0^x a(\eta)d\eta} + \frac{1}{\varepsilon}\int_0^x |f(\xi)| e^{-\frac{1}{\varepsilon}\int_\xi^x a(\eta)d\eta} d\xi \leq \frac{|A|}{\varepsilon} e^{\frac{-ax}{\varepsilon}}$$
$$+ a^{-1} \max_{0 \leq t \leq x} |f(t)|(1 - e^{\frac{-ax}{\varepsilon}}) \leq \frac{|A|}{\varepsilon} e^{\frac{-ax}{\varepsilon}} + a^{-1}\|f\|_\infty$$
$$\leq C\varepsilon^{-1} e^{\frac{-ax}{\varepsilon}} + C \leq C(1 + \varepsilon^{-1} e^{\frac{-ax}{\varepsilon}}).$$

Similarly,

$$|y''(x)| \leq C(1+\varepsilon^{-2} e^{\frac{-ax}{\varepsilon}}),\ |y^3(x)| \leq C(1+\varepsilon^{-3} e^{\frac{-ax}{\varepsilon}}),\ |y^4(x)| \leq C(1+\varepsilon^{-4} e^{\frac{-ax}{\varepsilon}}).$$

In general, for $k = 1, 2, 3, 4$.

$$|y^k(x)| \leq C(1 + \varepsilon^{-k} e^{\frac{-ax}{\varepsilon}}),$$

which implies Equation (3.2) and completes the proof of the lemma. □

4 Formulation of the method

The theoretical basis of non-standard discrete numerical method is based on the development of exact finite difference method. The author [30] presented techniques and rules for developing non-standard finite difference methods for different problem types. In [30], to develop a discrete scheme, denominator function for the discrete derivatives must be expressed in terms of more complicated functions of step sizes than those used in the standard procedures. These complicated functions constitute a general property of the schemes, which is useful while designing reliable schemes for such problems.

For the problem of the form in Equations (2.1)–(2.3), in order to construct exact finite difference scheme, we follow the procedures used in [2]. Let us consider the following singularly perturbed differential equation of the form

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x). \tag{4.1}$$

The constant coefficient homogeneous problems corresponding to Eq. (4.1)

$$\varepsilon y''(x) + ay'(x) + by(x) = 0, \tag{4.2}$$
$$\varepsilon y''(x) + ay'(x) = 0, \tag{4.3}$$

where $a(x) \geq a$ and $b(x) \geq b$. Two linear independent solutions of Equation (4.2) are $\exp(\lambda_1 x)$ and $\exp(\lambda_2 x)$, where

$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4\varepsilon b}}{2\varepsilon}.$$

We discretized the domain $[0, 1]$ using uniform mesh length $\varDelta x = h$ such that, $\varOmega^N = \{x_i = x_0 + ih, 1, 2, ..., N, x_0 = 0, x_N = 1, h = \frac{1}{N}\}$, where N denotes the number of mesh points. We denote the approximate solution to $y(x)$ at grid point x_i by Y_i. Now our main objective is to calculate a difference equation which has the same general solution as the differential equation Equation (4.2) has at the grid point x_i given by $Y_i = A_1 \exp(\lambda_1 x_i) + A_2 \exp(\lambda_2 x_i)$. Using the theory of difference equations and the procedures used in [2], we have

$$\det \begin{bmatrix} Y_{i-1} & \exp(\lambda_1 x_{i-1}) & \exp(\lambda_2 x_{i-1}) \\ Y_i & \exp(\lambda_1 x_i)) & \exp(\lambda_2 x_i) \\ Y_{i+1} & \exp(\lambda_1 x_{i+1}) & \exp(\lambda_2 x_{i+1}) \end{bmatrix} = 0. \tag{4.4}$$

Simplifying Equation (4.4), we obtain

$$-\exp\left(-\frac{ah}{2\varepsilon}\right) Y_{i-1} + 2\cosh\left(\frac{h\sqrt{a^2-4\varepsilon b}}{2\varepsilon}\right) Y_i - \exp\left(\frac{ah}{2\varepsilon}\right) Y_{i+1} = 0, \tag{4.5}$$

which is an exact difference scheme for Equation (4.2).
After doing the arithmetic manipulation and rearrangement on Equation (4.5), for the constant coefficient problem (4.3), we get

$$\varepsilon \frac{Y_{i-1} - 2Y_i + Y_{i+1}}{\frac{h\varepsilon}{a}(\exp(\frac{ah}{\varepsilon}) - 1)} + a\frac{Y_{i+1} - Y_i}{h} = 0.$$

