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A Study on the Solutions of Notable Engineering Models

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Abstract. In the commenced work, we establish some novel results concerning graph contractions in a more generalized setting. Furthermore, we deliver some examples to elaborate and explain the usability of the attained results. By virtue of nontrivial examples, we show our results improve, extend, generalize, and unify several noteworthy results in the existing state-of-art. We adopt computer simulation validating our results. To arouse further interest in the subject and to show its efficacy, we devote this work to recent applications which emphasize primarily the applications for the existence of the solution of various models related to engineering problems viz. fourth-order two-point boundary value problems describing deformations of an elastic beam, ascending motion of a rocket, and a class of integral equations. This approach is entirely new and will open up some new directions in the underlying graph structure.

Keywords: deformations of elastic beam, ascending motion of rocket, graphic contraction, fixed point.

AMS Subject Classification: 47H10; 34B15; 83C10.

1 Introduction

Differential equations and integral equations have many applications in science and engineering, in this direction, for some elegant results we refer [17, 18, 25]. Relevant literatures show that various investigations dealt with qualitative properties of differential equations described above like, convergence, boundedness, the existence of solutions. Banach's fixed point theory, popularly well known as the Banach contraction principle (BCP), plays a vital role in the theory of metric spaces [8, 15]. Employing an iteration process, one can locate the neighborhood of the fixed point. Therefore, it can be administered on a workstation, and finding the related fixed point of the contraction mappings becomes easier. BCP is extended and generalized by numerous authors in distinct ways. The ease of Banach's fixed point theorem particularly, for application point of view, made this theory more attractive. In this direction, researchers employed contraction mappings to establish the existence of solutions of differential equations, integral equations etc.

Ran and Reuring's extended the BCP to metric spaces equipped with a partial order [19]. To find a solution to some special matrix equations was also one of the great charms of the fixed-point theorists. To this end, the work of EL-Sayed and Ran [9] was a pioneer one. Later on, Nieto and Rodriguez Lopez [16] extended the work of [19] and applied their results to solve some differential equations. In 2008, Jachymiski [12] initiated a novel idea in fixed-point theory, where the author evoked graph structure on metric spaces instead of order structure. Some noteworthy efforts done on this concept can be seen in [4, 23, 25]. Recently Younis et al. [22,23,24,25] took the Banach's contraction principle for giving an association with graph theory, and with this amalgamation, they established some remarkable results in existing theory. In 1968, Kannan [14] gave a breakthrough in fixed point theory, Kannan's theorem was important as this establishes the completeness of the metric space involved, i.e., a metric space $\mathcal{S}$ is complete if and only if every mapping satisfying Kannan's inequality on $\mathcal{S}$ has a fixed point. The contraction in the sense of Banach does not have this property, the class of Kannan mappings is independent of Banach contraction.

Several fixed point theorists caught attention after the discovery of this remarkable result by Kannan (one can see [2, 3, 10, 13, 20]). Another glamour of such mappings is the characterization of metric completeness via fixed points. Several authors investigated Kannan mappings theoretically in different ways, but interestingly its application part has not been worked out. This article's main objective is to enunciate such mappings with an entirely different approach in a graph structure and then utilize the established results to the existence of solutions of some nonlinear problems viz. deformations of an elastic beam, first-order nonlinear differential equation representing the ascending motion of a rocket and some class of integral equations and ascending motion of a rocket.

To start with some basic notions and fundamental definitions are necessary to figure out.

For a non-void set $\mathcal{S}$, Δ denotes the diagonal $\mathcal{S} \times \mathcal{S}$. The notions $V(\mathcal{M})$ and $E(\mathcal{M})$ respectively denote the set of all vertices and edges for a digraph $\mathcal{M}$, where $E(\mathcal{M})$ accommodates all the loops of $\mathcal{M}$ (i.e., $\Delta \subset E(\mathcal{M})$). We represent the digraph by $\mathcal{M} = (V(\mathcal{M}), E(\mathcal{M}))$. By $\mathcal{M}^{-1}$, we represent the digraph $\mathcal{M}$ with reversed edges. Additionally the digraph $\mathcal{M}$ with symmetric edges is denoted by $\breve{\mathcal{M}}$.

Unambiguously, we write $E(\breve{\mathcal{M}}) := E(\mathcal{M}^{-1}) \cup E(\mathcal{M})$. A sequence $\{\varkappa_j\}_{j=0}^{h}$ consisting of $(h+1)$ vertices with $r = \varkappa_0$, $r' = \varkappa_h$ and $(\varkappa_{j-1}, \varkappa_j) \in E(\breve{\mathcal{M}})$ for $j = 1, 2, \ldots, h$ is called a directed path or simply a path. We say $\mathcal{M}$ to be a connected graph if there is a path between any of its vertices. However if $\mathcal{M}$ is undirected and there endures a path joining every two of its vertices, then call $\mathcal{M}$ to be a weakly connected graph. Moreover, a graph $\mathcal{M}^* = (V(\mathcal{M}^*), E(\mathcal{M}^*))$ is termed as a subgraph of $\mathcal{M} = (V(\mathcal{M}), E(\mathcal{M}))$ if $V(\mathcal{M}) \supseteq V(\mathcal{M}^*)$ and $E(\mathcal{M}) \supseteq E(\mathcal{M}^*)$.

The following are some important notions that will be carried out throughout the manuscript.

(i) $[r]_{\mathcal{M}}^{t} := \{r' \in \mathcal{S} : \exists$ a path directing from r to r' with length $t\}$.

(ii) $(r\mathcal{R}r')_{\mathcal{M}}$ denotes the relation $\mathcal{R}$ describing that there exists a path starting from r to r'.

(iii) If a point $\boldsymbol{r}$ lies on the path $(r\mathcal{R}r')_{\mathcal{M}}$, we denote it by $\boldsymbol{r} \in (r\mathcal{R}r')_{\mathcal{M}}$.

(iv) A sequence $\{\varkappa_h\} \in \mathcal{S}$ is called $\mathcal{M}$-term wise connected ($\mathcal{M}$-*twc*) if $(\varkappa_h \mathcal{R} \varkappa_{h+1})_{\mathcal{M}}$ for all $h \in \mathbb{N}$.

Very recently, Younis et al. [23, 25] employed some fixed point results based on graph structure to find the solutions of some nonlinear problems describing some physical models from science and engineering. They introduced the notion of graphical rectangular b-metric spaces [25] as an extension and generalization of b-metric spaces and rectangular metric spaces by the amalgamation of graph theory with metric spaces. Baradol et al. [6] discussed some open problems enunciated in [25] in a novel way in setting of graphical rectangular b-metric spaces. They went on to demonstrate how path length between two points in graphical rectangular b-metric spaces is important. The reader is referred to the notable publications [5, 6, 24, 26] for additional synthesis on this topic.

The following is the formal definition of graphical rectangular b-metric spaces.

Definition 1. [25] Let $\mathcal{M}$ be a graph endowing a non-void set $\mathcal{S}$ and let $\mathcal{M}_{b_r} : \mathcal{S} \times \mathcal{S} \to [0, \infty[$ be such that for $s \geq 1$, the following conditions are asserted: $(\mathcal{M}_1)$ $\mathcal{M}_{b_r}(r, r') = 0$ if and only if $r = r'$; $(\mathcal{M}_2)$ $\mathcal{M}_{b_r}(r, r') = \mathcal{M}_{b_r}(r', r)$ for all $r, r' \in \mathcal{S}$; $(\mathcal{M}_3)$ For $(r\mathcal{R}r')_{\mathcal{M}}, p, q \in (r\mathcal{R}r')_{\mathcal{M}}$, we have

$$\mathcal{M}_{b_r}(r, r') \leq s\left[\mathcal{M}_{b_r}(r, p) + \mathcal{M}_{b_r}(p, q) + \mathcal{M}_{b_r}(q, r')\right]$$

for all $r, r' \in \mathcal{S}$ and all distinct points (distinct from r and r') $p, q \in \mathcal{S}$. Then the doublet $(\mathcal{M}_{b_r}, \mathcal{S})$ is termed as a graphical generalized b-metric space or graphical rectangular b-metric space ($gr_b ms$) with coefficient $s \geq 1$.

Remark 1. [25]

(i) It may be noted that graphical rectangular b-metric spaces generalize graphical rectangular metric spaces because a graphical rectangular b-metric space reduces to a graphical rectangular metric space for $s = 1$.

(ii) A graphical b-metric space with coefficient s is a graphical rectangular b-metric space with coefficient s^2.
However, the converse need not be true in general.

