Mathematical Modelling and Analysis
Volume 27, Issue 4, 590–609, 2022
https://doi.org/10.3846/mma.2022.15311

A Composite Collocation Method Based on the Fractional Chelyshkov Wavelets for Distributed-Order Fractional Mobile-Immobile Advection-Dispersion Equation

Hamidreza Marasi and Mohammadhossein Derakhshan

Department of Applied Mathematics, Faculty of Mathematics, Statistics and Computer Science, University of Tabriz
29 Bahman Blvd, Tabriz, Iran
E-mail(*corresp.*): marasi@tabrizu.ac.ir
E-mail: m.h.derakhshan.20@gmail.com

Received August 10, 2021; revised September 18, 2022; accepted September 19, 2022

Abstract. In this study, an accurate and efficient composite collocation method based on the fractional order Chelyshkov wavelets is proposed for obtaining approximate solution of distributed-order fractional mobile-immobile advection-dispersion equation with initial and boundary conditions. Operational matrices based on the fractional Chelyshkov wavelets are constructed. The proposed method reduce the solution to a system of algebraic equations, which is solved by Newton's iterative method. Provided examples confirm the accuracy and applicability of the proposed method in line with the studied convergence analysis and error estimation. The obtained results of demonstrated numerical schemes illustrate that this approach is very accurate and efficient.

Keywords: mobile-immobile advection-dispersion, convergence, Chelyshkov wavelet, distributed order, composite collocation method.

AMS Subject Classification: 26A33; 34A08; 65M06; 65N12; 35R11.

1 Introduction

In recent years, fractional partial differential equations (FPDEs) with space or time fractional operators have become very favorite and a widespread consideration has been devoted to fractional operators [11, 13, 14, 16]. It has been found that these operators can model fluid dynamics, different physical processes and mathematical problems better than the corresponding non-integer order differential operators due to the non-local nature of fractional operators. In addition, FDEs are widely applied in various areas and disciplines as diffusion-wave phenomena, distributed order systems, dielectric materials [4] and electronic oscillator. A distributed-order differential model may be observed as a natural extension of the multi-term FDE [3, 23]. Also, distributed-order FDEs have been applied in [4] to expand the stress-strain relation in dielectrics. [18] used distributed-order FDEs to models of the filters in signal processing and the optimal distributed order for damping in control systems. These operators have important applications in modeling of super-slow diffusion where a mass of particles spreads at a logarithmic velocities. Specially, in the last decade the distributed-order differential equations have been attracted considerable interest of many researchers. Ford and Morgado [8] proved the uniqueness and existence of solution of distributed-order differential equations. Luchko [19] discussed the continuous dependence of solutions on initial conditions for the expanded distributed-order time-fractional diffusion equation on bounded domains. While the advection-dispersion model and its extensions such as the transient storage and mobile–immobile models based on a dispersion term of second order have been successfully applied in the past, recent studies highlights the requirement for transport models that can better demonstrate the connectivity and heterogeneity of spatial properties within a general lattice perspective of solute transport. The concept of the mobile–immobile has appear popular across hydrologists for describing transport in unsaturated and saturant zones, in granular and fractured media. The transport flow in porous medium is mainly controlled by the processes of dispersion and advection that will forecast a breakthrough curve. The mobile-immobile model is an intention of advection-dispersion equation, it can portray the movement and action of solute transport in both fractured media and porous. Analytical methods for solving mobile-immobile model are scarcely effective, so that we must resort to numerical methods. In recent years, many authors have proposed numerical methods to numerically solve the distributed-order FDEs, for example, Diethelm [6] proved effective numerical methods for solving linear and nonlinear distributed-order FDEs. In [24], Ye, Liu and Anh a compact difference scheme is presented to solve the distributed-order time-fractional diffusion-wave equation. Gao and Sun [9] derived the two difference schemes for both one-dimensional and two-dimensional distributed order differential equations. Alikhanov [1] proposed a difference scheme for the numerical approximation of the multi-term variable distributed order diffusion equation. Pourbabaee and Saadatmandi [21] derived a numerical method based on the shifted Legendre operational matrix of distributed order fractional derivative together with tau method for approximation of solutions of linear distributed-order FDEs. Also, other methods such as Legendre spectral element numerical method [5], linearized stable spectral method [12], hybrid fully spectral linearized scheme [15] and a new operational matrices based spectral method [10].

In this paper, we attempt to propose an efficient numerical scheme based on a composite collocation method using the fractional order Chelyshkov wavelets for solving distributed order mobile-immobile advection-dispersion equation. Consider the following time fractional distributed order mobile-immobile advection-dispersion equation

$$\alpha_1 Z_t(x,t) + \alpha_2 \mathbb{D}_t^{\mu} Z(x,t) = -\alpha_3 Z_x(x,t) + \alpha_4 Z_{xx}(x,t) + h(x,t), \tag{1.1}$$

subject to the following initial and boundary conditions:

$$\begin{aligned} Z(x,0) &= f_1(x),\ x \in [0,1], \\ Z(0,t) &= f_2(t),\ Z(1,t) = f_3(t),\ t \in [0,1], \end{aligned} \tag{1.2}$$

where $\alpha_1 \geq 0,\ \alpha_2 \geq 0,\ \alpha_3 > 0,\ \alpha_4 > 0$ and the distributed order fractional operator $\mathbb{D}_t^{\mu}$ is in the Caputo sense as the following:

$$\mathbb{D}_t^{\rho(\mu)} Z(x,t) = \int_0^1 \rho(\mu)^C \mathfrak{D}_t^{\mu} Z(x,t) d\mu,$$

where ${}^C\mathfrak{D}_t^{\mu} Z(x,t)$ denotes the time fractional derivative of the Caputo type:

$${}^C\mathfrak{D}_t^{\mu} Z(x,t) = \begin{cases} \dfrac{1}{\Gamma(n-\mu)} \int_0^t (t-\tau)^{n-\mu-1} \dfrac{\partial^n Z(x,\tau)}{\partial \tau^n} d\tau,\ \mu \in (n-1,n],\ n \in \mathbb{N}, \\ \dfrac{\partial^n Z(x,t)}{\partial t^n},\quad \mu = n, \end{cases}$$

and also, here we consider $n = 1$. The fundamental and essential advantage of applying orthogonal polynomials is that the considered equations and models can be converted to systems of algebraic equations. Therefore, the main aim of the presented work is to use the Chelyshkov wavelets of fractional order to solve Equations (1.1) and (1.2). First, we construct the Chelyshkov wavelets of fractional order, then the integration of Chelyshkov wavelet operators of fractional order is computed. The proposed numerical method reduces the distributed-order fractional mobile-immobile advection-dispersion equation to a system of algebraic equations. The derived system is solved by Newton's iterative method. High-dimension systems of algebraic equations can lead to large storage requirements and high computational complexity. However the Chelyshkov wavelet of fractional order is structurally sparse, which reduces the computational complexity of the resulting algebraic system. Also, Among the methods mentioned in the previous paragraph, fractional Chelyshkov wavelets have the simplest structures, high accuracy and fast convergence.

The rest of the paper is as follows: Section 2 presents some necessary and main definitions, lemmas and fundamental preliminaries of the fractional calculus. Also, we present the essential formulation of Chelyshkov wavelets and construct the Chelyshkov wavelets of fractional order, in this section. Section 3, proposes a numerical method to approximate the solution of Equations (1.1) and (1.2). In Section 4, the convergence analysis of the proposed numerical method is illustrated. In Section 5, some numerical examples are provided to show the efficiency and accuracy of the proposed approach. Finally, we have conclusions in Section 6.