The denominator function becomes $\Psi^2 = \frac{h\varepsilon}{a}\left(\exp\left(\frac{ha}{\varepsilon}\right) - 1\right)$. Adopting this denominator function for the variable coefficient problem, we write it as

$$\Psi_i^2 = \frac{h\varepsilon}{a_i}\left(\exp\left(\frac{ha_i}{\varepsilon}\right) - 1\right),$$

where Ψ_i^2 is the function of ε, a_i and h. By using the denominator function Ψ_i^2 in to the main scheme, we obtain the difference scheme as

$$L_\varepsilon^N Y_i \equiv \varepsilon \frac{Y_{i+1} - 2Y_i + Y_{i-1}}{\Psi_i^2} + a_i \frac{Y_{i+1} - Y_i}{h} = f_i.$$

This can be written as three term recurrence relation as

$$E_i Y_{i-1} + F_i Y_i + G_i Y_{i+1} = H_i, i = 1, 2, ..., N-1, \tag{4.6}$$

where $E_i = \frac{\varepsilon}{\Psi_i^2}$, $F_i = \frac{-2\varepsilon}{\psi_i^2} - \frac{a_i}{h}$, $G_i = \frac{\varepsilon}{\Psi_i^2} + \frac{a_i}{h}$ and $H_i = f_i$.

Since the problem involves of non-local boundary conditions, we considered the following cases, to obtain two equations at each end conditions.

For $i = 0$, Equation (4.6) becomes

$$E_0 Y_{-1} + F_0 Y_0 + G_0 Y_1 = H_0. \tag{4.7}$$

Here, in Equation (4.7) the term Y_{-1} is out of the domain, so that using Equation (2.2) we have

$$Y'(0) = \frac{\mu_0}{\varepsilon} = \frac{Y_1 - Y_{-1}}{2h} \Rightarrow Y_{-1} = Y_1 - 2hY'(0). \tag{4.8}$$

Putting Equation (4.8) into (4.7) gives

$$E_0 Y_0 + (E_0 + G_0) Y_1 = H_0 + 2hE_0 Y'(0). \tag{4.9}$$

For $i = N$, (4.6) becomes

$$E_N Y_{N-1} + F_N Y_N + G_N Y_{N+1} = H_N. \tag{4.10}$$

Here, in Equation (4.10) the term Y_{N+1} is out of the domain, so that using (2.3) we have

$$y_{N+1} = \frac{y_0}{B} + \frac{\gamma Y_{l_1}}{B} - \frac{d}{B}. \tag{4.11}$$

Putting Equation (4.11) into (4.10) gives

$$\frac{G_N}{B} Y_0 + \frac{G_N \gamma}{B} Y_{l_1} = H_N + \frac{G_N d}{B}. \tag{4.12}$$

Therefore, Equation (2.1) with the given boundary conditions (2.2) and (2.3), can be solved using the schemes in Equations (4.6), (4.9) and (4.12) which gives $N \times N$ system of algebraic equations.

5 Uniform convergence analysis

In this section, we need to show the discrete scheme in Equation (4.6), satisfy the discrete minimum principle, uniform stability estimates, and uniform convergence.

Lemma 3. *(Discrete Minimum Principle) Let Y_i be any mesh function that satisfies $Y_0 \geq 0$, $Y_N \geq 0$ and $L_\varepsilon^N Y_i \leq 0$, $i = 1, 2, 3, ..., N-1$, then $Y_i \geq 0$, for $i = 0, 1, 2, ..., N$.*