DEFINITION 2. [25] Let $\{\varkappa_h\}$ be a sequence in a $gr_b ms$:

(i) $\{\varkappa_h\}$ is said to be convergent if and only if there exists $r \in \mathcal{S}$ such that $\mathcal{M}_{b_r}(\varkappa_h, \varkappa) \to 0$ whenever $h \to \infty$;

(ii) $\{\varkappa_h\}$ is said to be Cauchy if and only if $\mathcal{M}_{b_r}(\varkappa_h, \varkappa_{h'}) \to 0$ as $h, h' \to \infty$. In other words $\{\varkappa_h\}$ is a Cauchy sequence, if given $\mu > 0$, there exists $h_0 \in \mathbb{N}$ such that $\mathcal{M}_{b_r}(\varkappa_h, \varkappa_{h'}) < \mu, \text{ for all } h, h' > h_0$.

Example 1. [25] Let $\mathcal{M}_{b_r}$ be the rectangular metric endowing the set $\mathcal{S} = \{e, d, c, b, a\}$. Define the metric by the following

$$\mathcal{M}_{b_r}(r, r') = \begin{cases} 0, & r = r', \\ \sqrt{\beta}/5, & r \ \ or\, r' \notin \{b, a\} \text{ and } r \neq r', \\ 3\sqrt{\beta} & r, r' \in \{b, a\} \text{ and } r \neq r', \end{cases}$$

with $\beta > 0$. Then, $(\mathcal{S}, \mathcal{M}_{b_r})$ is a $gr_b ms$ with coefficient $s = 5$ equipped with the graph $\mathcal{M}$, where $\mathcal{S} = V(\mathcal{M})$ and $E(\mathcal{M})$ describes all the edges shown in the adjacent figure (Figure 1).

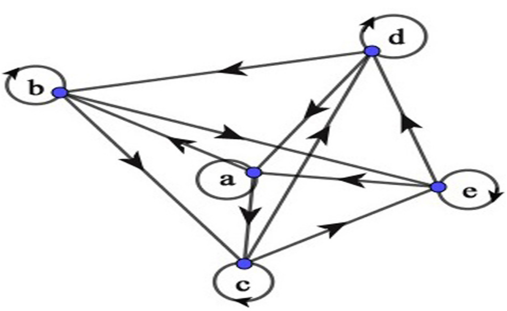

Figure 1. Graph $\mathcal{M}$ representing $gr_b ms$ $(\mathcal{S}, \mathcal{M}_{b_r}, s)$.

DEFINITION 3. [25] Let $(\mathcal{S}, \mathcal{M}_{b_r})$ be a $gr_b ms$. An open ball with center $r \in \mathcal{S}$ and radius $\mu > 0$ is defined by

$$B_{\mathcal{M}_{b_r}}(r, \mu) = \{r' \in \mathcal{S} : (r\mathcal{R}r')_{\mathcal{M}_{b_r}}, \mathcal{M}_{b_r}(r, r') < \mu\}.$$

2 New results concerning gr_bms

Let $\varkappa_0 \in \mathcal{S}$ be the starting value of the sequence $\{\varkappa_h\}$ with respect to the weighted graph $\mathcal{M}$. The sequence $\{\varkappa_h\}$ is said to be a J-Picard sequence (J-Ps) for a self map $J : \mathcal{S} \to \mathcal{S}$ if

$$\varkappa_h = J\varkappa_{h-1} \text{ for all } h \in \mathbb{N}.$$

The following is the principal definition of this article.

DEFINITION 4. Let $(\mathcal{S}, \mathcal{M}_{b_r})$ be a gr_bms. we call a map $J : \mathcal{S} \to \mathcal{S}$ to be $\mathcal{M}_{b_r}$-graph-Kannan (in short $\mathcal{M}_{b_r}$-gK) contraction on $(\mathcal{S}, \mathcal{M}_{b_r})$ if the following two conditions are contended:

($\spadesuit_1$) for all $p, q \in \mathcal{S}$ if $(p, q) \in E(\mathcal{M})$ implies $(Jp, Jq) \in E(\mathcal{M})$;

($\spadesuit_2$) there exists $\mu \in [0, \frac{1}{2}[$, for any $p, q \in \mathcal{S}$ with $(p, q) \in E(\mathcal{M})$, we have

$$\mathcal{M}_{b_r}(Jp, Jq) \leq \frac{\mu}{s}\left[\mathcal{M}_{b_r}(p, Jp) + \mathcal{M}_{b_r}(q, Jq)\right]. \tag{2.1}$$

Remark 2. Every Kannan contraction is a $\mathcal{M}_{b_r}$-gK contraction equipped with the graph $\mathcal{M} = (V(\mathcal{M}), E(\mathcal{M}))$, but the converse need not be true.

Note that, in view of Remark 1 and Remark 2, the findings in this article are useful extensions and generalizations of the corresponding results in various metric spaces concerning Kannan type mappings.

Under this new scenario, we prove our main result concerning $\mathcal{M}_{b_r}$-gK contraction in the context of graphical rectangular b–metric spaces as below.

Theorem 1. *Let $(\mathcal{S}, \mathcal{M}_{b_r})$ be a $\mathcal{M}$-complete gr_bms and $J : \mathcal{S} \to \mathcal{S}$ be a $\mathcal{M}_{b_r}$-gK contraction with respect to the graph $\mathcal{M}$. Suppose that the following conditions are fulfilled:*

(a) if there exists a limit $\boldsymbol{r} \in \mathcal{S}$ of a converging $\mathcal{M}$-twc J-Ps $\{\varkappa_h\}$ and $h_0 \in \mathbb{N}$ such that $(\varkappa_h, \boldsymbol{r}) \in E(\mathcal{M})$ or $(\boldsymbol{r}, \varkappa_h) \in E(\mathcal{M})$ for all $h > h_0$;

(b) for odd positive integers t and t', there exists $\varkappa_0 \in \mathcal{S}$ with $J\varkappa_0 \in [\varkappa_0]^t_{\mathcal{M}}$ and $J^2\varkappa_0 \in [\varkappa_0]^{t'}_{\mathcal{M}}$.

Then, there exists $\varkappa' \in \mathcal{S}$ such that the J-Ps $\{\varkappa_h\}$ with the starting value $\varkappa_0 \in \mathcal{S}$ is $\mathcal{M}$-twc and converges to $\varkappa'$.

Proof. Let $\varkappa_0 \in \mathcal{S}$ such that for odd positive integers t and t', we have $J\varkappa_0 \in [\varkappa_0]^t_{\mathcal{M}}$ and $J^2\varkappa_0 \in [\varkappa_0]^{t'}_{\mathcal{M}}$. By the hypothesis, there exist paths $\{\varkappa'_j\}^t_{j=0}$ and $\{v_j\}^{t'}_{j=0}$ such that

$$\begin{aligned} &\varkappa_0 = \varkappa'_0, \quad J\varkappa_0 = \varkappa'_t \quad \text{and} \quad (\varkappa'_{j-1}, \varkappa'_j) \in E(\mathcal{M}), \ \text{for all } j = 1, 2, \ldots, t, \\ &\varkappa_0 = v_0, \quad J^2\varkappa_0 = v_{t'} \quad \text{and} \quad (v_{j-1}, v_j) \in E(\mathcal{M}), \ \text{for all } j = 1, 2, \ldots, t'. \end{aligned}$$

Since, $(\varkappa'_{j-1}, \varkappa'_j) \in E(\mathcal{M})$, by ($\spadesuit_1$) we have

$$(J\varkappa'_{j-1}, J\varkappa'_j) \in E(\mathcal{M}) \quad \text{for} \quad j = 1, 2, \ldots, t.$$

Therefore, $\{J\varkappa'_j\}_{j=0}^t$ is a path from $J\varkappa'_0 = J\varkappa_0 = \varkappa_1$ to $J\varkappa'_t = J^2\varkappa_0 = \varkappa_2$ possessing length t. Similarly, for all $h \in \mathbb{N}$, $\{J^h\varkappa'_j\}_{j=0}^t$ is a path from $J^h\varkappa'_0 = J^h\varkappa_0 = \varkappa_h$ to $J^h\varkappa'_t = J^hJ\varkappa_0 = \varkappa_{h+1}$ of length t. Thus, $\{\varkappa_h\}$ is $\mathcal{M}$-termwise connected sequence. Additionally, for $j = 1, 2, \ldots, t$, and for all $h \in \mathbb{N}$, $(J^h\varkappa'_{j-1}, J^h\varkappa'_j) \in E(\mathcal{M})$. Utilizing ($\spadesuit_2$), for $j = 1, 2, \ldots t$, we will prove

$$\mathcal{M}_{b_r}(J^h\varkappa'_{j-1}, J^h\varkappa'_j) \leq \eta\mathcal{M}_{b_r}(J^{h-1}\varkappa'_{j-1}, J^{h-1}\varkappa'_j) \leq \ldots \leq \eta^h\mathcal{M}_{b_r}(\varkappa'_{j-1}, \varkappa'_j), \tag{2.2}$$

where $\eta \in \left[0, \frac{1}{s}\right)$. Indeed, we deduce that

$$\begin{aligned}\mathcal{M}_{b_r}(J^h\varkappa'_{j-1}, J^h\varkappa'_j) &= \mathcal{M}_{b_r}\big(J(J^{h-1}\varkappa'_{j-1}), J(J^{h-1}\varkappa'_j)\big)\\ &\leq \frac{\mu}{s}\Big\{\mathcal{M}_{b_r}\Big(J^{h-1}\varkappa'_{j-1}, J(J^{h-1}\varkappa'_{j-1})\Big) + \mathcal{M}_{b_r}\Big(J^{h-1}\varkappa'_j, J(J^{h-1}\varkappa'_j)\Big)\Big\}\\ &= \frac{\mu}{s}\left\{\mathcal{M}_{b_r}\Big(J^{h-1}\varkappa'_{j-1}, J^{h-1}\varkappa'_j\Big) + \mathcal{M}_{b_r}\Big(J^h\varkappa'_{j-1}, J^h\varkappa'_j\Big)\right\}.\end{aligned}$$