2 Notation and mathematical preliminaries

In this section, we provide some basic and fundamental definitions, lemmas and features of fractional integrals and derivatives which are applied in this paper.

Also, in this section, we express some basic concepts related to Chelyshkov wavelets, fractional order Chelyshkov wavelets and their properties.

DEFINITION 1. [25] A function $Z(t)$, $t > 0$ is said to be in the space $\mathbb{C}_\lambda$, where $\lambda \in \mathbb{R}$, if there exists a real number $\sigma > \lambda$ such that $Z(t) = t^\sigma Z_1(t), Z_1(t) \in \mathbb{C}[0,\infty)$. Also, the real function $Z(t)$, $t > 0$ is said to be in the space $\mathbb{C}^n_\lambda$ if and only if $Z^n(t) \in \mathbb{C}_\lambda$ where $n \in \mathbb{N}$.

DEFINITION 2. Suppose that $0 < \alpha \leq 1$ and $Z(t) \in \mathbb{C}_\lambda$ where $\lambda \geq -1$. Then, the Riemann-Liouville fractional integral of order α is given by:

$$\mathbb{I}_t^\alpha Z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} Z(u)\, du.$$

DEFINITION 3. Suppose that $0 < \alpha \leq 1$ and $Z(t) \in \mathbb{C}_{-1}$. Then, the Caputo fractional derivative of order α is given by:

$${}^C\mathbf{D}_t^\alpha Z(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-u)^{-\alpha} Z'(u)\, du = \mathbb{I}_t^{1-\alpha} Z'(t).$$

Lemma 1. *Suppose that $n-1 < \alpha \leq n$ and $Z(t) \in \mathbb{C}^n_{-1}$. Then, the Riemann-Liouville fractional integrals and Caputo fractional derivatives satisfy the following relations [7]:*

$$1.{}^C\boldsymbol{D}_t^\alpha \mathbb{I}_t^\alpha Z(t) = Z(t), \quad 2.\mathbb{I}_t^{\alpha C}\boldsymbol{D}_t^\alpha Z(t) = Z(t) - \sum_{j=0}^{n-1} \frac{t^j}{j!} Z^{(j)}(0),$$

$$3.{}^C\boldsymbol{D}_t^\alpha t^\gamma = \begin{cases} 0, & \varrho \in N_0,\, \gamma < \varrho, \\ \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma-\varrho+1)} t^{\gamma-\varrho}, & \textit{otherwise}. \end{cases}$$

2.1 Chelyshkov wavelets and fractional order Chelyshkov wavelets

Here, we briefly give some notations and preliminaries of Chelyshkov wavelets and fractional order Chelyshkov wavelets. Chelyshkov polynomials, $\mathfrak{P}_{\mathbf{N},\widehat{\mathbf{N}}}(t)$ can be shown through the following power relation [2]:

$$\mathfrak{P}_{\mathbf{N},\widehat{\mathbf{N}}}(t) = \sum_{i=\mathbf{N}}^{\widehat{\mathbf{N}}} (-1)^{i-\mathbf{N}} \begin{pmatrix} \widehat{\mathbf{N}} - \mathbf{N} \\ i - \mathbf{N} \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{N}} + i + 1 \\ \widehat{\mathbf{N}} - \mathbf{N} \end{pmatrix} t^i, \ \mathbf{N} = 0, 1, \ldots, \widehat{\mathbf{N}}.$$

Furthermore, the orthogonality condition on the interval $[0,1]$ is satisfied for these polynomials as follows:

$$\int_0^1 \mathfrak{P}_{\mathbf{N},\widehat{\mathbf{N}}}(t) \mathfrak{P}_{\mathbf{M},\widehat{\mathbf{N}}}(t) dt = \frac{\delta_{\mathbf{N},\mathbf{M}}}{\mathbf{N}+\mathbf{M}+1},$$

in which $\delta_{\mathbf{N},\mathbf{M}}$ is Kronecker delta. Now, we define Chelyshkov wavelets on $L^2[0,1)$ as follows [20]:

$$\varPsi_{\mathbf{M},\mathbf{N},\widehat{\mathbf{N}}}(t) = \begin{cases} \sqrt{2\mathbf{N}+1}2^{\frac{m-1}{2}} \mathfrak{P}_{\mathbf{N},\widehat{\mathbf{N}}}(2^{m-1}t - \hat{m}), & t \in \left[\frac{\hat{m}}{2^{m-1}}, \frac{\hat{m}+1}{2^{m-1}}\right), \\ 0, & \text{otherwise}, \end{cases}$$

in which $\mathbf{N} = 0, 1, \ldots, \widehat{\mathbf{N}}$, $\widehat{\mathbf{N}} = M - 1$, $\hat{m} = \mathbf{M} - 1$, $\mathbf{M} = 1, \ldots, 2^{m-1}$ and $\mathfrak{P}_{\mathbf{N},\widehat{\mathbf{N}}}(t)$ are the Chelyshkov polynomials.

Fractional-order Chelyshkov wavelets, $\Psi_{\mathbf{M},\mathbf{N},\widehat{\mathbf{N}}}(t)^{(\eta,\eta')}$ on $L^2[0,\eta')$ can be obtain through the following relation:

$$\Psi^{(\eta,\eta')}_{\mathbf{M},\mathbf{N},\widehat{\mathbf{N}}}(t)=\begin{cases}\sqrt{\eta(2\mathbf{N}+1)}2^{\frac{m-1}{2}}\mathfrak{P}^{(\eta)}_{\mathbf{N},\widehat{\mathbf{N}}}(\frac{2^{m-1}}{\eta'}t-\hat{m}), t\in\left[\frac{\hat{m}}{2^{m-1}}\eta', \frac{\hat{m}+1}{2^{m-1}}\eta'\right), \\ 0, \qquad \text{otherwise},\end{cases} \tag{2.1}$$

in which $\mathfrak{P}^{(\eta)}_{\mathbf{N},\widehat{\mathbf{N}}}(t)$ denotes the fractional order Chelyshkov polynomials on the interval $[0, 1)$. Furthermore, the analytic form of the fractional order Chelyshkov functions can be constructed by considering a change of variable as $t = t^\eta$,

$$\mathfrak{P}^{(\eta)}_{\mathbf{M},\mathbf{N},\widehat{\mathbf{N}}}(t)=\sum_{i=\mathbf{N}}^{\widehat{\mathbf{N}}}(-1)^{i-\mathbf{N}}\begin{pmatrix}\widehat{\mathbf{N}}-\mathbf{N}\\ i-\mathbf{N}\end{pmatrix}\begin{pmatrix}\widehat{\mathbf{N}}+i+1\\ \widehat{\mathbf{N}}-\mathbf{N}\end{pmatrix}t^{i\eta},\ \mathbf{N} = 0, 1, \ldots, \widehat{\mathbf{N}}. \tag{2.2}$$

Lemma 2. *Suppose that $\mathfrak{P}^{(\eta)}_{\mathbf{N},\widehat{\mathbf{N}}}(t)$ be the set of fractional order Chelyshkov polynomials in the interval $[0, 1]$ then, these polynomials are orthogonal with respect to the weight function $\mathbb{W}^{(\eta)}(t) = t^{(\eta-1)}$ in the interval $[0, 1]$, that is:*

$$\int_0^1 \mathfrak{P}^{(\eta)}_{\mathbf{N},\widehat{\mathbf{N}}}(t)\mathfrak{P}^{(\eta)}_{\mathbf{M},\widehat{\mathbf{N}}}(t)\mathbb{W}^{(\eta)}(t)dt = \frac{\delta_{\mathbf{N},\mathbf{M}}}{\eta(\mathbf{N}+\mathbf{M}+1)}.$$