Proof. The proof is by contradiction. Let j be such that $Y_j = \min_i Y_i$ and suppose that $Y_j \leq 0$. Clearly, $j \notin \{0, N\}$. $Y_{j+1} - Y_j \geq 0$ and $Y_j - Y_{j-1} \leq 0$. Therefore,

$$\begin{aligned} L_\varepsilon^N Y_j =& \varepsilon \left(\frac{Y_{j+1} - 2Y_j + Y_{j-1}}{\Psi_i^2} \right) + a_j \left(\frac{Y_{j+1} - Y_j}{h} \right) \\ =& \frac{\varepsilon}{\Psi_i^2} (Y_{j+1} - 2Y_j + Y_{j-1}) + \frac{a_j}{h} (Y_{j+1} - Y_j) \\ =& \frac{\varepsilon}{\Psi_i^2} ((Y_{j+1} - Y_j) - (Y_j - Y_{j-1})) + \frac{a_j}{h} (Y_{j+1} - Y_j) \geq \ \ 0, \end{aligned}$$

where the strict inequality holds if $Y_{j+1} - Y_j > 0$. This is a contradiction and therefore $Y_j \geq 0$. Since j is arbitrary, we have $Y_i \geq 0$, for $i = 0, 1, 2, ..., N$. From the discrete minimum principle, we obtain an $\varepsilon-$ uniform stability property for the operator L_ε^N. □

Lemma 4. *(Uniform stability estimate) If ϕ_j is any mesh function such that*

$$\phi_0 = \phi_N = 0. \text{ Then, } | \phi_j | \leq \frac{1}{a} \max_{1 \leq i \leq N-1} | L_\varepsilon^N \phi_i |, \ \ j = 0, 1, 2, ..., N.$$

Proof. We introduce two mesh functions ψ_j^+, ψ_j^- defined by

$$\psi_j^{\pm} = \left(\frac{1}{a}\max_{1\le i\le N-1} \mid L_\varepsilon^N \phi_i \mid\right) \pm \phi_j.$$

It follows that

$$\begin{aligned}\psi^{\pm}(0) = \left(\frac{1}{a}\max_{1\le i\le N-1} \mid L_\varepsilon^N \phi_i \mid\right) \pm \phi_0 = \frac{1}{a}\max_{1\le i\le N-1} \mid \varepsilon\delta^2\phi_i + a_i D^+\phi_i \mid \pm\phi_0 \\ = \frac{1}{a}\max_{1\le i\le N-1} \mid \varepsilon\delta^2\phi_i + a_i D^+\phi_i \mid \ge \ 0,\end{aligned}$$

and

$$\begin{aligned}\psi^{\pm}(N) = \left(\frac{1}{a}\max_{1\le i\le N-1} \mid L_\varepsilon^N \phi_i \mid\right) \pm \phi_N = \frac{1}{a}\max_{1\le i\le N-1} \mid \varepsilon\delta^2\phi_i + a_i D^+\phi_i \mid \pm\phi_N \\ = \frac{1}{a}\max_{1\le i\le N-1} \mid \varepsilon\delta^2\phi_i + a_i D^+\phi_i \mid \ge \ 0,\end{aligned}$$

and, for all $j = 1, 2, ..., N-1$,

$$L_\varepsilon^N \psi_j^{\pm} = \left(\frac{1}{a}\max_{1\le i\le N-1} \mid L_\varepsilon^N \phi_i \mid\right) \pm L_\varepsilon^N \phi_j \le \ 0.$$

From discrete minimum principle, if $\psi_0 \ge 0, \psi_N \ge 0$ and $L_\varepsilon^N \psi_j \le 0, \ \forall\, 0 < j < N$ then, $\psi_j^{\pm} \ge 0, \ \ 0 \le j \le N.$ □

We proved above the discrete operator L_ε^N satisfy the minimum principle. Next, we analyze the uniform convergence analysis. Using Taylor series expansion, the bound for $y(x_{i-1})$ and $y(x_{i+1})$ at x_i as

$$\begin{cases} y(x_{i-1}) = y(x_i) - hy'(x_i) + \frac{h^2}{2!}y''(x_i) - \frac{h^3}{3!}y^{(3)}(x_i) + \frac{h^4}{4!}y^{(4)}(x_i) + O(h^5), \\ y(x_{i+1}) = y(x_i) + hy'(x_i) + \frac{h^2}{2!}y''(x_i) + \frac{h^3}{3!}y^{(3)}(x_i) + \frac{h^4}{4!}y^{(4)}(x_i) + O(h^5). \end{cases}$$