That is, we have

$$\mathcal{M}_{b_r}(J^h\varkappa'_{j-1}, J^h\varkappa'_j) \leq \left(\frac{\mu}{s-\mu}\right)\mathcal{M}_{b_r}\big(J^{h-1}\varkappa'_{j-1}, J^{h-1}\varkappa'_j\big). \tag{2.3}$$

Put $\eta = \mu/(s-\mu)$ and observe that $\eta \in [0, 1/s)$ by assuming distinct values of μ and s. Hence the inequality (2.3) is equivalent to

$$\mathcal{M}_{b_r}(J^h\varkappa'_{j-1}, J^h\varkappa'_j) \leq \eta\,\mathcal{M}_{b_r}\big(J^{h-1}\varkappa'_{j-1}, J^{h-1}\varkappa'_j\big);\ \text{for all } \eta \in \big[0, 1/s\big).$$

Repeating above process, we get

$$\mathcal{M}_{b_r}(J^h\varkappa'_{j-1}, J^h\varkappa'_j) \leq \eta\mathcal{M}_{b_r}(J^{h-1}\varkappa'_{j-1}, J^{h-1}\varkappa'_j) \leq \ldots \leq \eta^h\mathcal{M}_{b_r}(\varkappa'_{j-1}, \varkappa'_j). \tag{2.4}$$

Thus our case of establishing the Equation (2.2) is set up. On the similar lines, we have

$$\mathcal{M}_{b_r}(J^hv_{j-1}, J^hv_j) \leq \eta\,\mathcal{M}_{b_r}(J^{h-1}v_{j-1}, J^{h-1}v_j) \leq \ldots \leq \eta^h\mathcal{M}_{b_r}(v_{j-1}, v_j). \tag{2.5}$$

By using graphical rectangular property, we acquire

$$\begin{aligned}&\mathcal{M}_{b_r}(\varkappa_0, \varkappa_1) = \mathcal{M}_{b_r}(\varkappa'_0, \varkappa'_t) \leq s[\mathcal{M}_{b_r}(\varkappa'_0, \varkappa'_1) + \mathcal{M}_{b_r}(\varkappa'_1, \varkappa'_2) + \mathcal{M}_{b_r}(\varkappa'_2, \varkappa'_t)]\\ &\leq s[\mathcal{M}_{b_r}(\varkappa'_0, \varkappa'_1) + \mathcal{M}_{b_r}(\varkappa'_1, \varkappa'_2)] + s^2[\mathcal{M}_{b_r}(\varkappa'_2, \varkappa'_3) + \mathcal{M}_{b_r}(\varkappa'_3, \varkappa'_4) + \mathcal{M}_{b_r}(\varkappa'_4, \varkappa'_t)]\\ &\quad\vdots\\ &\leq s[\mathcal{M}_{b_r}(\varkappa'_0, \varkappa'_1) + \mathcal{M}_{b_r}(\varkappa'_1, \varkappa'_2)] + s^2[\mathcal{M}_{b_r}(\varkappa'_2, \varkappa'_3) + \mathcal{M}_{b_r}(\varkappa'_3, \varkappa'_4)] + \ldots\\ &+ s^{\frac{t-1}{2}}[\mathcal{M}_{b_r}(\varkappa'_{t-3}, \varkappa'_{t-2}) + \mathcal{M}_{b_r}(\varkappa'_{t-2}, \varkappa'_{t-1}) + \mathcal{M}_{b_r}(\varkappa'_{t-1}, \varkappa'_t)] = \mathcal{D}^t_{r_b}.\end{aligned} \tag{2.6}$$

Also

$$\begin{aligned}&\mathcal{M}_{b_r}(\varkappa_0, \varkappa_2) = \mathcal{M}_{b_r}(v_0, v_{t'}) \leq s[\mathcal{M}_{b_r}(v_0, v_1) + \mathcal{M}_{b_r}(v_1, v_2) + \mathcal{M}_{b_r}(v_2, v_{t'})]\\ &\leq s[\mathcal{M}_{b_r}(v_0, v_1) + \mathcal{M}_{b_r}(v_1, v_2)] + s^2[\mathcal{M}_{b_r}(v_2, v_3) + \mathcal{M}_{b_r}(v_3, v_4) + \mathcal{M}_{b_r}(v_4, v_{t'})]\end{aligned}$$

$$\begin{aligned}&\vdots\\ &\leq s[\mathcal{M}_{b_r}(v_0,v_1)+\mathcal{M}_{b_r}(v_1,v_2)]+s^2[\mathcal{M}_{b_r}(v_2,v_3)+\mathcal{M}_{b_r}(v_3,v_4)]+\ldots\\ &\quad+s^{\frac{t'-1}{2}}[\mathcal{M}_{b_r}(v_{t'-3},v_{t'-2})+\mathcal{M}_{b_r}(v_{t'-2},v_{t'-1})+\mathcal{M}_{b_r}(v_{t'-1},v_{t'})]\\ &=\mathcal{D}^{t'}_{r_b}\ \text{(say)}. \end{aligned}\tag{2.7}$$

Proceeding in the same way and using the inequalities (2.4) and (2.6), we have

$$\begin{aligned}\mathcal{M}_{b_r}(\varkappa_h,\varkappa_{h+1})&=\mathcal{M}_{b_r}(J^h\varkappa_0,J^h\varkappa_t)=\mathcal{M}_{b_r}(J^h\varkappa'_0,J^h\varkappa'_t)\\ &\leq s[\mathcal{M}_{b_r}(J^h\varkappa'_0,J^h\varkappa'_1)+\mathcal{M}_{b_r}(J^h\varkappa'_1,J^h\varkappa'_2)]+s^2[\mathcal{M}_{b_r}(J^h\varkappa'_2,J^h\varkappa'_3)\\ &+\mathcal{M}_{b_r}(J^h\varkappa'_3,J^h\varkappa'_4)]+\ldots+s^{\frac{t-1}{2}}[\mathcal{M}_{b_r}(J^h\varkappa'_{t-3},J^h\varkappa'_{t-2})\\ &+\mathcal{M}_{b_r}(J^h\varkappa'_{t-2},J^h\varkappa'_{t-1})+\mathcal{M}_{b_r}(J^h\varkappa'_{t-1},J^h\varkappa'_t)]=\eta^h\mathcal{D}^t_{r_b}.\end{aligned}\tag{2.8}$$

Also, using the inequalities (2.5) and (2.7), we have

$$\begin{aligned}\mathcal{M}_{b_r}(\varkappa_h,\varkappa_{h+2})&=\mathcal{M}_{b_r}(J^h\varkappa_0,J^h\varkappa_2)=\mathcal{M}_{b_r}(J^hv_0,J^hv_{t'})\\ &\leq s[\mathcal{M}_{b_r}(J^hv_0,J^hv_1)+\mathcal{M}_{b_r}(J^hv_1,J^hv_2)]+s^2[\mathcal{M}_{b_r}(J^hv_2,J^hv_3)\\ &+\mathcal{M}_{b_r}(J^hv_3,J^hv_4)]+\ldots+s^{\frac{t'-1}{2}}[\mathcal{M}_{b_r}(J^hv_{t'-3},J^hv_{t'-2})\\ &+\mathcal{M}_{b_r}(J^hv_{t'-2},J^hv_{t'-1})+\mathcal{M}_{b_r}(J^hv_{t'-1},J^hv_{t'})]=\eta^h\mathcal{D}^{t'}_{r_b}.\end{aligned}\tag{2.9}$$