Proof. By considering the change of variable $t = t^\eta$ and applying features of the fractional order Chelyshkov functions, we obtain:

$$\begin{aligned}\int_0^1 \mathfrak{P}_{\mathbf{N},\widehat{\mathbf{N}}}(t)\mathfrak{P}_{\mathbf{M},\widehat{\mathbf{N}}}(t)dt &= \int_0^1 \mathfrak{P}_{\mathbf{N},\widehat{\mathbf{N}}}(t^\eta)\mathfrak{P}_{\mathbf{M},\widehat{\mathbf{N}}}(t^\eta)\eta t^{\eta-1}dt \\ &= \eta\int_0^1 \mathfrak{P}^{(\eta)}_{\mathbf{N},\widehat{\mathbf{N}}}(t)\mathfrak{P}^{(\eta)}_{\mathbf{M},\widehat{\mathbf{N}}}(t)\mathbb{W}^{(\eta)}(t)dt.\end{aligned}$$

Therefore, by applying orthogonality condition of Chelyshkov polynomials, the proof is completed. □

3 A composite collocation method based on the fractional Chelyshkov wavelets

A function $Z(t) \in L^2[0,\eta')$ can be expanded in terms of fractional order Chelyshkov wavelets as:

$$Z(t) \simeq \sum_{\mathbf{M}=1}^{2^{m-1}}\sum_{\mathbf{N}=0}^{\widehat{\mathbf{N}}} d_{\mathbf{M},\mathbf{N}}\Psi^{(\eta,\eta')}_{\mathbf{M},\mathbf{N},\widehat{\mathbf{N}}}(t) = \Delta^T\Lambda^{(\eta,\eta')}(t),$$

in which

$$d_{\mathbf{M},\mathbf{N}} = \int_0^{\eta'} \Psi^{(\eta,\eta')}_{\mathbf{M},\mathbf{N},\widehat{\mathbf{N}}}(t) Z(t) \mathbb{W}^{(\eta)}(t) dt.$$

Also, Δ and Λ are $2^{m-1}M$-vectors such that:

$$\Delta = [d_{1,0}, d_{1,1}, d_{1,2}, \dots, d_{2^{m-1},M-1}]^T,$$
$$\Lambda^{(\eta,\eta')}(t) = [\Psi^{(\eta,\eta')}_{1,0,\widehat{\mathbf{N}}}(t), \Psi^{(\eta,\eta')}_{1,1,\widehat{\mathbf{N}}}(t), \Psi^{(\eta,\eta')}_{1,2,\widehat{\mathbf{N}}}(t), \dots, \Psi^{(\eta,\eta')}_{2^{m-1},M-1,\widehat{\mathbf{N}}}(t)]^T.$$

For the two-variable functions $Z(x,t) \in L^2([0,\eta') \times [0,\eta'))$, we can get its approximation by applying fractional order Chelyshkov wavelets as:

$$\begin{aligned} Z(x,t) &\simeq \sum_{\mathbf{M}=1}^{2^{m-1}} \sum_{\mathbf{N}=0}^{M-1} \sum_{\mathbf{M}'=1}^{2^{m'-1}} \sum_{\mathbf{N}'=0}^{M'-1} d'_{\mathbf{M},\mathbf{N},\mathbf{M}',\mathbf{N}'} \Psi^{(\eta,\eta')}_{\mathbf{M},\mathbf{N},\widehat{\mathbf{N}}}(x) \Psi^{(\eta,\eta')}_{\mathbf{M}',\mathbf{N}',\widehat{\mathbf{N}}'}(t) \\ &= (\Lambda^{(\eta,\eta')}(x))^T \Omega \Lambda^{(\eta,\eta')}(t), \end{aligned}$$

in which Ω is $(2^{m-1}M) \times (2^{m'-1}M')$-matrix and

$$d'_{\mathbf{M},\mathbf{N},\mathbf{M}',\mathbf{N}'} = \int_0^{\eta'} \int_0^{\eta'} Z(x,t) \Psi^{(\eta,\eta')}_{\mathbf{M},\mathbf{N},\widehat{\mathbf{N}}}(x) \Psi^{(\eta,\eta')}_{\mathbf{M},\mathbf{N},\widehat{\mathbf{N}}}(t) \mathbb{W}^{(\eta)}(x) \mathbb{W}^{(\eta)}(t) dx dt.$$

Now, we will get a relation to calculate the Riemann-Liouville fractional integral $\mathbb{I}^{\alpha}_t$ of the function $\Lambda^{(\eta,\eta')}(t)$ as follows:

$$\mathbb{I}^{\alpha}_t \Lambda^{(\eta,\eta')}(t) = \mathbb{Q}(t,\alpha,\eta).$$

For this aim, we apply the Laplace transform, which is also suitable to the fractional order Chelyskov wavelets and to their Riemann-Liouville integral transforms, since these functions belong to $L^2[0,\eta)$. From (2.1) we get:

$$\begin{aligned} \Psi^{(\eta,\eta')}_{\mathbf{M},\mathbf{N},\widehat{\mathbf{N}}}(t) = \quad & \nu_{\frac{\hat{m}}{2^{m-1}}\eta'}(t) \sqrt{\eta(2\mathbf{N}+1)} 2^{\frac{m-1}{2}} \mathfrak{P}^{(\eta)}_{\mathbf{N},\widehat{\mathbf{N}}}(\tfrac{2^{m-1}}{\eta'} t - \hat{m}) \\ & -\nu_{\frac{\hat{m}+1}{2^{m-1}}\eta'}(t) \sqrt{\eta(2\mathbf{N}+1)} 2^{\frac{m-1}{2}} \mathfrak{P}^{(\eta)}_{\mathbf{N},\widehat{\mathbf{N}}}(\tfrac{2^{m-1}}{\eta'} t - \hat{m}), \end{aligned} \tag{3.1}$$

in which $\nu_{\xi}(t)$ is unit step function given by:

$$\nu_{\xi}(t) = \begin{cases} 1, & t \geq \xi, \\ 0, & t < \xi. \end{cases}$$

Next, we use the Laplace transform to the both sides of relation (3.1), we obtain:

$$\begin{aligned} &\mathcal{L}\left\{\Psi^{(\eta,\eta')}_{\mathbf{M},\mathbf{N},\widehat{\mathbf{N}}}(t); s\right\} = \sqrt{\eta(2\mathbf{N}+1)} 2^{\frac{m-1}{2}} \mathcal{L}\Big\{\nu_{\frac{\hat{m}}{2^{m-1}}\eta'}(t) \mathfrak{P}^{(\eta)}_{\mathbf{N},\widehat{\mathbf{N}}}(\frac{2^{m-1}}{\eta'} t - \hat{m}); s\Big\} \\ &\quad - \sqrt{\eta(2\mathbf{N}+1)} 2^{\frac{m-1}{2}} \mathcal{L}\Big\{\nu_{\frac{\hat{m}+1}{2^{m-1}}\eta'}(t) \mathfrak{P}^{(\eta)}_{\mathbf{N},\widehat{\mathbf{N}}}(\frac{2^{m-1}}{\eta'} t - \hat{m}); s\Big\} \\ &= \sqrt{\eta(2\mathbf{N}+1)} 2^{\frac{m-1}{2}} e^{-\frac{\hat{m}}{2^{m-1}}\eta' s} \mathcal{L}\Big\{\mathfrak{P}^{(\eta)}_{\mathbf{N},\widehat{\mathbf{N}}}(\frac{2^{m-1}}{\eta'} t); s\Big\} \\ &\quad - \sqrt{\eta(2\mathbf{N}+1)} 2^{\frac{m-1}{2}} e^{-\frac{\hat{m}+1}{2^{m-1}}\eta' s} \mathcal{L}\Big\{\mathfrak{P}^{(\eta)}_{\mathbf{N},\widehat{\mathbf{N}}}(\frac{2^{m-1}}{\eta'} t + 1); s\Big\}. \end{aligned}$$

