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Existence Results for Fractional p-Laplacian Systems via Young Measures

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Abstract. In this paper, we show the existence result of the following fractional p-Laplacian system

$$(-\varDelta)_p^s u = f(x,u) \quad \text{in } \varOmega, \;\; u = 0 \quad \text{in } \mathbb{R}^n \backslash \varOmega,$$

for a given datum f. The existence of weak solutions is obtained by using the theory of Young measures.

Keywords: fractional p-Laplacian system, weak solution, Galerkin method, Young measure.

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1 Introduction and main result

Recently, a great deal of attention has been paid to the study of problems involving fractional and nonlocal operators. This type of problems arises in continuum mechanics, population dynamics, game theory and in many other different applications. The literature on nonlocal operators and their applications is very interesting and large. We refer to [1,10,11,12,13,18,19,20,25] and other references therein.

In this paper, we are interested in the existence of solutions for the fractional p-Laplacian system of the form:

$$(-\varDelta)_p^s u = f(x,u) \quad \text{in } \varOmega, \;\; u = 0 \quad \text{in } \mathbb{R}^n \backslash \varOmega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded set with Lipschitz boundary $\partial\Omega$, $u : \Omega \to \mathbb{R}^m$, $m \in \mathbb{N}$, is the unknown vector-valued function and f is a given datum assumed to satisfy some conditions (see below). The term $(\Delta)_p^s u$ in (1.1) is the fractional p-Laplacian operator which will be detailed in Section 2.

Note that problems of type (1.1) have been treated in several papers. For example, Qui and Xiang [22] proved the existence of nonnegative solutions by using Leray-Schauder's nonlinear altenrative. When $p = 2$, problem (1.1) reduces to the fractional Laplacian problem

$$(-\Delta)^s u = f(x,u) \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \mathbb{R}^n \backslash \Omega. \tag{1.2}$$

In [24], the authors get the existence of nontrivial weak solutions of problem (1.2) by using the mountain pass theorem. See also [2, 18, 20] for more details and results.

In the present paper, we study the existence of weak solutions for problem (1.1) involving nonlocal fractional operator by using the tool of Young measures. To the best of our knowledge, problem (1.1) has never been studied by the theory Young measures. We refer the reader to see [3, 4, 5, 6, 7, 8] where we have applied such a theory for some quasilinear elliptic systems.

In this paper, we suppose that $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is a Carathéodory function satisfying:
(F1) there exist $c > 0$ and $0 \le \beta < p - 1$ such that

$$|f(x,\xi)| \le d(x) + c|\xi|^\beta \quad \text{for all } \xi \in \mathbb{R}^m \text{ and a.e. } x \in \Omega,$$

where $d \in L^{p'}(\Omega)$, with $d \ge 0$ a.e. in Ω.

We first give the definition of weak solutions for problem (1.1).

Definition 1. We say that $u \in W_0$ is a weak solution of the problem (1.1) for the datum f if

$$\int\int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+ps}} dxdy = \int_\Omega f(x,u)\varphi(x)dx$$

for any $\varphi \in W_0$, where W_0 and Q will be introduced in Section 2.

Now we are in a position to state the main result as follows:

Theorem 1. *Suppose that the assumption (F1) is satisfied. Then there exists a weak solution for problem* (1.1).

2 Preliminaries

In this section, we first give some basic results of fractional Sobolev space that will be used in the sequel (see [11, 14, 16, 20]). To this end, a brief review on Young measure will be presented (see [9, 15, 17]).

Let $0 < s < 1 < p < \infty$ and p_s^* the fractional critical exponent

$$p_s^* = \begin{cases} np/(n-ps) & \text{if } ps < n, \\ \infty & \text{if } ps \ge n. \end{cases}$$

The fractional p-Laplacian operator $(-\Delta)_p^s u$ is defined as follows

$$(-\Delta)_p^s u(x) = P.V. \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+ps}} dy, \quad x \in \mathbb{R}^n,$$

where $x \in \mathbb{R}^n$, P.V. is a commonly used abbreviation for "in the principal value sense".