In order to prove $\{\varkappa_h\}$ is a Cauchy sequence, we consider the following cases:
Case-1: For an odd integer k, we have

$$\begin{aligned}\mathcal{M}_{b_r}(\varkappa_h,\varkappa_{h+k})&\leq s[\mathcal{M}_{b_r}(\varkappa_h,\varkappa_{h+1})+\mathcal{M}_{b_r}(\varkappa_{h+1},\varkappa_{h+2})]+s^2[\mathcal{M}_{b_r}(\varkappa_{h+2},\varkappa_{h+3})\\ &+\mathcal{M}_{b_r}(\varkappa_{h+3},\varkappa_{h+4})]+\ldots+s^{\frac{k-1}{2}}[\mathcal{M}_{b_r}(\varkappa_{h+k-3},\varkappa_{h+k-2})\\ &+\mathcal{M}_{b_r}(\varkappa_{h+k-2},\varkappa_{h+k-1})+\mathcal{M}_{b_r}(\varkappa_{h+k-1},\varkappa_{h+k})].\end{aligned}$$

By using the inequality (2.8), we have

$$\begin{aligned}\mathcal{M}_{b_r}(\varkappa_h,\varkappa_{h+k})&\leq s[\eta^h\mathcal{D}^t_{r_b}+\eta^{h+1}\mathcal{D}^t_{r_b}]+s^2[\eta^{h+2}\mathcal{D}^t_{r_b}+\eta^{h+3}\mathcal{D}^t_{r_b}]\\ &+\ldots+s^{\frac{k-1}{2}}[\eta^{h+k-3}\mathcal{D}^t_{r_b}+\eta^{h+k-2}\mathcal{D}^t_{r_b}+\eta^{h+k-1}\mathcal{D}^t_{r_b}]\to 0\ \text{ as }\ h\to\infty.\end{aligned}$$

Case-2: For an even integer k, we have

$$\begin{aligned}&\mathcal{M}_{b_r}(\varkappa_h,\varkappa_{h+k})\leq s[\mathcal{M}_{b_r}(\varkappa_h,\varkappa_{h+1})+\mathcal{M}_{b_r}(\varkappa_{h+1},\varkappa_{h+2})]\\ &+s^2[\mathcal{M}_{b_r}(\varkappa_{h+2},\varkappa_{h+3})+\mathcal{M}_{b_r}(\varkappa_{h+3},\varkappa_{h+4})]+\ldots+s^{\frac{k-2}{2}}\\ &\times[\mathcal{M}_{b_r}(\varkappa_{h+k-4},\varkappa_{h+k-3})+\mathcal{M}_{b_r}(\varkappa_{h+k-3},\varkappa_{h+k-2})+\mathcal{M}_{b_r}(\varkappa_{h+k-2},\varkappa_{h+k})].\end{aligned}$$

By using the inequalities (2.8) and (2.9), we have

$$\begin{aligned}\mathcal{M}_{b_r}(\varkappa_h,\varkappa_{h+k})&\leq s[\eta^h\mathcal{D}^t_{r_b}+\eta^{h+1}\mathcal{D}^t_{r_b}]+s^2[\eta^{h+2}\mathcal{D}^t_{r_b}+\eta^{h+3}\mathcal{D}^t_{r_b}]\\ &+\ldots+s^{\frac{k-2}{2}}[\eta^{h+k-4}\mathcal{D}^t_{r_b}+\eta^{h+k-3}\mathcal{D}^t_{r_b}+\eta^{h+k-2}\mathcal{D}^{t'}_{r_b}]\to 0\ \text{ as }\ h\to\infty.\end{aligned}$$

From Case-1 and Case-2, we observe that $\{\varkappa_h\}$ is the Cauchy sequence in $\mathcal{S}$. Also, $\mathcal{S}$ being $\mathcal{M}$-complete, implies that the sequence $\{\varkappa_h\}$ is convergent in $\mathcal{S}$.

By virtue of (a), there exist some $\varkappa' \in \mathcal{S}$, $h' \in \mathbb{N}$ such that $(\varkappa_h, \varkappa') \in E(\mathcal{M})$ or $(\varkappa', \varkappa_h) \in E(\mathcal{M})$ for $h > h'$ and

$$\lim_{h\to\infty} \mathcal{M}_{b_r}(\varkappa_h, \varkappa') = 0,$$

which guarantees that $\{\varkappa_h\}$ converges to $\varkappa'$. □

Remark 3. Theorem 1 provides an answer to the open problem posed in [25].

Corollary 1. In addition to the hypotheses contained in Theorem 1, we suppose $\mathcal{M}$ to be weakly connected, then J has a unique fixed point $\varkappa'$.

We provide the following example to validate our findings.

Example 2. Consider a graph $\mathcal{M}$ endowing a set $\mathcal{S} = \{3^{-h} : h \in \mathbb{N}\} \cup \{0\}$ with $V(\mathcal{M}) = \mathcal{S}$ and $E(\mathcal{M}) = \Delta \cup \{(b_1, b_2) \in \mathcal{S} \times \mathcal{S} : (b_1 \mathcal{R} b_2), b_1 \leq b_2\}$. Let the metric $\mathcal{M}_{b_r}$ be defined by the following

$$\mathcal{M}_{b_r}(b_1, b_2) = \begin{cases} |b_1 - b_2|^2 & \text{if} \quad b_1 \neq b_2, \\ 0 & \text{if} \quad b_1 = b_2. \end{cases}$$

For $s = 2$, $(\mathcal{M}_{b_r}, \mathcal{S}, s)$ is a $gr_b ms$. Now define the map $J : \mathcal{S} \to \mathcal{S}$ by

$$Jy^* = y^*/3, \text{ for all } y^* \in \mathcal{S}.$$

There exists $b_0 = \frac{1}{3}$ such that $J(\frac{1}{3}) = \frac{1}{9} \in [\frac{1}{3}]^1_{\mathcal{M}}$, i.e., $\left(\frac{1}{3}\mathcal{R}\frac{1}{9}\right)_{\mathcal{M}}$. Similarly $\left(\frac{1}{3}\mathcal{R}\frac{1}{27}\right)$ and the contractive condition (2.1) is satisfied for $\mu = \frac{3}{7}$. This endorses that J is a $\mathcal{M}_{b_r}$-gK contraction defined on the map J in the context of $gr_b ms$.

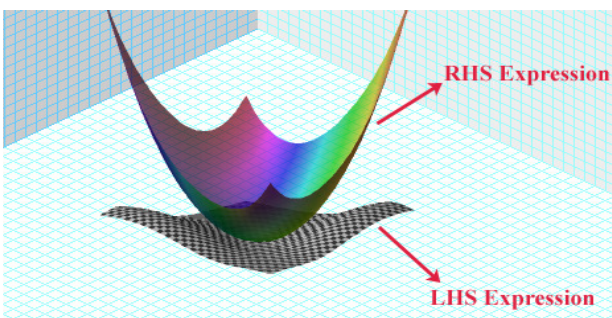

Figure 2. Validation graph–Kannan mapping.

Figure 2 validates that the right hand side of $\mathcal{M}_{b_r}$-gK contraction (2.1) controls its left hand side by taking $\mu = \frac{3}{7}$ in the framework of $gr_b ms$ defined above. With easy computation, it follows that all the assertions of Theorem 1 are fulfilled, and 0 is the required fixed point of J. The following is the underlying weighted graph (Figure 3) for the set of vertices $V'(\mathcal{M}) = \{0, \frac{1}{243}, \frac{1}{81}, \frac{1}{27}, \frac{1}{9}, \frac{1}{3}\} \subseteq V(\mathcal{M})$, where the weight of edge (b_1, b_2)=$\mathcal{M}_{b_r}(b_1, b_2)$.

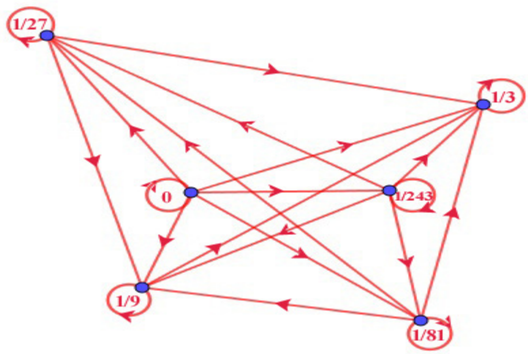

Figure 3. Related graph in which $\mathcal{M}_{b_r}(b_1, b_2)$= weight of edge (b_1, b_2).

Remark 4.

- In view of Example 2 and Remark 2, one can easily verify that main results of Bojor [7] and Kannan [14] do not satisfy the contractive the condition (2.1). Hence one can not apply the results of Bojor [7] and Kannan [14] on the mapping J.
- Thus the results obtained in this article are useful generalizations and extensions of some celebrated results of Bojor [7] and Kannan [14], in the framework of $gr_b ms$.