We obtain the bound for

$$\begin{cases} |D^+D^-y(x_i)| \le C|y''(x_i)|, \\ |y''(x_i) - D^+D^-y(x_i)| \le Ch^2|y^{(4)}(x_i)|. \end{cases}$$

Similarly, for the first derivative term,

$$|y'(x_i) - D^+y(x_i)| \le Ch|y''(x_i)|.$$

Theorem 1. *Let the coefficients functions $a(x)$ and the source function $f(x)$ in Equations (2.1)–(2.3) of the domain Ω be sufficiently smooth, so that $y(x) \in C^4[0,1]$. Then, the discrete solution Y_i satisfies*

$$|L^N(y_i - Y_i)| \le Ch\Big(1 + \sup_{x\in(0,1)}\Big(\exp\big(\frac{-ax_i}{\varepsilon}\big)/\varepsilon^3\Big)\Big).$$

Proof. We consider the truncation error discretization as

$$\begin{aligned}
|L^N(y_i - Y_i)| &= |L^N y_i - L^N Y_i| \\
&\leq C|\varepsilon y_i'' + a_i y_i' - \{\varepsilon \frac{D^+D^-h^2}{\Psi_i^2} y_i + a_i D^+ y_i\}| \\
&\leq C|\varepsilon \left(y_i'' - \frac{D^+D^-h^2}{\Psi_i^2} y_i \right) + a_i(y_i' - D^+ y_i)| \\
&\leq C\varepsilon |y_i'' - D^+D^- y_i| + C\varepsilon |\left(h^2/\Psi_i^2 - 1\right) D^+D^- y_i| + Ch|y_i''| \\
&\leq C\varepsilon h^2 |y_i^{(4)}| + Ch|y_i''| + Ch|y_i''| \leq C\varepsilon h^2 |y_i^{(4)}| + Ch|y_i''|.
\end{aligned}$$

We used the estimate $\varepsilon |\frac{h^2}{\Psi^2} - 1| \leq Ch$ which can be derived from Equation (4.2). Indeed, define $\rho = \dfrac{a_i h}{\varepsilon}, \rho \in (0, \infty)$. Then,

$$\varepsilon \left| \frac{h^2}{\Psi^2} - 1 \right| = a_i h \left| \frac{1}{\exp(\rho) - 1} - \frac{1}{\rho} \right| =: a_i h Q(\rho).$$

By simplifying and writing explicitly, we obtain

$$Q(\rho) = \frac{\exp(\rho) - \rho - 1}{\rho(\exp(\rho) - 1)},$$

and we obtain the limit is bounded as

$$\lim_{\rho \longrightarrow 0} Q(\rho) = \frac{1}{2}, \quad \lim_{\rho \longrightarrow \infty} Q(\rho) = 0.$$

Hence, for all $\rho \in (0, \infty)$ we have $Q(\rho) \leqslant C$. So, the error estimate in the discretization is bounded as

$$|L^N(y_i - Y_i)| \leqslant C\varepsilon h^2 |y_i^{(4)}| + Ch|y_i''|. \tag{5.1}$$

From Equation (5.1) and boundedness of derivatives of solution in Lemma 2, we obtain

$$\begin{aligned}
&|L^N(y(x_i) - Y_i)| \\
&\leq C\varepsilon h^2 \left| \left(1 + \varepsilon^{-4} \exp\left(\frac{-a x_i}{\varepsilon}\right)\right) \right| + Ch \left| \left(1 + \varepsilon^{-2} \exp\left(\frac{-a x_i}{\varepsilon}\right)\right) \right| \\
&\leq Ch^2 \left| \left(\varepsilon + \varepsilon^{-3} \exp\left(\frac{-a x_i}{\varepsilon}\right)\right) \right| + Ch \left| \left(1 + \varepsilon^{-2} \exp\left(\frac{-a x_i}{\varepsilon}\right)\right) \right| \\
&\leq Ch \left(1 + \sup_{x \in (0,1)} \left(\frac{\exp(\frac{-a x_i}{\varepsilon})}{\varepsilon^3} \right) \right),
\end{aligned}$$

since $\varepsilon^{-3} > \varepsilon^{-2}$. □

Most of the time during analysis, one encounters with exponential terms involving divided by the power function in ε which are always the main cause of worry. For their careful consideration while proving the ε -uniform convergence, we prove as follows.