In the following, we denote $Q = \mathbb{R}^{2n} \backslash \mathcal{O}$, where $\mathcal{O} = \complement(\Omega) \times \complement(\Omega) \subset \mathbb{R}^{2n}$, and $\complement(\Omega) = \mathbb{R}^n \backslash \Omega$. W is a linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ such that the restriction to Ω of any function u in W belongs to $L^p(\Omega;\mathbb{R}^m)$ and

$$\int\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy < \infty.$$

The space W is equipped with the norm

$$\|u\|_W = \|u\|_{L^p(\Omega;\mathbb{R}^m)} + \Big(\int\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy\Big)^{\frac{1}{p}}.$$

We will work in the closed linear space

$$W_0 = \big\{u \in W : \ u(x) = 0 \quad \text{a.e. in } \mathbb{R}^n \backslash \Omega\big\},$$

equipped with the norm

$$\|u\|_{W_0} := [u]_{sp} = \Big(\int\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy\Big)^{\frac{1}{p}}.$$

Then $(W_0, \|.\|_{W_0})$ is a uniformly convex Banach space (see [26]). Moreover, $\mathcal{C}_0^\infty(\Omega;\mathbb{R}^m)$ is dense in W_0 (see [16]). It is readily seen that the embedding $W_0 \hookrightarrow L^\theta(\Omega;\mathbb{R}^m)$ is continuous for all $1 \leq \theta \leq p_s^*$, and compact for all $1 \leq \theta < p_s^*$ (see [14]). The dual space of $(W_0, \|.\|_{W_0})$ is denoted by $(W_0^*, \|.\|_{W_0^*})$.

In the last step of the existence proof we use the Young measure properties. In the following $\mathcal{C}_0(\mathbb{R}^m)$ denote the space of continuous real-valued functions on $\mathbb{R}^m$ with compact support with respect to the $\|.\|_\infty$-norm. Its dual $\mathcal{M}(\mathbb{R}^m)$ is the space of signed Radon measures with finit mass. The related duality pairing is given by

$$\langle \nu, \varphi \rangle = \int_{\mathbb{R}^m} \varphi(\lambda) d\nu(\lambda).$$

Definition 2. [15] Let $\{z_j\}_{j\geq 1}$ be a bounded sequence in $L^\infty(\Omega;\mathbb{R}^m)$. Then there exists a subsequence $\{z_k\} \subset \{z_j\}$ and a Borel probability measure ν_x on $\mathbb{R}^m$ for a.e. $x \in \Omega$, such that for almost each $\varphi \in \mathcal{C}(\mathbb{R}^m)$ we have

$$\varphi(z_k) \rightharpoonup^* \overline{\varphi} \quad \text{weakly in } L^\infty(\Omega),$$

where $\overline{\varphi}(x) = \langle \nu_x, \varphi \rangle = \int_{\mathbb{R}^m} \varphi(\lambda) d\nu_x(\lambda) \quad$ for a.e. $x \in \Omega$.

The fundamental theorem on Young measure can be stated in the following lemma:

Lemma 1. *[17] Let $\Omega \subset \mathbb{R}^n$ be Lebesgue measurable, let $K \subset \mathbb{R}^m$ be closed, and let $z_j : \Omega \to \mathbb{R}^m$, $j \in \mathbb{N}$, be a sequence of Lebesgue measurable functions satisfying $z_j \to K$ in measure as $j \to \infty$, i.e., given any open neighbourhood U of K in $\mathbb{R}^m$*

$$\lim_{j\to\infty} |\{x \in \Omega : \ z_j(x) \notin U\}| = 0.$$

Then there exists a subsequence z_k and a family $\{\nu_x\}_{x\in\Omega}$ of non-negative Radon measures on $\mathbb{R}^m$, such that
(i) $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^m)} := \int_{\mathbb{R}^m} d\nu_x(\lambda) \le 1$ for almost $x \in \Omega$.
(ii) $\varphi(z_k) \rightharpoonup^* \overline{\varphi}$ weakly in $L^\infty(\Omega)$ for all $\mathcal{C}_0(\mathbb{R}^m)$, where $\overline{\varphi} = \langle \nu_x, \varphi\rangle$.
(iii) If for all $R > 0$