From the definition of $\mathfrak{P}^{(\eta)}_{\mathbf{N},\widehat{\mathbf{N}}}(t)$ in Equation (2.2), we obtain:

$$\begin{aligned}
&\mathcal{L}\Big\{\Psi^{(\eta,\eta')}_{\mathbf{M},\mathbf{N},\widehat{\mathbf{N}}}(t);s\Big\} = \sqrt{\eta(2\mathbf{N}+1)}2^{\frac{m-1}{2}}e^{-\frac{\hat{m}}{2^{m-1}}\eta' s}\\
&\quad\mathcal{L}\Big\{\sum_{i=\mathbf{N}}^{\widehat{\mathbf{N}}}(-1)^{i-\mathbf{N}}\left(\begin{array}{c}\widehat{\mathbf{N}}-\mathbf{N}\\ i-\mathbf{N}\end{array}\right)\left(\begin{array}{c}\widehat{\mathbf{N}}+i+1\\ \widehat{\mathbf{N}}-\mathbf{N}\end{array}\right)\frac{2^{(m-1)i\eta}}{\eta'^{i\eta}}t^{i\eta};s\Big\} - \sqrt{\eta(2\mathbf{N}+1)}\\
&\quad\times 2^{\frac{m-1}{2}}e^{-\frac{\hat{m}+1}{2^{m-1}}\eta' s}\mathcal{L}\Big\{\sum_{i=\mathbf{N}}^{\widehat{\mathbf{N}}}\sum_{j=0}^{\lfloor i\eta\rfloor}(-1)^{i-\mathbf{N}}\left(\begin{array}{c}\widehat{\mathbf{N}}-\mathbf{N}\\ i-\mathbf{N}\end{array}\right)\left(\begin{array}{c}\widehat{\mathbf{N}}+i+1\\ \widehat{\mathbf{N}}-\mathbf{N}\end{array}\right)\\
&\quad\times\left(\begin{array}{c}i\eta\\ j\end{array}\right)\frac{2^{(m-1)(i\eta-j)}}{\eta'^{i\eta-j}}t^{i\eta-j};s\Big\} = \sqrt{\eta(2\mathbf{N}+1)}2^{\frac{m-1}{2}}e^{-\frac{\hat{m}}{2^{m-1}}\eta' s}\\
&\quad\times\sum_{i=\mathbf{N}}^{\widehat{\mathbf{N}}}(-1)^{i-\mathbf{N}}\left(\begin{array}{c}\widehat{\mathbf{N}}-\mathbf{N}\\ i-\mathbf{N}\end{array}\right)\left(\begin{array}{c}\widehat{\mathbf{N}}+i+1\\ \widehat{\mathbf{N}}-\mathbf{N}\end{array}\right)\frac{2^{(m-1)i\eta}\Gamma(i\eta+1)}{\eta'^{i\eta}s^{i\eta+1}}\\
&\quad-\sqrt{\eta(2\mathbf{N}+1)}2^{\frac{m-1}{2}}e^{-\frac{\hat{m}+1}{2^{m-1}}\eta' s}\sum_{i=\mathbf{N}}^{\widehat{\mathbf{N}}}\sum_{j=0}^{\lfloor i\eta\rfloor}(-1)^{i-\mathbf{N}}\left(\begin{array}{c}\widehat{\mathbf{N}}-\mathbf{N}\\ i-\mathbf{N}\end{array}\right)\left(\begin{array}{c}\widehat{\mathbf{N}}+i+1\\ \widehat{\mathbf{N}}-\mathbf{N}\end{array}\right)\\
&\quad\times\left(\begin{array}{c}i\eta\\ j\end{array}\right)\frac{2^{(m-1)(i\eta-j)}\Gamma(i\eta-j+1)}{\eta'^{i\eta-j}s^{i\eta-s+1}}.
\end{aligned}\tag{3.2}$$

By using the properties of the Laplace transform and the Riemann-Liouville fractional integral and applying Equation (3.2), we obtain:

$$\begin{aligned}
&\mathcal{L}\Big\{\mathbb{I}^{\alpha}_{t}\Psi^{(\eta,\eta')}_{\mathbf{M},\mathbf{N},\widehat{\mathbf{N}}}(t);s\Big\} = \mathcal{L}\Big\{\frac{t^{\alpha-1}}{\Gamma(\alpha)} * \Psi^{(\eta,\eta')}_{\mathbf{M},\mathbf{N},\widehat{\mathbf{N}}}(t);s\Big\}\\
&\quad= \mathcal{L}\Big\{\frac{t^{\alpha-1}}{\Gamma(\alpha)};s\Big\}\mathcal{L}\Big\{\Psi^{(\eta,\eta')}_{\mathbf{M},\mathbf{N},\widehat{\mathbf{N}}}(t);s\Big\} = \sqrt{\eta(2\mathbf{N}+1)}2^{\frac{m-1}{2}}e^{-\frac{\hat{m}}{2^{m-1}}\eta' s}\\
&\quad\times\sum_{i=\mathbf{N}}^{\widehat{\mathbf{N}}}(-1)^{i-\mathbf{N}}\left(\begin{array}{c}\widehat{\mathbf{N}}-\mathbf{N}\\ i-\mathbf{N}\end{array}\right)\left(\begin{array}{c}\widehat{\mathbf{N}}+i+1\\ \widehat{\mathbf{N}}-\mathbf{N}\end{array}\right)\frac{2^{(m-1)i\eta}\Gamma(i\eta+1)}{\eta'^{i\eta}s^{i\eta+\alpha+1}}\\
&\quad-\sqrt{\eta(2\mathbf{N}+1)}2^{\frac{m-1}{2}}e^{-\frac{\hat{m}+1}{2^{m-1}}\eta' s}\sum_{i=\mathbf{N}}^{\widehat{\mathbf{N}}}\sum_{j=0}^{\lfloor i\eta\rfloor}(-1)^{i-\mathbf{N}}\left(\begin{array}{c}\widehat{\mathbf{N}}-\mathbf{N}\\ i-\mathbf{N}\end{array}\right)\left(\begin{array}{c}\widehat{\mathbf{N}}+i+1\\ \widehat{\mathbf{N}}-\mathbf{N}\end{array}\right)\\
&\quad\times\left(\begin{array}{c}i\eta\\ j\end{array}\right)\frac{2^{(m-1)(i\eta-j)}\Gamma(i\eta-j+1)}{\eta'^{i\eta-j}s^{i\eta-s+\alpha+1}}.
\end{aligned}\tag{3.3}$$