3 Some applications

This section deals with some practicing of our results concerning $\mathcal{M}_{b_r}$-gK contractions in the structure of $gr_b ms$. We establish the existence of solution to some class of integral equations and fourth–order two–point boundary value problem. For $N > 0$, consider the set of real-valued continuous functions $([0, N])_c$. Let $\mathcal{U} = \{\curlywedge \in \mathcal{S} : \inf_{0 \le h \le N} \curlywedge(h) > 0$ and $\curlywedge(h) \le 1, h \in [0, N], N > 0\}$. Let $\mathcal{M}$ be the graph with $V(\mathcal{M}) = \mathcal{S}$ and

$$E(\mathcal{M}) = \Delta \cup \{(\curlywedge, \curlywedge^*) \in \mathcal{S} \times \mathcal{S} : \curlywedge, \curlywedge^* \in \mathcal{U}, \curlywedge(h) \le \curlywedge^*(h), \text{ for all } h \in [0, N]\}.$$

3.1 Existence of solution to two-point fourth-order boundary value problem

The multi-point boundary value problems, especially two-point fourth–order boundary value problems, are crucial in describing an expansive elastic deflection class. Such boundary value problems for ordinary differential equations make an appearance in several distinct dimensions of applied physics and engineering science. These nonlinear fourth-order two-point boundary value problems have been investigated by various researchers using different techniques, for example, the Leray-Schauder Continuation theorem and coincidence degree theory. Our aim is to illustrate sufficient conditions for the existence of solution of the two–point fourth–order BVP portraying the *deformations of an elastic* *beam*, which plays a significant role to the problems related to mechanics and engineering. More details on these problems, one can refer to [1, 11].

Consider the two–point fourth–order BVP governing the following functional equation:

$$\begin{cases} ⅄''''(h) = \mathcal{P}(h, ⅄(h), ⅄'(h)), & h \in [0,1], \\ ⅄(0) = ⅄'(0) = ⅄''(1) = ⅄'''(1) = 0, & \end{cases} \tag{3.1}$$

where $\mathcal{P} : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a function that is continuous, and employed in the investigation of malleable beams. For convenience, we reduce the aforementioned two-point BVP (3.1) via Green's function to its equivalent integral equation stated below:

$$⅄(h) = \int_0^1 \mathcal{L}^*(h, \mathfrak{d}) \mathcal{P}(\mathfrak{d}, ⅄(\mathfrak{d}), ⅄'(\mathfrak{d})) d\mathfrak{d}, \quad h \in [0,1], \tag{3.2}$$

where the corresponding Green's function $\mathcal{L}^*(h, \mathfrak{d})$ is given by

$$\mathcal{L}^*(h, \mathfrak{d}) = \frac{1}{6} \begin{cases} \mathfrak{d}^2(3h - \mathfrak{d}), & \text{if } 0 \le \mathfrak{d} \le h \le 1; \\ h^2(3\mathfrak{d} - h), & \text{if } 0 \le h \le \mathfrak{d} \le 1. \end{cases}$$

Let the graphical b-rectangular metric $\mathcal{M}_{b_r} : \mathcal{S} \times \mathcal{S} \to [0, \infty[$ be defined by the following:

$$\mathcal{M}_{b_r}(⅄, ⅄^*) = \sup_{0 \le h \le 1} |\, ⅄\,(h) - ⅄^*(h)|^2,$$

for all $⅄, ⅄^* \in \mathcal{S}$. Clearly $(\mathcal{S}, \mathcal{M}_{b_r})$ is a $\mathcal{M}$-complete graphical b-rectangular space.

The next result furnishes us sufficient assertions for the uniqueness and existence of the solution of the two-point BVP (3.1).

Theorem 2. *Consider the underneath assertions:*

(1) $\gamma(h) \le \int_0^1 \mathcal{L}^*(h, \mathfrak{d}) \mathcal{P}(\mathfrak{d}, \gamma(\mathfrak{d}), \gamma'(\mathfrak{d})) d\mathfrak{d}, \ \ \gamma \in ([0,1], \mathbb{R})_c;$

(2) Select $\kappa > 0$ *suitably such that*

$$\inf_{0 \le h \le 1} \mathcal{L}^*(h, \mathfrak{d}) > 0, \ 0 \le \sup_{0 \le h \le 1} (\mathcal{Q}(\kappa, h))^2 < \frac{1}{2} \text{ and } \mathcal{P}(\mathfrak{d}, 1, 0) \le 1,$$

where $\mathcal{Q}(\kappa, h) = \frac{h^4 - 4h^3 + 6h^2}{24\kappa}$.

(3) For each $h \in [0,1]$, $⅄^*, ⅄ \in \mathcal{S}$, *and* J *is defined in (3.3), we have*

$$\begin{aligned} &|\mathcal{P}(h, ⅄^*(h), ⅄^{*\prime}(h)) - \mathcal{P}(h, ⅄(h), ⅄'(h))| \\ &\qquad \le \frac{1}{\kappa} \sqrt{|⅄^*(h) - J\,⅄^*\,(h)|^2 + |⅄(h) - J\,⅄\,(h)|^2}. \end{aligned}$$

Then the integral equation (3.2) possesses a unique solution, and accordingly there is a unique solution of the two point BVP depicting the deformation of an elastic beam (3.1).

Proof. Let $J : \mathcal{S} \to \mathcal{S}$ be the operator on the set $\mathcal{S}$ defined by

$$J \curlywedge (h) = \int_0^1 \mathcal{L}^*(h, \mathfrak{d})\mathcal{P}(\mathfrak{d}, \curlywedge(\mathfrak{d}), \curlywedge'(\mathfrak{d}))d\mathfrak{d}, \text{ for all } \curlywedge \in \mathcal{S}. \tag{3.3}$$

Obviously, J is well defined. Now, for $(\curlywedge^*, \curlywedge) \in E(\mathcal{M})$ with $\curlywedge^*, \curlywedge \in \mathcal{S}$, we have

$$\begin{aligned}
&|J \curlywedge^* (h) - J \curlywedge (h)| \\
&= \left| \int_0^1 \mathcal{L}^*(h, \mathfrak{d})\mathcal{P}(\mathfrak{d}, \curlywedge^*(\mathfrak{d}), \curlywedge^{*\prime}(\mathfrak{d}))d\mathfrak{d} - \int_0^1 \mathcal{L}^*(h, \mathfrak{d})\mathcal{P}(\mathfrak{d}, \curlywedge(\mathfrak{d}), \curlywedge'(\mathfrak{d}))d\mathfrak{d} \right| \\
&\leq \int_0^1 \mathcal{L}^*(h, \mathfrak{d}) \left|\mathcal{P}(\mathfrak{d}, \curlywedge^*(\mathfrak{d}), \curlywedge^{*\prime}(\mathfrak{d})) - \mathcal{P}(\mathfrak{d}, \curlywedge(\mathfrak{d}), \curlywedge'(\mathfrak{d}))\right| d\mathfrak{d} \\
&\leq \sup_{0\leq h\leq 1} |\mathcal{P}(\mathfrak{d}, \curlywedge^*(h), \curlywedge^{*\prime}(h)) - \mathcal{P}(\mathfrak{d}, \curlywedge(h), \curlywedge'(h))| \int_0^1 \mathcal{L}^*(h, \mathfrak{d})d\mathfrak{d} \\
&\leq \frac{1}{\kappa} \sup_{0\leq h\leq 1} \sqrt{\left|\curlywedge^*(h) - J \curlywedge^* (h)\right|^2 + \left|\curlywedge(h) - J \curlywedge (h)\right|^2} \int_0^1 \mathcal{L}(h, \mathfrak{d})d\mathfrak{d} \\
&= \left(\frac{h^4 - 4h^3 + 6h^2}{24\kappa}\right) \sup_{0\leq h\leq 1} \sqrt{\left|\curlywedge^*(h) - J \curlywedge^* (h)\right|^2 + \left|\curlywedge(h) - J \curlywedge (h)\right|^2}.
\end{aligned}$$

From here we can write

$$\begin{aligned}
&\sup_{0\leq h\leq 1} |J \curlywedge^* (h) - J \curlywedge (h)|^2 \\
&\leq \left\{\sup_{0\leq h\leq 1} (\mathcal{Q}(\kappa, h))^2\right\} \sup_{0\leq h\leq 1} \left\{\left|\curlywedge^*(h) - J \curlywedge^* (h)\right|^2 + \left|\curlywedge(h) - J \curlywedge (h)\right|^2\right\}.
\end{aligned}$$

In light of the given assertions, we take $\left\{\sup_{0\leq h\leq 1} (\mathcal{Q}(\kappa, h))^2\right\} = \frac{\mu}{s} \in [0, \frac{1}{2})$, and hence we have

$$\mathcal{M}_{b_r}(J\curlywedge^*, J\curlywedge) \leq \frac{\mu}{s} \left[\mathcal{M}_{b_r}(\curlywedge^*, J\curlywedge^*) + \mathcal{M}b_r(\curlywedge, J\curlywedge)\right].$$