$$\lim_{L\to\infty} \sup_{k\in\mathbb{N}} \big|\{x \in \Omega \cap B_R(0) : \ |z_k(x)| \ge L\}\big| = 0, \tag{2.1}$$

then $\|\nu_x\| = 1$ for almost every $x \in \Omega$, and for any measurable $\Omega' \subset \Omega$ we have $\varphi(z_k) \rightharpoonup \overline{\varphi} = \langle \nu_x, \varphi\rangle$ weakly in $L^1(\Omega')$ for continuous function φ provided the sequence $\varphi(z_k)$ is weakly precompact in $L^1(\Omega')$.

3 Proof of the main result

In this section, we prove the existence of solutions to problem (1.1). Our method is based on the Galerkin approximation and again the tool of Young measures.

Let $V_1 \subset V_1 \subset ... \subset V_2 \subset W_0$ be a sequence of finite dimensional subspaces with the property that $\underset{k\ge1}{\cup} V_k$ is dense in W_0. Note that (V_k) exists since W_0 is a uniformly convex Banach space, thus separable. To construct the approximating solutions, we define the operator $T(u) : W_0 \to W_0^*$ by

$$\langle T(u), \varphi\rangle = \int\int_Q \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+ps}} dxdy - \int_\Omega f(x, u)\varphi(x)dx.$$

Lemma 2. *1) $T(u) : W_0 \to W_0^*$ is well defined and bounded.*
2) The restriction of T to a finite subspace of W_0 is continuous.
3) T is coercive.

Proof. 1) By the Hölder inequality and (F1), it follows (without loss of generality, we may assume that $\beta = p - 1$) that

$$\big|\langle T(u), \varphi\rangle\big| = \Big|\int\int_Q \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+ps}} dxdy - \int_\Omega f(x, u)\varphi(x)dx\Big| \le \|u\|_{W_0}^{p-1}\|\varphi\|_{W_0} + \big(\|d\|_{p'} + c\|u\|_p^{p-1}\big)\|\varphi\|_p \le C\|\varphi\|_{W_0}$$

for all $u, \varphi \in W_0$, where we have used the embedding $W_0 \hookrightarrow L^p(\Omega; \mathbb{R}^m)$.
2) Let $\{u_k\} \subset W_0$ such that $u_k \to u$ in $V_k = \mathrm{span}\,\{e_1, \ldots, e_k\}$ where V_k is a

finite subspace of W_0 and $\{e_i\}_{i=1}^k$ is a basis of V_k. Since $u_k \to u$ in V_k, then on the one hand $u_k \to u$ almost everywhere for a subsequence, on the other hand $\{u_k\}$ is bounded in W_0. It follows for all $\varphi \in W_0$, $\|\varphi\|_{W_0} \leq 1$, that (without loss of generality, we can assume that $\beta = p-1$)

$$\begin{aligned}&\big|\langle T(u_k),\varphi\rangle - \langle T(u),\varphi\rangle\big| = \bigg| \int\int_Q \big[|u_k(x)-u_k(y)|^{p-2}(u_k(x)-u_k(y))\\ &\quad - |u(x)-u(y)|^{p-2}(u(x)-u(y))\big]/|x-y|^{n+ps} \times (\varphi(x)-\varphi(y))dxdy\\ &- \int_\Omega \big(f(x,u_k)-f(x,u)\big)\varphi(x)dx\bigg| \leq \Big(\int\int_Q \big||u_k(x)-u_k(y)|^{p-2}(u_k(x)-u_k(y))\\ &- |u(x)-u(y)|^{p-2}(u(x)-u(y))\big|^{\frac{p}{p-1}}/|x-y|^{(n+ps)\frac{p}{p-1}}dxdy\Big)^{\frac{p-1}{p}}\\ &+ \Big(\int_\Omega \big|f(x,u_k)-f(x,u)\big|^{\frac{p}{p-1}}dx\Big)^{\frac{p-1}{p}}.\end{aligned}$$