Lemma 5. *For a fixed mesh and for $\varepsilon \to 0$, it holds*

$$\lim_{\varepsilon\to 0} \max_{1\le i\le N-1} \left(\frac{\exp(\frac{-ax_i}{\varepsilon})}{\varepsilon^m} \right) = 0, \quad m = 1, 2, 3, \ldots,$$

$$\lim_{\varepsilon\to 0} \max_{1\le i\le N-1} \left(\frac{\exp(\frac{-a(1-x_i)}{\varepsilon})}{\varepsilon^m} \right) = 0, \quad m = 1, 2, 3, \ldots,$$

where $x_i = ih, h = \frac{1}{N}, i = 1, 2, ..., N-1.$

Proof. Consider the partition $[0,1] := \{0 = x_0 < x_1 < .... < x_{N-1} < x_N = 1\}$ for the interior grid points, we have

$$\max_{1\le i\le N-1} \frac{\exp\left(\frac{-ax_i}{\varepsilon}\right)}{\varepsilon^m} \le \frac{\exp\left(\frac{-ax_1}{\varepsilon}\right)}{\varepsilon^m} = \frac{\exp\left(\frac{-ah}{\varepsilon}\right)}{\varepsilon^m} \quad \text{and}$$

$$\max_{1\le i\le N-1} \frac{\exp\left(\frac{-a(1-x_i)}{\varepsilon}\right)}{\varepsilon^m} \le \frac{\exp\left(\frac{-a(1-x_{N-1})}{\varepsilon}\right)}{\varepsilon^m} = \frac{\exp\left(\frac{-ah}{\varepsilon}\right)}{\varepsilon^m},$$

as $x_1 = 1 - x_{N-1} = h$. Then by the application of L'Hospital's rule m times gives

$$\lim_{\varepsilon\longrightarrow 0} \frac{\exp(-ah/\varepsilon)}{\varepsilon^m} = \lim_{r=\frac{1}{\varepsilon}\longrightarrow\infty} \frac{r^m}{\exp(ahr)} = \lim_{r=\frac{1}{\varepsilon}\longrightarrow\infty} \frac{m!}{(ah)^m \exp(ahr)} = 0.$$

Hence, the proof is completed. □

Theorem 2. *Under the hypothesis of boundness of discrete solution (i.e., it satisfies the discrete minimum principle), Lemma 5 and Theorem 1, the discrete solution satisfies the following bound.*

$$\sup_{0\le\varepsilon\le 1} \max_i |y_i - Y_i| \le CN^{-1}.$$

Proof. Results from boundness of solution, Lemma 5 and Theorem 1 gives the required estimates. Hence the proof. □

6 Numerical example and results

To validate the established theoretical results, we perform numerical experiments using the model problems of the form in Equations (2.1)–(2.3) from [8].

Example 1. Consider the model singularly perturbed boundary value problem:

$$\varepsilon y''(x) + 2y'(x) = (\varepsilon - 2)e^{-x}, \;\; 0 < x < 1,$$

subject to the boundary conditions

$$y'(0) = \frac{1}{\varepsilon}, \;\; \text{and} \;\; y(0) + \frac{1}{3} y\left(\frac{1}{4}\right) + y(1) = 1.$$

Its exact solution is

$$y(x) = d_1 + d_2 e^{-\frac{2x}{\varepsilon}} + e^{-x},$$

where $d_1 = -\frac{3}{7}[e^{-1} + \frac{1}{3}e^{-\frac{1}{4}} + (1 + e^{-\frac{2x}{\varepsilon}} + \frac{1}{3}e^{-\frac{1}{2\varepsilon}})d_2]$, $d_2 = -\frac{1+\varepsilon}{2}$.