Consequently the assertion ($\spadesuit_2$) of Theorem 1 is established. Next, consider $(\curlywedge^*, \curlywedge) \in E(\mathcal{M})$ with $\curlywedge^*, \curlywedge \in \mathcal{S}$. We notice that $\curlywedge^*, \curlywedge \in \mathcal{U}$ and $\curlywedge^*(h) \leq \curlywedge(h)$ for all $h \in [0, 1]$; and by condition 2, we obtain $\inf_{0\leq h\leq 1} J(\curlywedge^*)(h) > 0$,

$$\begin{aligned}
J(\curlywedge^*)(h) &= \int_0^1 \mathcal{L}^*(h, \mathfrak{d})\mathcal{P}(\mathfrak{d}, \curlywedge^*(\mathfrak{d}), \curlywedge^{*\prime}(\mathfrak{d}))d\mathfrak{d} \leq \int_0^1 \mathcal{L}^*(h, \mathfrak{d})\mathcal{P}(\mathfrak{d}, 1, 0)d\mathfrak{d} \leq 1, \\
J(\curlywedge^*)(h) &= \int_0^1 \mathcal{L}^*(h, \mathfrak{d})\mathcal{P}(\mathfrak{d}, \curlywedge^*(\mathfrak{d}), \curlywedge^{*\prime}(\mathfrak{d}))d\mathfrak{d} \\
&\leq \int_0^1 \mathcal{L}^*(h, \mathfrak{d})\mathcal{P}(\mathfrak{d}, \curlywedge(\mathfrak{d}), \curlywedge'(\mathfrak{d}))d\mathfrak{d} = J(\mathfrak{d})(h).
\end{aligned}$$

This adds up to state that $J \curlywedge^* (h) \in \mathcal{U}$ and $\big(J(\curlywedge^*)(h), J(\curlywedge)(h)\big) \in E(\mathcal{M})$. In compliance with the assertion ($\spadesuit_1$), there exists a solution say $\gamma \in \mathcal{U}$ such that $J(\gamma) \in [\gamma]^1_{\mathcal{M}}$ and $J^2(\gamma) \in [\gamma]^1_{\mathcal{M}}$, and thus the goal that the condition (b) of

Theorem 1 is checked. Subsequently, by elementary calculations, one can see easily that the rest of the assertions of Theorem 1 and Corollary 1 are satisfied. Consequently, J has a unique fixed point, and hence the two-point BVP (3.1) has a unique solution in $\mathcal{S}$. $\square$

3.2 An application to a class of integral equations

In this subsection we find the existence of solution to the integral equation of the form

$$\curlywedge(h) = \int_0^N \mathcal{L}(h,\mathfrak{d})\mathcal{F}(\mathfrak{d},\curlywedge(\mathfrak{d}))d\mathfrak{d} + g(h),\ h \in [0,N], \tag{3.4}$$

where $\mathcal{L} : [0,N] \times [0,N] \to \mathbb{R}$ is the Kernel of the given integral equation, $\mathcal{F} : [0,N] \times \mathbb{R}^+ \to \mathbb{R}$ and $g : [0,N] \to \mathbb{R}$ are continuous functions.
Define graphical metric $\mathcal{M}_{b_r} : \mathcal{S} \times \mathcal{S} \to [0,\infty)$ by

$$\mathcal{M}_{b_r}(\curlywedge,\curlywedge^*) = \sup_{0\le h\le N} |\curlywedge(h) - \curlywedge^*(h)|^2,$$

for all $\curlywedge,\curlywedge^* \in \mathcal{S}$. Then $(\mathcal{S},\mathcal{M}_{b_r})$ is a $\mathcal{M}$-complete $gr_b ms$.
Let the operator $J : \mathcal{S} \to \mathcal{S}$ be given by the following

$$J\curlywedge(h) = \int_0^N \mathcal{L}(h,\mathfrak{d})\mathcal{F}(\mathfrak{d},\curlywedge(\mathfrak{d}))d\mathfrak{d} + g(h),\ \text{for all } \curlywedge \in \mathcal{S}.$$

Then $\curlywedge$ is a fixed point of J if and only if $\curlywedge$ is a solution of the IE (3.4).
Let the subsequent assertions take place:
(k_1) $\beta \in ([0,N],\mathbb{R})_c$ is a lower solution of the IE (3.4), i.e.,

$$\beta(h) \le \int_0^N \mathcal{L}(h,\mathfrak{d})\mathcal{F}(\mathfrak{d},\curlywedge(\mathfrak{d}))d\mathfrak{d} + g(h).$$

(k_2) For every $\curlywedge \in [0,N]$, $\mathcal{F}(\mathfrak{d},.)$ is a nondecreasing function on $]0,1]$, and pick μ appropriately such that

$$\inf_{0\le h\le N} \mathcal{L}(h,\mathfrak{d}) > 0,\ 0 \le \sup_{0\le h\le N} (\mathcal{W}(\mu,h))^2 < \frac{1}{2} \text{ and } \mathcal{L}(h,\mathfrak{d})\mathcal{F}(\mathfrak{d},1) \le N^{-1},$$

where $\mathcal{W}(\mu,h) = \frac{1+h\mu-e^{h\mu}}{\mu^2}$.
(k_3) For each $h \in [0,N]$ and $\curlywedge^*,\curlywedge \in \mathcal{S}$, we have

$$|\mathcal{F}(h,\curlywedge^*(h)) - \mathcal{F}(h,\curlywedge(h))| \le \sqrt{|\curlywedge^*(h) - J\curlywedge^*(h)|^2 + |\curlywedge(h) - J\curlywedge(h)|^2}.$$

Theorem 3. *Under the assertions* (k_1)*–*(k_3)*, the integral equation (3.4) possesses a unique solution.*

Proof. Using the theory of the hypothesis under consideration, for $(⅄^*, ⅄) \in E(\mathcal{M})$ and $⅄^*, ⅄ \in \mathcal{S}$, we have

$$
\begin{aligned}
&|J\, ⅄^*\,(h) - J\, ⅄\,(h)| = \left| \int_0^N \mathcal{L}(h,\mathfrak{d})\mathcal{F}(\mathfrak{d}, ⅄^*(\mathfrak{d}))d\mathfrak{d} + g(h) \right. \\
&\left. - \int_0^N \mathcal{L}(h,\mathfrak{d})\mathcal{F}(\mathfrak{d}, ⅄(\mathfrak{d}))d\mathfrak{d} - g(h) \right| \le \int_0^N \mathcal{L}(h,\mathfrak{d}) \left|\mathcal{F}(\mathfrak{d}, ⅄^*(\mathfrak{d})) - \mathcal{F}(\mathfrak{d}, ⅄(\mathfrak{d}))\right| d\mathfrak{d} \\
&\le \int_0^N \mathcal{L}(h,\mathfrak{d}) \sup_{0\le h\le N} \left|\mathcal{F}(h, ⅄^*(h)) - \mathcal{F}(h, ⅄(h))\right| d\mathfrak{d} \\
&= \sup_{0\le h\le N} \left|\mathcal{F}(h, ⅄^*(h)) - \mathcal{F}(h, ⅄(h))\right| \int_0^N \mathcal{L}(h,\mathfrak{d})d\mathfrak{d} \\
&\le \sup_{0\le h\le N} \sqrt{\left|⅄^*(h) - J\, ⅄^*\,(h)\right|^2 + \left|⅄(h) - J\, ⅄\,(h)\right|^2} \int_0^N \mathcal{L}(h,\mathfrak{d})d\mathfrak{d} \\
&= \left(\frac{1 + h\mu - e^{h\mu}}{\mu^2}\right) \sup_{0\le h\le N} \sqrt{\left|⅄^*(h) - J\, ⅄^*\,(h)\right|^2 + \left|⅄(h) - J\, ⅄\,(h)\right|^2}.
\end{aligned}
$$

We can write

$$
\begin{aligned}
\sup_{0\le h\le N} |J\, ⅄^*\,(h) - J\, ⅄\,(h)|^2 &\le \left\{ \sup_{0\le h\le N} \left(\mathcal{W}(\mu,h)\right)^2 \right\} \\
&\times \sup_{0\le h\le N} \left\{ \left|⅄^*(h) - J\, ⅄^*\,(h)\right|^2 + \left|⅄(h) - J\, ⅄\,(h)\right|^2 \right\}.
\end{aligned}
$$

In view of hypothesis, taking $\sup_{0\le h\le N} \left(\mathcal{W}(\mu,h)\right)^2 = \frac{\mu}{s} \in [0, \frac{1}{2})$, we get

$$
\mathcal{M}_{b_r}(J⅄^*, J⅄) \le \frac{\mu}{s}\left[\mathcal{M}_{b_r}(⅄^*, J⅄^*) + \mathcal{M}_{b_r}(⅄, J⅄)\right].
$$