Taking the inverse Laplace transform on Equation (3.3) yields:

$$\begin{aligned}
&\mathbb{I}^{\alpha}_{t}\Psi^{(\eta,\eta')}_{\mathbf{M},\mathbf{N},\widehat{\mathbf{N}}}(t)\\
&\quad= \sqrt{\eta(2\mathbf{N}+1)}2^{\frac{m-1}{2}}\sum_{i=\mathbf{N}}^{\widehat{\mathbf{N}}}(-1)^{i-\mathbf{N}}\left(\begin{array}{c}\widehat{\mathbf{N}}-\mathbf{N}\\ i-\mathbf{N}\end{array}\right)\left(\begin{array}{c}\widehat{\mathbf{N}}+i+1\\ \widehat{\mathbf{N}}-\mathbf{N}\end{array}\right)\\
&\quad\times\frac{2^{(m-1)i\eta}\Gamma(i\eta+1)}{\eta'^{i\eta}\Gamma(i\eta+\alpha+1)}(t-\frac{\hat{m}}{2^{m-1}}\eta')^{i\eta+\alpha}\nu_{\frac{\hat{m}}{2^{m-1}}\eta'}(t)
\end{aligned}$$

$$- \sqrt{\eta(2\mathbf{N}+1)}2^{\frac{m-1}{2}} \sum_{i=\mathbf{N}}^{\widehat{\mathbf{N}}} \sum_{j=0}^{\lfloor i\eta \rfloor} (-1)^{i-\mathbf{N}} \begin{pmatrix} \widehat{\mathbf{N}}-\mathbf{N} \\ i-\mathbf{N} \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{N}}+i+1 \\ \widehat{\mathbf{N}}-\mathbf{N} \end{pmatrix}$$

$$\times \begin{pmatrix} i\eta \\ j \end{pmatrix} \frac{2^{(m-1)(i\eta-j)}\Gamma(i\eta-j+1)}{\eta'^{i\eta-j}\Gamma(i\eta-j+\alpha+1)}(t-\frac{\hat{m}+1}{2^{m-1}}\eta')^{i\alpha-j+\alpha}\nu_{\frac{\hat{m}+1}{2^{m-1}}\eta'}(t). \quad (3.4)$$

Thus, by applying Equation (3.4), we obtain the following formula:

$$\mathbb{I}_t^{\alpha}\Psi_{\mathbf{M},\mathbf{N},\widehat{\mathbf{N}}}^{(\eta,\eta')}(t) = \begin{cases} 0, \quad t \in [0, \frac{\hat{m}}{2^{m-1}\eta'}), \\ \sqrt{\eta(2\mathbf{N}+1)}2^{\frac{m-1}{2}}\sum_{i=\mathbf{N}}^{\widehat{\mathbf{N}}}(-1)^{i-\mathbf{N}}\begin{pmatrix} \widehat{\mathbf{N}}-\mathbf{N} \\ i-\mathbf{N} \end{pmatrix}\begin{pmatrix} \widehat{\mathbf{N}}+i+1 \\ \widehat{\mathbf{N}}-\mathbf{N} \end{pmatrix} \\ \times\frac{2^{(m-1)i\eta}\Gamma(i\eta+1)}{\eta'^{i\eta}\Gamma(i\eta+\alpha+1)}(t-\frac{\hat{m}}{2^{m-1}}\eta')^{i\eta+\alpha}, \quad t\in[\frac{\hat{m}}{2^{m-1}\eta'}, \frac{\hat{m}+1}{2^{m-1}\eta'}), \\ \sqrt{\eta(2\mathbf{N}+1)}2^{\frac{m-1}{2}}\sum_{i=\mathbf{N}}^{\widehat{\mathbf{N}}}(-1)^{i-\mathbf{N}}\begin{pmatrix} \widehat{\mathbf{N}}-\mathbf{N} \\ i-\mathbf{N} \end{pmatrix}\begin{pmatrix} \widehat{\mathbf{N}}+i+1 \\ \widehat{\mathbf{N}}-\mathbf{N} \end{pmatrix} \\ \times\frac{2^{(m-1)i\eta}\Gamma(i\eta+1)}{\eta'^{i\eta}\Gamma(i\eta+\alpha+1)}(t-\frac{\hat{m}}{2^{m-1}}\eta')^{i\eta+\alpha} \\ -\sqrt{\eta(2\mathbf{N}+1)}2^{\frac{m-1}{2}}\sum_{i=\mathbf{N}}^{\widehat{\mathbf{N}}}\sum_{j=0}^{\lfloor i\eta\rfloor}(-1)^{i-\mathbf{N}}\begin{pmatrix} \widehat{\mathbf{N}}-\mathbf{N} \\ i-\mathbf{N} \end{pmatrix} \\ \times\begin{pmatrix} \widehat{\mathbf{N}}+i+1 \\ \widehat{\mathbf{N}}-\mathbf{N} \end{pmatrix}\begin{pmatrix} i\eta \\ j \end{pmatrix}\frac{2^{(m-1)(i\eta-j)}\Gamma(i\eta-j+1)}{\eta'^{i\eta-j}\Gamma(i\eta-j+\alpha+1)} \\ \times(t-\frac{\hat{m}+1}{2^{m-1}}\eta')^{i\alpha-j+\alpha}\nu_{\frac{\hat{m}+1}{2^{m-1}}\eta'}(t), \quad t\in[\frac{\hat{m}+1}{2^{m-1}\eta'}, \eta'). \end{cases}$$

3.1 Description of the proposed numerical method

In this section, we describe the using of the fractional order Chelyshkov wavelets for solving Equation (1.1) with the initial and boundary conditions (1.2). For this purpose, first we extend $\frac{\partial^3 Z(x,t)}{\partial^2 \partial t}$ as follows:

$$\frac{\partial^3 Z(x,t)}{\partial x^2 \partial t} = (\Lambda^{(\eta,\eta')}(x))^T \Omega \Lambda^{(\eta_1,\eta')}(t). \quad (3.5)$$

By successively integrating the above equation with respect to t and x, respectively, we get:

$$\frac{\partial^2 Z(x,t)}{\partial x^2} = (\Lambda^{(\eta,\eta')}(x))^T \Omega \mathbb{Q}(t,1,\eta_1) + \left.\frac{\partial^2 Z(x,t)}{\partial x^2}\right|_{t=0}$$
$$= (\Lambda^{(\eta,\eta')}(x))^T \Omega \mathbb{Q}(t,1,\eta_1) + \frac{\partial^2 f_1(x)}{\partial x^2}, \quad (3.6)$$

$$\frac{\partial Z(x,t)}{\partial x} = \mathbb{Q}^T(x,1,\eta)\Omega\mathbb{Q}(t,1,\eta_1) + \frac{\partial f_1(x)}{\partial x} - \left.\frac{\partial f_1(x)}{\partial x}\right|_{x=0} + \left.\frac{\partial Z(x,t)}{\partial x}\right|_{x=0}. \quad (3.7)$$

For approximating of the unknown function $\left.\frac{\partial Z(x,t)}{\partial x}\right|_{x=0}$ by integrating Equation (3.7) from 0 to 1 we have:

$$\left.\frac{\partial Z(x,t)}{\partial x}\right|_{x=0} \simeq f_3(t) - f_2(t) - \Big[\int_0^1 \mathbb{Q}^T(x,1,\eta)dx\Big]\Omega\mathbb{Q}(t,1,\eta_1)$$
$$+ f_1(1) - f_1(0) - \left.\frac{\partial f_1(x)}{\partial x}\right|_{x=0}.$$

Integrating Equation (3.5) with respect to x and substituting boundary conditions (1.2), yields:

$$\begin{aligned}\frac{\partial Z(x,t)}{\partial t} &= \mathbb{Q}^T(x,2,\eta)\Omega\Lambda^{(\eta_1,\eta')}(t) - x\mathbb{Q}^T(1,2,\eta)\Omega\Lambda^{(\eta_1,\eta')}(t)\\ &\quad + (1-x)\frac{\partial f_2(t)}{\partial t} + x\frac{\partial f_3(t)}{\partial t}.\end{aligned} \tag{3.8}$$

Also, by integrating the Equation (3.8) with respect to t, we get:

$$\begin{aligned}Z(x,t) &= \mathbb{Q}^T(x,2,\eta)\Omega\mathbb{Q}(t,1,\eta_1) - x\mathbb{Q}^T(1,2,\eta)\Omega\mathbb{Q}(t,1,\eta_1)\\ &\quad + (1-x)(f_2(t)-f_2(0)) + x(f_3(t)-f_3(0)) + f_1(x).\end{aligned} \tag{3.9}$$