Example 2. Consider the model singularly perturbed boundary value problem

$$\varepsilon y''(x) + 2y'(x) = (\varepsilon - 2)e^{-x}, \quad 0 < x < 1,$$

subject to the boundary conditions

$$y'(0) = \frac{1}{\varepsilon}, \quad \text{and} \quad y(0) + \frac{2}{3}y\left(\frac{3}{4}\right) + y(1) = 1.$$

Its exact solution is

$$y(x) = d_1 + d_2 e^{-\frac{2x}{\varepsilon}} + e^{-x},$$

where $d_1 = -\frac{3}{8}[e^{-1} + \frac{2}{3}e^{-\frac{3}{4}} + (1 + e^{-\frac{2x}{\varepsilon}} + \frac{2}{3}e^{-\frac{3}{2\varepsilon}})d_2]$, $d_2 = -\frac{1+\varepsilon}{2}$.

We define the pointwise absolute errors E_ε^N and the computed ε-uniform maximum pointwise error E^Nas follows

$$E_\varepsilon^N = \|Y - y\|, \quad E^N = \max_\varepsilon E_\varepsilon^N,$$

where Y is the numerical approximation to y for various values of N and ε. We also define the computed ε-uniform convergence rate

$$R^N = \log_2(E^N/E^{2N}).$$

Figure 1 indicates the behavior of the numerical solution for Examples 1 and 2 respectively, and display an existing boundary layers. We observed that for small values of ε the solution of test problem exhibit a boundary layer at $x = 0$.

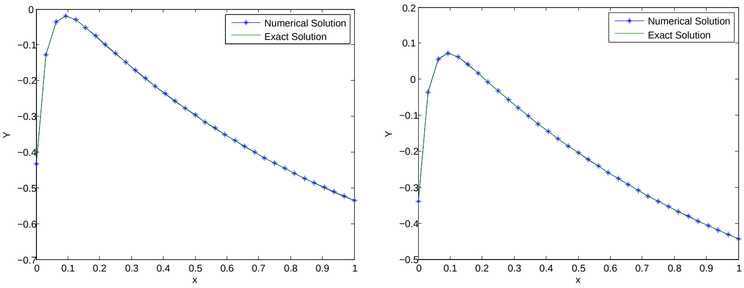

Figure 1. The behavior of the numerical solution at $\varepsilon = 2^{-4}$ and $N = 32$ of Example 1 and Example 2 respectively.

We can also observe from Figure 2 that, the point wise error are decreased as the number of mesh points increase.

Tables 1 and 2 indicate ε-uniform maximum point wise error E^N and the rate of convergence R^N for both Examples 1 and 2 respectively.

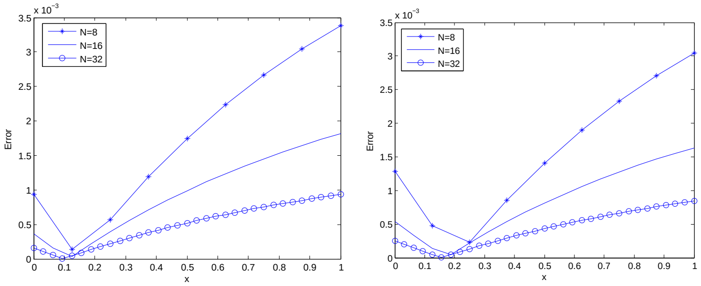

Figure 2. Point wise absolute error of Example 1 and Example 2 respectively at $\varepsilon = 10^{-20}$ with different mesh points N.

Table 1. Maximum absolute error and rate of convergence for different values of ε and number of mesh points, N with nonstandard FDM for Example 1.

ε	N=16	N=32	N=64	N=128	N=256
10^{-4}	4.5340e-03	2.2114e-03	1.0919e-04	5.4256e-04	2.7046e-04
10^{-8}	4.5342e-03	2.2115e-03	1.0920e-03	5.4258e-04	2.7044e-04
10^{-12}	4.5342e-03	2.2115e-03	1.0920e-03	5.4258e-04	2.7044e-04
10^{-16}	4.5342e-03	2.2115e-03	1.0920e-03	5.4258e-04	2.7044e-04
10^{-20}	4.5342e-03	2.2115e-03	1.0920e-03	5.4258e-04	2.7044e-04
E^N	4.5342e-03	2.2115e-03	1.0920e-03	5.4258e-04	2.7044e-04
R^N	1.0358	1.0181	1.0091	1.0045	

Table 2. Maximum absolute error and rate of convergence for different values of ε and number of mesh point, N with NSFDM for Example 2.