Since

$$\big||a|^{p-2}a - |b|^{p-2}b\big| \leq 2^{p-2}(p-1)|a-b|\big(|a|+|b|\big)^{p-2},$$

thus

$$\begin{aligned}&\big|\langle T(u_k),\varphi\rangle - \langle T(u),\varphi\rangle\big| \leq C\Big(\int\int_Q \big|(u_k(x)-u_k(y))-(u(x)-u(y))\big|^{\frac{p}{p-1}}\\ &\quad\times \big(|u_k(x)-u_k(y)|+|u(x)-u(y)|\big)^{\frac{p(p-2)}{p-1}}/|x-y|^{(n+ps)\frac{p}{p-1}}dxdy\Big)^{\frac{p-1}{p}}\\ &\quad+\Big(\int_\Omega |f(x,u_k)-f(x,u)|^{\frac{p}{p-1}}dx\Big)^{\frac{p-1}{p}}\\ &\leq C\|u_k-u\|_{W_0}\big(\|u_k\|_{W_0}^{p-2}+\|u\|_{W_0}^{p-2}\big)+\Big(\int_\Omega |f(x,u_k)-f(x,u)|^{\frac{p}{p-1}}dx\Big)^{\frac{p-1}{p}}.\end{aligned} \tag{3.1}$$

We have

$$|a-b|^p \leq 2^{p-1}\big(|a|^p+|b|^p\big), \quad 1<p,$$

and since $1 < \frac{p}{p-1}$, it follows that

$$\int_\Omega |f(x,u_k)|^{\frac{p}{p-1}}dx \leq 2^{\frac{1}{p-1}}\int_\Omega \big(|d(x)|^{p'}+|u_k|^p\big)dx \leq C \tag{3.2}$$

by the boundedness of u_k in $L^p(\Omega;\mathbb{R}^m)$. It follows by (3.2) that the sequence $\big\{|f(x,u_k)-f(x,u)|^{p'}\big\}$ is uniformly bounded and equiintegrable in $L^1(\Omega)$. The Vitali Convergence Theorem (see [23]) implies

$$\lim_{k\to\infty}\int_\Omega \big|f(x,u_k)-f(x,u)\big|^{p'}dx = 0.$$

By virtue of (3.1) and the definition of u_k, we deduce that

$$\big|\langle T(u_k),\varphi\rangle - \langle T(u),\varphi\rangle\big| \to 0 \quad \text{as} \quad k\to\infty.$$

3) We have

$$\langle T(u),u\rangle = \int\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}dxdy - \int_\Omega f(x,u).udx$$
$$\geq \|u\|_{W_0}^p - \|d\|_{p'}\|u\|_p - \|u\|_p^{\beta+1} \geq \|u\|_{W_0}^p - c_1\|d\|_{p'}\|u\|_{W_0} - c_1^{\beta+1}\|u\|_{W_0}^{\beta+1},$$

where c_1 is the constant of the embedding $W_0 \hookrightarrow L^p(\Omega;\mathbb{R}^m)$. Hence

$$\lim_{\|u\|_{W_0}\to\infty} \frac{\langle T(u),u\rangle}{\|u\|_{W_0}} = \infty \quad \text{since} \quad p > \max\{1,\beta+1\}.$$

□

Now we can construct the approximating solutions.