Hence the assertion ($♠_2$) of Theorem 1 is contended. Next, consider $(⅄^*, ⅄) \in E(\mathcal{M})$ with $⅄^*, ⅄ \in \mathcal{S}$, we acquire $⅄^*, ⅄ \in \mathcal{U}$ and $⅄^*(h) \le ⅄(h)$ for all $h \in [0, N]$. By condition (k_2), we deduce $\inf_{0\le h\le N} J(⅄^*)(h) > 0$,

$$
J(⅄^*)(h) = \int_0^N \mathcal{L}(h,\mathfrak{d})\mathcal{F}(\mathfrak{d}, ⅄^*(\mathfrak{d}))\, d\mathfrak{d} \le \int_0^N \mathcal{L}(h,\mathfrak{d})\mathcal{F}(\mathfrak{d}, 1)\, d\mathfrak{d} \le 1,
$$

and

$$
J(⅄^*)(h) = \int_0^N \mathcal{L}(h,\mathfrak{d})\mathcal{F}(\mathfrak{d}, ⅄^*(\mathfrak{d}))\, d\mathfrak{d} \le \int_0^N \mathcal{L}(h,\mathfrak{d})\mathcal{F}(\mathfrak{d}, ⅄(\mathfrak{d}))\, d\mathfrak{d} = J(⅄)(h).
$$

It follows that $J\, ⅄^*\,(h) \in \mathcal{U}$ and $\big(J(⅄^*)(h), J(⅄)(h)\big) \in E(\mathcal{M})$. In consistence with the affirmation ($♠_1$), there exists $\lambda \in \mathcal{U}$ such that $J(\lambda) \in [\lambda]^1_{\mathcal{M}}$ and $J^2(\lambda) \in [\lambda]^1_{\mathcal{M}}$, and hence the assertion (b) of Theorem 1 is contended. Other assertions of Theorem 1 and Corollary 1 can easily be verified. Therefore, J has a unique fixed point and hence the integral equation (3.4) has a unique solution in $\mathcal{S}$. $\square$

4 An application of BCP in gr_bms

In [25], the authors instigated the concept of gr_bms along with its topological properties, and proved analogous of the Banach's theorem in this structure. The result was further polished in [26], where the authors extend the range of the Lipschitz constant ξ to the case $\left(\frac{1}{s},1\right]$. Before enunciating the application part of this section, the following notations and definitions are worth mentioning.

DEFINITION 5. A graph $\mathcal{M}=(V(\mathcal{M}),E(\mathcal{M}))$ is said to own the property $(\mathcal{P})$, if there exists a limit $\boldsymbol{r}\in\mathcal{S}$ of a converging $\mathcal{M}$-*twc J-Ps* $\{\varkappa_h\}$ and $h_0\in\mathbb{N}$ such that $(\varkappa_h,\boldsymbol{r})\in E(\mathcal{M})$ or $(\boldsymbol{r},\varkappa_h)\in E(\mathcal{M})$ for all $h>h_0$.

The following definition is due to [26] where the range of the Lipschitz constant ξ is extended to the case $\left(\frac{1}{s},1\right]$.

DEFINITION 6. Let $(\mathcal{S},\mathcal{M}_{b_r})$ be a gr_bms and $\mathcal{M}$ be the subgraph of a graph M associated with $(\mathcal{S},\mathcal{M}_{b_r})$ accommodating all the loops. We say $J:\mathcal{S}\to\mathcal{S}$ is an $\mathcal{M}$–Banach contraction (or graphical $(M,\mathcal{M})$-contraction) on $\mathcal{S}$ if the following two conditions are contended:

$(\mathcal{M}_{b_r}C1)$ For all $(\varkappa_1,\varkappa_2)\in E(\mathcal{M})$, we have $(J\varkappa_1,J\varkappa_2)\in E(\mathcal{M})$;

$(\mathcal{M}_{b_r}C2)$ $\exists\ \xi\in[0,1)$, for all $\varkappa_1,\varkappa_2\in\mathcal{S}$ with $(\varkappa_1,\varkappa_2)\in E(\mathcal{M})$, we have $\mathcal{M}_{b_r}(J\varkappa_1,J\varkappa_2)\le\xi\mathcal{M}_{b_r}(\varkappa_1,\varkappa_2)$.

Theorem 4. *[26] Let $J:\mathcal{S}\to\mathcal{S}$ be an $\mathcal{M}$–Banach contraction on an $\mathcal{M}$-complete graphical rectangular b-metric space. Suppose the underneath assertions are satisfied:*

(i) $\mathcal{M}$ holds the property $(\mathcal{P})$;

(ii) For odd positive integers ℓ and ρ, there exists $\varkappa_0\in\mathcal{S}$ with $J\varkappa_0\in[\varkappa_0]^{\ell}_{\mathcal{M}}$ and $J^2\varkappa_0\in[\varkappa_0]^{\rho}_{\mathcal{M}}$.
Then, $\exists\ \varkappa'\in\mathcal{S}$ so that the J-PicardSequence $\{\varkappa_{\wp}\}$ with initial value $\varkappa_0\in\mathcal{S}$ is $\mathcal{M}$-Term wise connected and converges to both $\varkappa'$ and $J\varkappa'$.

4.1 An application to the ascending motion of a rocket

In light of notions used in the Section 3, this subsection is devoted to establish the existence of solution of the equation representing ascending motion of rocket.
Consider a rocket possessing an initial mass $\mathfrak{m}_{\mathfrak{o}}$ (comprising propellant and shell) operating in an ascending motion. The rocket ingests the fuel at a constant rate $\mathfrak{r}=\frac{-d\mathfrak{m}}{dt}$. Relative to the motion of the rocket, fuel is expelled at a constant speed $\mathfrak{u}$. The mass of the rocket at any instant time $\mathfrak{t}$ is $\mathfrak{m}(\mathfrak{t})=\mathfrak{m}_{\mathfrak{o}}-\mathfrak{r}\mathfrak{t}$. During the driving stage, the equation of motion of the underlying rocket with aerodynamic drag force $\mathfrak{F}_{\mathfrak{d}}=\zeta\mathfrak{v}^2$ (ζ being the damping coefficient and $\mathfrak{v}$ is the velocity at time $\mathfrak{t}$) governing the upward motion at an excessive speed is given by the following first-order nonlinear differential equation with variable coefficients (see [21]).

$$\mathfrak{m}(\mathfrak{t})\frac{d\mathfrak{v}(\mathfrak{t})}{d\mathfrak{t}}+\zeta\mathfrak{v}^2(\mathfrak{t})+\mathfrak{m}(\mathfrak{t})\mathfrak{g}-\mathfrak{q}\mathfrak{u}=0, \tag{4.1}$$

where $\mathfrak{g}$ is the force of gravity acting on the system.
If we set the velocity $\mathfrak{v}(\mathfrak{t}) = \frac{\mathfrak{m}(\mathfrak{t})\mathfrak{f}(\mathfrak{t})}{\zeta\mathfrak{f}(\mathfrak{t})}$, where $\mathfrak{f}$ is the converted velocity, and substitute the derived equation into the equation of motion while altering the time to the dimensionless variable η, then the Equation (4.1) reduces to the following Bessel type differential equation:

$$\eta^2 \frac{d^2\Lambda(\eta)}{d\eta^2} + \eta\frac{d\Lambda(\eta)}{d\eta} - \mathcal{P}(\eta, \Lambda(\eta)) = 0, \qquad (4.2)$$
$$\Lambda(0) = 0, \quad \Lambda(1) = 0,$$

where $\mathcal{P} : [0, 1] \times R^+ \to R$ a function which is continuous.

In order to make our obtained fixed point results applicable, we reduce the equation (4.2) via Green's function to its equivalent integral equation stated below:

$$\Lambda(\eta) = \int_0^1 \mathcal{A}(\eta, \mathfrak{b})\, \mathcal{P}(\mathfrak{b}, \Lambda(\mathfrak{b}))d\mathfrak{b}, \quad \eta \in [0, 1], \qquad (4.3)$$

where the corresponding Green's function $\mathcal{A}(\eta, \mathfrak{b})$ is the following

$$\mathcal{A}(\eta, \mathfrak{b}) = \begin{cases} \frac{\eta}{2\mathfrak{b}}(1 - \mathfrak{b}^2), & 0 \le \eta < \mathfrak{b} \le 1; \\ \frac{\mathfrak{b}}{2\eta}(1 - \eta^2), & 0 \le \mathfrak{b} < \eta \le 1. \end{cases}$$

Let the graphical b-rectangular metric $\mathcal{M}_{b_r} : \mathcal{S} \times \mathcal{S} \to [0, \infty[$ be defined as in the Subsection 3.1. Consider the operator $J : \mathcal{S} \to \mathcal{S}$ given by the following

$$J\Lambda(\eta) = \int_0^1 \mathcal{A}(\eta, \mathfrak{b})\, \mathcal{P}(\mathfrak{b}, \Lambda(\mathfrak{b}))d\mathfrak{b}, \quad \text{for all } \eta \in \mathcal{S}. \qquad (4.4)$$

Then Λ is a fixed point of J if and only if Λ is a solution of the integral equation (4.3).

Succeeding theorem establishes the sufficient conditions for the existence of solution of the equation constituting the ascending motion of a rocket.