Taking the Caputo fractional derivative of order μ with respect to t on both sides of Equation (3.9), we have:

$$\begin{aligned}{}^C\mathfrak{D}_t^{\mu}Z(x,t) &= \mathbb{Q}^T(x,2,\eta)\Omega\mathbb{Q}(t,1-\mu,\eta_1) - x\mathbb{Q}^T(1,2,\eta)\Omega\mathbb{Q}(t,1-\mu,\eta_1)\\ &\quad + (1-x){}^C\mathfrak{D}_t^{\mu}f_2(t) + x{}^C\mathfrak{D}_t^{\mu}f_3(t).\end{aligned} \tag{3.10}$$

Substituting (3.6) to (3.8) and (3.10) into (1.1), yields:

$$\begin{aligned}&\alpha_1\Big[\mathbb{Q}^T(x,2,\eta)\Omega\Lambda^{(\eta_1,\eta')}(t) - x\mathbb{Q}^T(1,2,\eta)\Omega\Lambda^{(\eta_1,\eta')}(t) + (1-x)\frac{\partial f_2(t)}{\partial t}\\ &\quad + x\frac{\partial f_3(t)}{\partial t}\Big] + \alpha_2\int_0^1 \rho(\mu)\big[\mathbb{Q}^T(x,2,\eta)\Omega\mathbb{Q}(t,1-\mu,\eta_1) - x\mathbb{Q}^T(1,2,\eta)\\ &\quad \times\Omega\mathbb{Q}(t,1-\mu,\eta_1) + (1-x){}^C\mathfrak{D}_t^{\mu}f_2(t) + x{}^C\mathfrak{D}_t^{\mu}f_3(t)\big]d\mu\\ &= -\alpha_3\Big[\mathbb{Q}^T(x,1,\eta)\Omega\mathbb{Q}(t,1,\eta_1) + \frac{\partial f_1(x)}{\partial x} - \frac{\partial f_1(x)}{\partial x}\Big|_{x=0} + f_3(t) - f_2(t)\\ &\quad - \Big[\int_0^1 \mathbb{Q}^T(x,1,\eta)dx\Big]\Omega\mathbb{Q}(t,1,\eta_1) + f_1(1) - f_1(0) - \frac{\partial f_1(x)}{\partial x}\Big|_{x=0}\Big]\\ &\quad + \alpha_4\Big[(\Lambda^{(\eta,\eta')}(x))^T\Omega\mathbb{Q}(t,1,\eta_1) + \frac{\partial^2 f_1(x)}{\partial x^2}\Big] + h(x,t).\end{aligned} \tag{3.11}$$

Now, we apply the Gauss-Legendre numerical integration to approximate the integral term in (3.11):

$$\begin{aligned}&\alpha_1\Big[\mathbb{Q}^T(x,2,\eta)\Omega\Lambda^{(\eta_1,\eta')}(t) - x\mathbb{Q}^T(1,2,\eta)\Omega\Lambda^{(\eta_1,\eta')}(t) + (1-x)\frac{\partial f_2(t)}{\partial t}\\ &\quad + x\frac{\partial f_3(t)}{\partial t}\Big] + \alpha_2\Big(\sum_{k=0}^{\bar{k}}\zeta_k\rho(\frac{1+\tau_k}{2})\Big[\mathbb{Q}^T(x,2,\eta)\Omega\mathbb{Q}(t,1-\frac{1+\tau_k}{2},\eta_1)\\ &\quad - x\mathbb{Q}^T(1,2,\eta)\Omega\mathbb{Q}(t,1-\frac{1+\tau_k}{2},\eta_1) + (1-x){}^C\mathfrak{D}_t^{\frac{1+\tau_k}{2}}f_2(t) + x{}^C\mathfrak{D}_t^{\frac{1+\tau_k}{2}}f_3(t)\Big]\Big)\\ &= -\alpha_3\Big[\mathbb{Q}^T(x,1,\eta)\Omega\mathbb{Q}(t,1,\eta_1) + \frac{\partial f_1(x)}{\partial x} - \frac{\partial f_1(x)}{\partial x}\Big|_{x=0}\\ &\quad + f_3(t) - f_2(t) - \Big[\int_0^1\mathbb{Q}^T(x,1,\eta)dx\Big]\Omega\mathbb{Q}(t,1,\eta_1) + f_1(1) - f_1(0) - \frac{\partial f_1(x)}{\partial x}\Big|_{x=0}\Big]\\ &\quad + \alpha_4\Big[(\Lambda^{(\eta,\eta')}(x))^T\Omega\mathbb{Q}(t,1,\eta_1) + \frac{\partial^2 f_1(x)}{\partial x^2}\Big] + h(x,t),\end{aligned} \tag{3.12}$$

where τ_k and ζ_k are nods and weights of the Gauss-Legendre formula, respectively. Using the following collocation points in (3.12)

$$x_{\mathbf{M},\mathbf{N}} = \frac{x_{\mathbf{N}} + \mathbf{M} - 1}{2^{m-1}},\ \mathbf{M} = 1, 2, \dots, 2^{m-1},\ \mathbf{N} = 0, 1, \dots, \widehat{\mathbf{N}},$$
$$t_{\mathbf{M}',\mathbf{N}'} = \frac{t_{\mathbf{N}'} + \mathbf{M}' - 1}{2^{m'-1}},\ \mathbf{M}' = 1, 2, \dots, 2^{m'-1},\ \mathbf{N}' = 0, 1, \dots, \widehat{\mathbf{N}'},$$

where $x_{\mathbf{N}}$ and $t_{\mathbf{N}}$ are zeros of shifted Legendre polynomials, we get:

$$\begin{aligned}
&\alpha_1\Big[\mathbb{Q}^T(x_{\mathbf{M},\mathbf{N}}, 2, \eta)\varOmega\varLambda^{(\eta_1,\eta')}(t_{\mathbf{M}',\mathbf{N}'}) - x_{\mathbf{M},\mathbf{N}}\mathbb{Q}^T(1, 2, \eta)\varOmega\varLambda^{(\eta_1,\eta')}(t_{\mathbf{M}',\mathbf{N}'})\\
&+ (1 - x_{\mathbf{M},\mathbf{N}})\frac{\partial f_2(t_{\mathbf{M}',\mathbf{N}'})}{\partial t} + x_{\mathbf{M},\mathbf{N}}\frac{\partial f_3(t_{\mathbf{M}',\mathbf{N}'})}{\partial t}\Big]\\
&+ \alpha_2\Big(\sum_{k=0}^{\bar{k}}\zeta_k\rho(\frac{1+\tau_k}{2})\Big[\mathbb{Q}^T(x_{\mathbf{M},\mathbf{N}}, 2, \eta)\varOmega\mathbb{Q}(t_{\mathbf{M}',\mathbf{N}'}, 1 - \frac{1+\tau_k}{2}, \eta_1)\\
&- x_{\mathbf{M},\mathbf{N}}\mathbb{Q}^T(1, 2, \eta)\varOmega\mathbb{Q}(t_{\mathbf{M}',\mathbf{N}'}, 1 - \frac{1+\tau_k}{2}, \eta_1) + (1 - x_{\mathbf{M},\mathbf{N}})\\
&\quad\times {}^C\mathfrak{D}_t^{\frac{1+\tau_k}{2}} f_2(t_{\mathbf{M}',\mathbf{N}'}) + x_{\mathbf{M},\mathbf{N}}{}^C\mathfrak{D}_t^{\frac{1+\tau_k}{2}} f_3(t_{\mathbf{M}',\mathbf{N}'})\Big]\Big)\\
&= -\alpha_3\Big[\mathbb{Q}^T(x_{\mathbf{M},\mathbf{N}}, 1, \eta)\varOmega\mathbb{Q}(t_{\mathbf{M}',\mathbf{N}'}, 1, \eta_1) + \frac{\partial f_1(x_{\mathbf{M},\mathbf{N}})}{\partial x} - \frac{\partial f_1(x)}{\partial x}\Big|_{x=0}\\
&\quad+ f_3(t_{\mathbf{M}',\mathbf{N}'}) - f_2(t_{\mathbf{M}',\mathbf{N}'}) - \Big[\int_0^1 \mathbb{Q}^T(x, 1, \eta)dx\Big]\varOmega\mathbb{Q}(t_{\mathbf{M}',\mathbf{N}'}, 1, \eta_1)\\
&\quad+ f_1(1) - f_1(0) - \frac{\partial f_1(x)}{\partial x}\Big|_{x=0}\Big]\\
&+ \alpha_4\Big[(\varLambda^{(\eta,\eta')}(x_{\mathbf{M},\mathbf{N}}))^T\varOmega\mathbb{Q}(t_{\mathbf{M}',\mathbf{N}'}, 1, \eta_1) + \frac{\partial^2 f_1(x_{\mathbf{M},\mathbf{N}})}{\partial x^2}\Big] + h(x_{\mathbf{M},\mathbf{N}}, t_{\mathbf{M}',\mathbf{N}'}).
\end{aligned}$$