Lemma 3. *For all $k\in\mathbb{N}$ there exists $u_k \in V_k$ such that*

$$\langle T(u_k),\varphi\rangle = 0 \quad \textit{for all}\, \varphi \in V_k \tag{3.3}$$

and there exists a constant $R>0$ such that

$$\|u_k\|_{W_0} \leq R \quad \textit{for all}\, k\in\mathbb{N}. \tag{3.4}$$

Proof. Let fix k and assume that dim $V_k = r$. For simplicity, we write $\sum_{i=1}^k a^i e_i = a^i e_i$, where $(e_i)_{i=1}^r$ is a basis of V_k. Define the map

$$S:\mathbb{R}^r \longrightarrow \mathbb{R}^r, \qquad (a^1,...,a^r) \to \Big(\langle T(a^i e_i), e_j\rangle\Big)_{j=1}^r.$$

By Lemma 2, S is continuous. Let $u = a^i e_i$, we have then $\|a\|_{\mathbb{R}^r} \to \infty$ is equivalent to $\|u\|_{W_0} \to \infty$ and $S(a).a = \langle T(u),u\rangle$. Hence

$$S(a).a \to \infty \quad \text{as} \quad \|a\|_{\mathbb{R}^r} \to \infty.$$

Consequently, there is $R>0$ such that for all $a \in \partial B_R(0) \subset \mathbb{R}^r$ we have $S(a).a > 0$. According to [21, Lemma 4.3, p. 53], there exists $x \in B_R(0)$ solution of $S(x)=0$. Therefore, for all $k\in\mathbb{N}$ there exists $u_k\in V_k$ such that

$$\langle T(u_k),\varphi\rangle = 0 \quad \text{for all}\, \varphi\in V_k.$$

Remark that if $\|u_k\|_{W_0} \to \infty$, then $\langle T(u_k),u_k\rangle \to \infty$ by Lemma 2. This is a contradiction with (3.3). Hence $\{u_k\}$ is uniformly bounded, i.e. there exists $R>0$ such that $\|u_k\|_{W_0} \leq R$ for all $k\in\mathbb{N}$. □

As mentioned in the introduction, the tool we use to prove the existence of a weak solution is the Young measure. This tool permits to identify weak limit as described in the following important lemma:

Lemma 4. *Assume that* (3.4) *holds. Then there exists a Young measure $\nu_{(x,y)}$ generated by $u_k \in L^p(Q;\mathbb{R}^m)$ such that*
1) $\nu_{(x,y)}$ is a probability measure, i.e. $\|\nu_{(x,y)}\|_{\mathcal{M}(\mathbb{R}^m)} = 1$ for almost every $(x,y)\in Q$.
2) The weak L^1-limit of u_k is given by $\langle\nu_{(x,y)}, id\rangle = \int_{\mathbb{R}^m}\lambda d\nu_{(x,y)}(\lambda)$.
3) $\nu_{(x,y)}$ satisfies $\langle\nu_{(x,y)}, id\rangle = u(x,y)$ for almost every $(x,y)\in Q$.

Proof. 1) Let us consider

$$u_k(x,y) = \frac{v_k(x) - v_k(y)}{|x-y|^{\frac{n}{p}+s}} \in L^p(Q;\mathbb{R}^m) \quad \text{for every } v_k \in W_0.$$

We know for any $R > 0$, that $(\Omega \cap B_R)^2 \subseteq \Omega \times \Omega \subsetneq Q$, where $B_R = B(0,R)$ is the ball centered in 0 with radius R. Let $L \in \mathbb{R}$ such that $Q_L \equiv \{(x,y) \in (\Omega \cap B_R)^2 : \, |u_k(x,y)| \geq L\}$. We have

$$\|u_k\|_{L^p(Q;\mathbb{R}^m)} = \Big(\int\int_Q \frac{|v_k(x) - v_k(y)|^p}{|x-y|^{n+ps}} dxdy \Big)^{\frac{1}{p}} = \|v_k\|_{W_0} \leq R$$

by (3.4), which implies that $\{u_k\}$ is bounded in $L^p(Q;\mathbb{R}^m)$. Hence, there exists $c \geq 0$ such that

$$c \geq \int\int_Q |u_k(x,y)|^p dxdy \geq \int\int_{Q_L} |u_k(x,y)|^p dxdy \geq L^p |Q_L|,$$