ε	N=16	N=32	N=64	N=128	N=256
10^{-4}	4.1080e-03	2.0018e-03	9.8802e-04	4.9081e-04	2.4426e-04
10^{-8}	4.1082e-03	2.0019e-03	9.8807e-04	4.9083e-04	2.4463e-04
10^{-12}	4.1082e-03	2.0019e-03	9.8807e-04	4.9083e-04	2.4462e-04
10^{-16}	4.1082e-03	2.0019e-03	9.8807e-04	4.9083e-04	2.4462e-04
10^{-20}	4.1082e-03	2.0019e-03	9.8807e-04	4.9083e-04	2.4462e-04
E^N	4.1082e-03	2.0019e-03	9.8807e-04	4.9083e-04	2.4462e-04
R^N	1.0371	1.0187	1.0094	1.0047	

In Figure 3, the log-log plot of the maximum absolute error verses N are given for singular perturbation parameter ranging from $\varepsilon = 10^{-4}$ to 10^{-20}. In this figure the graphs are parallel and overlapped as ε goes small, this indicate that the proposed scheme converges independent of the values of perturbation parameter.

The comparison of maximum absolute error and rate of convergence for Examples 1 and 2 are given in Tables 3 and 4 respectively, and indicate that, the developed numerical method is more accurate parameter uniform than results in [8].

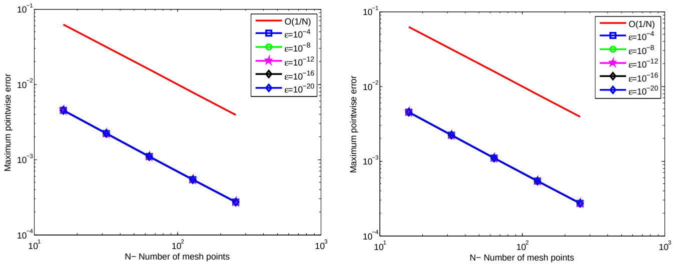

Figure 3. ε-uniform convergence with nonstandard fitted operator method in log-log scale for Example 1 and Example 2 respectively.

Table 3. Comparison of ε-uniform maximum absolute errors and rate of convergence for Example 1.

	N=16	N=32	N=64	N=128	N=256
Present method					
E^N	4.5342e-03	2.2115e-03	1.0920e-03	5.4258e-03	2.7044e-04
R^N	1.0358	1.0181	1.0091	1.0045	
Method in [8]					
E^N	0.0104076	0.0051770	0.0025818	0.0012892	0.000644
R^N	1.01	1.00	1.00	1.00	

Table 4. Comparison of ε-uniform maximum absolute errors and rate of convergence for Example 2.

	N=16	N=32	N=64	N=128	N=256
Present method					
E^N	4.1082e-03	2.0019e-03	9.8807e-04	4.9083e-04	2.4462e-04
R^N	1.0371	1.0187	1.0094	1.0047	
Method in [8]					
E^N	0.0116499	0.0057949	0.0028900	0.0014431	0.0007211
R^N	1.01	1.00	1.00	1.00	

7 Discussion and conclusions

This study introduces uniformly convergent numerical method based on nonstandard finite difference method for solving singularly perturbed boundary value problems with non-local boundary conditions. The behavior of the continuous solution of the problem is studied and shown that it satisfies the continuous stability estimate and the derivatives of the solution are also bounded. The numerical scheme is developed on uniform mesh. The nonlocal boundary condition is treated using finite difference formula; and the results are compared accordingly. The stability of the developed scheme is established and its uniform convergence is proved. To validate the applicability of the method, two model problems are considered for numerical experimentation for different values of the perturbation parameter and mesh points. Unlike other fitted operator finite difference methods constructed in standard ways, the method that we presented in this paper is fairly simple to construct. Moreover, the method is more accurate and gives good result where existing numerical methods fails (That is for the values where the perturbation parameter, ε is much less than the mesh size, h).

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