Theorem 5. *We suppose that the following assumptions take place:*

(1) $\delta(\eta) = \int_0^1 \mathcal{A}(\eta, \mathfrak{b})\, \mathcal{P}(\mathfrak{b}, \Lambda(\mathfrak{b}))d\mathfrak{b}$, $\delta \in ([0, 1], \mathbb{R})_c$;

(2) $\inf_{0\le h\le 1} \mathcal{A}(\eta, \mathfrak{b}) > 0$, $0 \le \sup_{0\le h\le 1} \left(\frac{\eta \log(1/16+\eta)}{2}\right)^2 < 1$ and $\mathcal{P}(\mathfrak{b}, 1) \le 1$;

(3) For each $\mathfrak{b} \in [0, 1]$, $\Lambda^*, \Lambda \in \mathcal{S}$, *and* J *is defined in (4.4), we have*

$$|\mathcal{P}(\mathfrak{b}, \Lambda^*(\eta)) - \mathcal{P}(\mathfrak{b}, \Lambda(\eta))| \le |\Lambda^*(\eta) - \Lambda(\eta)|.$$

Then there is a unique solution of integral equation (4.3), and consequently the first-order nonlinear differential equation (4.1) depicting the ascending motion of the rocket has a solution.

Proof. Utilizing Theorem 4, maintaining the same procedure, as described in the proof of the theorems in Section 3, we can easily get the desired result. □

Open Problems: 1) Can the results proved in this article be extended to multivalued mappings?

2) Can $\mathcal{M}_{b_r}$-gK type contractions be employed to set up the existence of solution of the following beam equation

$$\begin{aligned} &\curlywedge''''(r)g(\curlywedge)\curlywedge' + \mathcal{P}(r,\curlywedge,\curlywedge',\curlywedge'') = e(r), \qquad r\in[0,1],\\ &\curlywedge(0) = \curlywedge'(0) = 0, \curlywedge'(1) = \curlywedge'''(1) = 0, \end{aligned}$$

where $\mathcal{P} : [0,1]\times\mathbb{R}^3 \to \mathbb{R}$ is a continuous function and $e(r) \in L^1[0,1]$.

5 Conclusions

Based on the graph structure, $\mathcal{M}_{b_r}$-gK contraction is established, which is a generally new expansion to the current writing in the context of gr_bms. In this structure, some novel results are established, from which several existing results can be extracted. We propounded an appropriate example endowed with a graph for the accuracy of this paper's obtained results. Moreover, BCP is enunciated with modified assertions in the setting of gr_bms. Adorning the utilizations of the obtained results, some nonlinear problems having practical significance are considered: two-point boundary value problem describing deformations of an elastic beam, ascending motion of a rocket and a class of integral equations. The outcomes are significant both practically and hypothetically for the researchers working on fixed point theory applications and well as the scientists dealing with mechanical and engineering problems.

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References

[1] A.R. Aftabizadeh. Existence and uniqueness theorems for fourth-order boundary value problems. *Journal of Mathematical Analysis and Applications*, **116**(2):415–426, 1986. https://doi.org/10.1016/S0022-247X(86)80006-3.

[2] A. Aghanians and K. Nourouzi. Fixed points for Kannan type contractions in uniform spaces endowed with a graph. *Nonlinear Analysis: Modelling and Control*, **21**(1):103–113, 2016. https://doi.org/10.15388/NA.2016.1.7.

[3] C. Alegre, H. Dağ, S. Romaguera and P. Tirado. Characterizations of quasi-metric completeness in terms of Kannan-type fixed point theorems. *Hacettepe Journal of Mathematics and Statistics*, **46**(1):67–76, 2017. https://doi.org/10.15672/HJMS.2016.395.

[4] M.R. Alfuraidan. Remarks on Caristis fixed point theorem in metric spaces with a graph. *Fixed Point Theory and Applications*, **2014**(1):1–8, 2014. https://doi.org/10.1186/1687-1812-2014-240.

[5] P. Baradol, D. Gopal and S. Radenović. Computational fixed points in graphical rectangular metric spaces with application. *Journal of Computational and Applied Mathematics*, **375**:112805, 2020. https://doi.org/10.1016/j.cam.2020.112805.

[6] P. Baradol, J. Vujaković, D. Gopal and S. Radenović. On some new results in graphical rectangular b-metric spaces. *Mathematics*, **8**(4):488, 2020. https://doi.org/10.3390/math8040488.

[7] F. Bojor. Fixed points of Kannan mappings in metric spaces endowed with a graph. *Analele Universitatii" Ovidius" Constanta-Seria Matematica*, **20**(1):31–40, 2012. https://doi.org/10.2478/v10309-012-0003-x.

[8] L.B. Ćirić and B. Fisher. *Fixed Point Theory: contraction mapping principle.* FMS Press, Beograd, 2003.

[9] S.M. El-Sayed and A.C.M. Ran. On an iteration method for solving a class of nonlinear matrix equations. *SIAM Journal on Matrix Analysis and Applications*, **23**(3):632–645, 2002. https://doi.org/10.1137/S0895479899345571.

[10] J. Górnicki. Fixed point theorems for Kannan type mappings. *Journal of Fixed Point Theory and Applications*, **19**(3):2145–2152, 2017. https://doi.org/10.1007/s11784-017-0402-8.

[11] C.P. Gupta. Existence and uniqueness theorems for the bending of an elastic beam equation. *Applicable Analysis*, **26**(4):289–304, 1988. https://doi.org/10.1080/00036818808839715.

[12] J. Jachymski. The contraction principle for mappings on a metric space with a graph. *Proceedings of the American Mathematical Society*, **136**(4):1359–1373, 2008. https://doi.org/10.1090/S0002-9939-07-09110-1.

[13] S.B. Kaliaj. A Kannan-type fixed point theorem for multivalued mappings with application. *The Journal of Analysis*, **27**(3):837–849, 2019. https://doi.org/10.1007/s41478-018-0135-0.

[14] R. Kannan. Some results on fixed points. *Bull. Cal. Math. Soc.*, **60**:71–76, 1968.

[15] M.A. Khamsi and W.A. Kirk. *An introduction to metric spaces and fixed point theory.* John Wiley & Sons, 2011.

[16] J.J. Nieto and R. Rodríguez-López. Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. *Order*, **22**(3):223–239, 2005. https://doi.org/10.1007/s11083-005-9018-5.

[17] S. Radenović, T. Došenović, T. Aleksić Lampert and Z. Golubović. A note on some recent fixed point results for cyclic contractions in b-metric spaces and an application to integral equations. *Applied Mathematics and Computation*, **273**:155–164, 2016. https://doi.org/10.1016/j.amc.2015.09.089.

[18] S. Radenović, K. Zoto, N. Dedović, V. Šešum-Cavic and A.H. Ansari. Bhaskar-Guo-Lakshmikantam-Ćirić type results via new functions with applications to integral equations. *Applied Mathematics and Computation*, **357**:75–87, 2019. https://doi.org/10.1016/j.amc.2019.03.057.

[19] A.C.M. Ran and M.C.B. Reurings. A fixed point theorem in partially ordered sets and some applications to matrix equations. *Proceedings of the American Mathematical Society*, **132**:1435–1443, 2004. https://doi.org/10.1090/S0002-9939-03-07220-4.

[20] S. Reich. Some remarks concerning contraction mappings. *Canadian Mathematical Bulletin*, **14**(1):121–124, 1971. https://doi.org/10.4153/CMB-1971-024-9.

[21] W.C. Xie. *Differential equations for engineers.* Cambridge University Press, 2010.

[22] M. Younis and D. Singh. On the existence of the solution of Hammerstein integral equations and fractional differential equations. *Journal of Applied Mathematics and Computing*, **68**:1087–1105, 2022. https://doi.org/10.1007/s12190-021-01558-1.

[23] M. Younis, D. Singh and A.A.N. Abdou. A fixed point approach for tuning circuit problem in dislocated b-metric spaces. *Mathematical Methods in the Applied Sciences*, **45**(4):2234–2253, 2020. https://doi.org/10.1002/mma.7922.

[24] M. Younis, D. Singh, I. Altun and V. Chauhan. Graphical structure of extended b-metric spaces: an application to the transverse oscillations of a homogeneous bar. *International Journal of Nonlinear Sciences and Numerical Simulation*, 2021. https://doi.org/10.1515/ijnsns-2020-0126.

[25] M. Younis, D. Singh and A. Goyal. A novel approach of graphical rectangular b-metric spaces with an application to the vibrations of a vertical heavy hanging cable. *Journal of Fixed Point Theory and Applications*, **21**(33):1–17, 2019. https://doi.org/10.1007/s11784-019-0673-3.

[26] M. Younis, D. Singh and L. Shi. Revisiting graphical rectangular b-metric spaces. *Asian-European Journal of Mathematics*, **15**(4):2250072, 2022. https://doi.org/10.1142/S1793557122500723.