where $|Q_L|$ is the Lebesgue measure of Q_L. Therefore, (u_k) satisfies equation (2.1) in Lemma 1, thus there is a Young measure noted by $\nu_{(x,y)}$ associated to u_k such that $\|\nu_{(x,y)}\|_{\mathcal{M}(\mathbb{R}^m)} = 1$ for almost every $(x,y) \in Q$.
2) Since $L^p(Q;\mathbb{R}^m)$ is reflexive $(p > 1)$, it follows by (3.4), the existence of a subsequence (still denoted by u_k) weakly convergent in $L^p(Q;\mathbb{R}^m)$. Moreover, weakly convergent in $L^1(Q;\mathbb{R}^m)$, since $1 < p$. By Lemma 1(iii), taking φ as the idenity mapping I, we have

$$u_k \rightharpoonup \langle \nu_{(x,y)}, id \rangle = \int_{\mathbb{R}^m} \lambda d\nu_{(x,y)}(\lambda) \quad \text{weakly in } L^1(Q;\mathbb{R}^m).$$

3) By (3.4), we have $v_k \rightharpoonup v$ in W_0 and $v_k \rightharpoonup v$ in $L^p(\Omega;\mathbb{R}^m)$ (for a subsequence). Thus $u_k \rightharpoonup u$ in $L^p(Q;\mathbb{R}^m)$ where $u(x,y) = \frac{v(x)-v(y)}{|x-y|^{\frac{n}{p}+s}}$. Owing to (2), the uniqueness of limits implies that

$$\langle \nu_{(x,y)}, id \rangle = u(x,y) = \frac{v(x) - v(y)}{|x-y|^{\frac{n}{p}+s}} \quad \text{for almost every } (x,y) \in Q.$$

□

Now, we are in a position to prove Theorem 1.

Proof. Let $\{u_k\}$ be the sequence defined in the proof of Lemma 4, i.e.

$$u_k(x,y) = \frac{v_k(x) - v_k(y)}{|x-y|^{\frac{n}{p}+s}} \quad \text{for every } v_k \in W_0.$$

We have

$$\int\int_Q |u_k(x,y)|^p dxdy = \int\int_Q \frac{|v_k(x) - v_k(y)|^p}{|x-y|^{n+ps}} dxdy = \int_\Omega f(x,v_k) v_k dx.$$

By (3.4), up to a subsequence,

$$v_k \to v \quad \text{strongly in } L^p(\Omega;\mathbb{R}^m) \quad \text{and a.e. in } \Omega.$$

It follows from the continuity condition in (F1) that $f(x,v_k)(v_k-v)\to 0$ a.e. in Ω as $k\to\infty$.
By the growth condition in (F1), $\{f(x,v_k)(v_k-v)\}$ is uniformly bounded and equiintegrable in $L^1(\Omega)$. Hence, the Vitali Convergence Theorem implies

$$\lim_{k\to\infty}\int_\Omega f(x,v_k)(v_k-v)dx = 0.$$

Owing to a weak convergence defined in Lemma 4, we get

$$\int\int_Q |u_k(x,y)|^{p-2}u_k(x,y)dxdy \rightharpoonup \int\int_Q\int_{\mathbb{R}^m}|\lambda|^{p-2}\lambda d\nu_{(x,y)}(\lambda)dxdy$$
$$= \int\int_Q |u(x,y)|^{p-2}u(x,y)dxdy = \int\int_Q \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))}{|x-y|^{n+ps}}dxdy$$

weakly in $L^1(Q;\mathbb{R}^m)$. Changing the role of u_k and v_k, we obtain

$$\lim_{k\to\infty}\int\int_Q \frac{|u_k(x)-u_k(y)|^{p-2}(u_k(x)-u_k(y))(v(x)-v(y))}{|x-y|^{n+ps}}dxdy$$
$$= \int\int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+ps}}dxdy$$

for every $v\in W_0$ by density of $\underset{k\geq 1}{\cup} V_k$ in W_0. Then, the proof of Theorem 1 is complete. □

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