This is a a system of $(2^{m-1}M) \times (2^{m'-1}M')$ algebraic equations which can be solved for the unknown $\varOmega$ applying the Newton's iterative method.

4 Convergence analysis

In this section, we introduce convergence analysis and error estimation of the presented method. In order to study the error bound of the proposed method, we demonstrate the following theorem.

Theorem 1. *Let* ${}^C\mathfrak{D}_x^{n\eta}\Big({}^C\mathfrak{D}_t^{n'\eta_1}Z(x,t)\Big) \in \mathbb{C}\Big([0,1) \times [0,1)\Big)$ *where* $n = 0, 1, \dots, M$ *and* $n' = 0, 1, \dots, M'$. *Also, assume that*

$$\varUpsilon_{M,M'}^{\eta,\eta_1} = \langle\mathfrak{P}_{\mathbf{N},\widehat{\mathbf{N}}}^{(\eta)}, \mathfrak{P}_{\mathbf{N}',\widehat{\mathbf{N}}}^{(\eta_1)}\rangle_{\mathbf{N},\mathbf{N}'=0}^{M-1,M'-1},$$

and let $P_{\mathbf{M},\mathbf{M}'}^{\mathbf{N},\mathbf{N}'}Z(x,t)$ *be the best approximation of* $Z(x,t)$ *based on the fractional order Chelyshkov wavelets out of* $\varUpsilon_{M,M'}^{\eta,\eta_1}$ *on the interval* $\varTheta = \Big[\frac{\mathbf{M}-1}{2^{m-1}}, \frac{\mathbf{M}}{2^{m-1}}\Big) \times$

$\left[\frac{\mathbf{M'}-1}{2^{m'-1}}, \frac{\mathbf{M'}}{2^{m'-1}}\right)$. *Then, the error bound of applying fractional order Chelyshkov wavelets series for* $M'\eta_1 > \mu$ *would be calculated as:*

$$
\begin{aligned}
&\|{}^C\mathfrak{D}_t^{\mu} Z - {}^C\mathfrak{D}_t^{\mu} P_{\mathbf{M},\mathbf{M'}}^{\mathbf{N},\mathbf{N'}} Z\|_{L^2([0,1)\times[0,1))} \\
&\leq \frac{\mathbb{R}_{M,M'}^{\eta,\eta_1}}{\Gamma(M\eta+1)\Gamma(M'\eta_1-\mu+1)\sqrt{(2M\eta+1)(2M'\eta_1-2\mu+1)}},
\end{aligned}
$$

in which $\mathbb{R}_{M,M'}^{\eta,\eta_1} = \sup_{(x,t)\in[0,1)\times[0,1)} \left|{}^C\mathfrak{D}_x^{M\eta}\left({}^C\mathfrak{D}_t^{M'\eta_1} Z(x,t)\right)\right|$.

Proof. Let

$$
\mathbb{H}_{M,M'}^{\eta,\eta_1}(x,t) = \sum_{n=0}^{M-1}\sum_{n'=0}^{M'-1} \frac{x^{n\eta} t^{n'\eta_1}}{\Gamma(n\eta+1)\Gamma(n'\eta_1+1)} {}^C\mathfrak{D}_x^{n\eta}\left({}^C\mathfrak{D}_t^{n'\eta_1} Z(x,t)\right)\Big|_{x=t=0},
$$

then considering the multi-variable Taylor expansion [17] and the generalized Taylor's expansion [22], we have:

$$
\left|Z(x,t) - \mathbb{H}_{M,M'}^{\eta,\eta_1}(x,t)\right| \leq \frac{x^{M\eta} t^{M'\eta_1}}{\Gamma(M\eta+1)\Gamma(M'\eta_1+1)} r_{M,M'}^{\eta,\eta_1}, \tag{4.1}
$$

in which $r_{M,M'}^{\eta,\eta_1} = \sup_{(x,t)\in\Theta} \left|{}^C\mathfrak{D}_x^{M\eta}\left({}^C\mathfrak{D}_t^{M'\eta_1} Z(x,t)\right)\right|$. By means of the properties of Caputo fractional derivative and Equation (4.1), we obtain:

$$
\left|{}^C\mathfrak{D}_t^{\mu} Z(x,t) - {}^C\mathfrak{D}_t^{\mu}\mathbb{H}_{M,M'}^{\eta,\eta_1}(x,t)\right| \leq \frac{x^{M\eta} t^{M'\eta_1-\mu}}{\Gamma(M\eta+1)\Gamma(M'\eta_1-\mu+1)} r_{M,M'}^{\eta,\eta_1}. \tag{4.2}
$$

Then, with the help of Equation (4.2), we obtain:

$$
\begin{aligned}
&\|{}^C\mathfrak{D}_t^{\mu} Z - {}^C\mathfrak{D}_t^{\mu} P_{\mathbf{M},\mathbf{M'}}^{\mathbf{N},\mathbf{N'}} Z\|_{L^2([0,1)\times[0,1))}^2 = \sum_{\mathbf{N}=1}^{2^{m-1}}\sum_{\mathbf{N'}=1}^{2^{m'-1}} \|{}^C\mathfrak{D}_t^{\mu} Z - {}^C\mathfrak{D}_t^{\mu} P_{\mathbf{M},\mathbf{M'}}^{\mathbf{N},\mathbf{N'}} Z\|_{L^2(\Theta)}^2 \\
&\leq \sum_{\mathbf{N}=1}^{2^{m-1}}\sum_{\mathbf{N'}=1}^{2^{m'-1}} \|{}^C\mathfrak{D}_t^{\mu} Z - {}^C\mathfrak{D}_t^{\mu}\mathbb{H}_{M,M'}^{\eta,\eta_1}\|_{L^2(\Theta)}^2 \leq \sum_{\mathbf{N}=1}^{2^{m-1}}\sum_{\mathbf{N'}=1}^{2^{m'-1}} \int_{\left[\frac{\mathbf{M}-1}{2^{m-1}}, \frac{\mathbf{M}}{2^{m-1}}\right)} \\
&\times \int_{\left[\frac{\mathbf{M'}-1}{2^{m'-1}}, \frac{\mathbf{M'}}{2^{m'-1}}\right)} \left[\frac{x^{M\eta} t^{M'\eta_1-\mu}}{\Gamma(M\eta+1)\Gamma(M'\eta_1-\mu+1)} r_{M,M'}^{\eta,\eta_1}\right]^2 dt\,dx \\
&\leq \int_0^1\int_0^1 \left[\frac{x^{M\eta} t^{M'\eta_1-\mu}}{\Gamma(M\eta+1)\Gamma(M'\eta_1-\mu+1)} \mathbb{R}_{M,M'}^{\eta,\eta_1}\right]^2 dt\,dx \\
&\leq \frac{(\mathbb{R}_{M,M'}^{\eta,\eta_1})^2}{\Gamma(M\eta+1)^2\Gamma(M'\eta_1-\mu+1)^2(2M\eta+1)(2M'\eta_1-2\mu+1)}.
\end{aligned} \tag{4.3}
$$

Therefore, by taking the square root of Equation (4.3) in to account the proof is complete. □

By utilizing Theorem 1, the following proposition is obtained.

Proposition 1. *Under the assumptions of Theorem 1, we have:*

$$\begin{aligned}
&\|{}^{C}\mathfrak{D}_x^2 Z - {}^{C}\mathfrak{D}_x^2 P_{\mathbf{M},\mathbf{M}'}^{\mathbf{N},\mathbf{N}'} Z\|_{L^2([0,1)\times[0,1))} \\
&\quad = \frac{\mathbb{R}_{M,M'}^{\eta,\eta_1}}{\Gamma(M\eta-1)\Gamma(M'\eta_1+1)\sqrt{(2M\eta-3)(2M'\eta_1+1)}}, \ M\eta > 2, \\
&\|{}^{C}\mathfrak{D}_x^1 Z(x,t) - {}^{C}\mathfrak{D}_x^1 P_{\mathbf{M},\mathbf{M}'}^{\mathbf{N},\mathbf{N}'} Z(x,t)\|_{L^2([0,1)\times[0,1))} \\
&\quad = \frac{\mathbb{R}_{M,M'}^{\eta,\eta_1}}{\Gamma(M\eta)\Gamma(M'\eta_1+1)\sqrt{(2M\eta-1)(2M'\eta_1+1)}}, \ M\eta > 1, \\
&\|{}^{C}\mathfrak{D}_t^1 Z - {}^{C}\mathfrak{D}_t^1 P_{\mathbf{M},\mathbf{M}'}^{\mathbf{N},\mathbf{N}'} Z\|_{L^2([0,1)\times[0,1))} \\
&\quad = \frac{\mathbb{R}_{M,M'}^{\eta,\eta_1}}{\Gamma(M\eta+1)\Gamma(M'\eta_1)\sqrt{(2M\eta+1)(2M'\eta_1-1)}}, \ M'\eta_1 > 1,
\end{aligned}$$

Theorem 2. *Under the assumptions of Theorem 1 and Proposition 1, the error bound $\mathbb{E}_{\mathbf{M},\mathbf{M}'}^{\mathbf{N},\mathbf{N}'}$ is obtained as:*

$$\begin{aligned}
\|\mathbb{E}_{\mathbf{M},\mathbf{M}'}^{\mathbf{N},\mathbf{N}'}\|_{L^2([0,1)\times[0,1))} &\le \alpha_1 \frac{\mathbb{R}_{M,M'}^{\eta,\eta_1}}{\Gamma(M\eta+1)\Gamma(M'\eta_1)\sqrt{(2M\eta+1)(2M'\eta_1-1)}} \\
&+ \alpha_2 \frac{\mathbb{M}\mathbb{R}_{M,M'}^{\eta,\eta_1}}{\Gamma(M\eta+1)\Gamma(M'\eta_1-\mu+1)\sqrt{(2M\eta+1)(2M'\eta_1-2\mu+1)}} \\
&+ \alpha_3 \frac{\mathbb{R}_{M,M'}^{\eta,\eta_1}}{\Gamma(M\eta)\Gamma(M'\eta_1+1)\sqrt{(2M\eta-1)(2M'\eta_1+1)}} \\
&+ \alpha_4 \frac{\mathbb{R}_{M,M'}^{\eta,\eta_1}}{\Gamma(M\eta-1)\Gamma(M'\eta_1+1)\sqrt{(2M\eta-3)(2M'\eta_1+1)}},
\end{aligned}$$

where $\|\rho(\mu)\| \le \mathbb{M}$ and

$$\mathbb{E}_{\mathbf{M},\mathbf{M}'}^{\mathbf{N},\mathbf{N}'} = \alpha_1 Z_t(x,t) + \alpha_2 \mathbb{D}_t^{\mu} Z(x,t) + \alpha_3 Z_x(x,t) - \alpha_4 Z_{xx}(x,t) - h(x,t).$$

Proof. We know that,

$$\begin{aligned}
\|\mathbb{E}_{\mathbf{M},\mathbf{M}'}^{\mathbf{N},\mathbf{N}'}\|_{L^2([0,1)\times[0,1))} &= \|\alpha_1 \frac{\partial P_{\mathbf{M},\mathbf{M}'}^{\mathbf{N},\mathbf{N}'} Z(x,t)}{\partial t} + \alpha_2 \mathbb{D}_t^{\mu} Z(x,t) \\
&+ \alpha_3 \frac{\partial P_{\mathbf{M},\mathbf{M}'}^{\mathbf{N},\mathbf{N}'} Z(x,t)}{\partial x} - \alpha_4 \frac{\partial^2 P_{\mathbf{M},\mathbf{M}'}^{\mathbf{N},\mathbf{N}'} Z(x,t)}{\partial x^2} - h(x,t)\|_{L^2([0,1)\times[0,1))}.
\end{aligned}$$

By applying Equation (1.2), we have:

$$\begin{aligned}
\|\mathbb{E}_{\mathbf{M},\mathbf{M}'}^{\mathbf{N},\mathbf{N}'}\|_{L^2([0,1)\times[0,1))} &= \|\alpha_1 \frac{\partial P_{\mathbf{M},\mathbf{M}'}^{\mathbf{N},\mathbf{N}'} Z(x,t)}{\partial t} + \alpha_2 \mathbb{D}_t^{\mu} P_{\mathbf{M},\mathbf{M}'}^{\mathbf{N},\mathbf{N}'} Z(x,t) \\
&+ \alpha_3 \frac{\partial P_{\mathbf{M},\mathbf{M}'}^{\mathbf{N},\mathbf{N}'} Z(x,t)}{\partial x} - \alpha_4 \frac{\partial^2 P_{\mathbf{M},\mathbf{M}'}^{\mathbf{N},\mathbf{N}'} Z(x,t)}{\partial x^2} - \alpha_1 \frac{\partial Z(x,t)}{\partial t} \\
&- \alpha_2 \mathbb{D}_t^{\mu} Z(x,t) - \alpha_3 \frac{\partial Z(x,t)}{\partial x} + \alpha_4 \frac{\partial^2 Z(x,t)}{\partial x^2}\|_{L^2([0,1)\times[0,1))}.
\end{aligned}$$

By applying Theorem 1 and Proposition 1, we obtain:

$$
\begin{aligned}
\|\mathbb{E}^{\mathbf{N},\mathbf{N}'}_{\mathbf{M},\mathbf{M}'}\|_{L^2([0,1)\times[0,1))} \leq & \Big\|\alpha_1\Big(\frac{\partial Z(x,t)}{\partial t}-\frac{\partial P^{\mathbf{N},\mathbf{N}'}_{\mathbf{M},\mathbf{M}'}Z(x,t)}{\partial t}\Big)\Big\|_{L^2([0,1)\times[0,1))}\\
&+\|\alpha_2(\mathbb{D}^{\mu}_t Z(x,t)-\mathbb{D}^{\mu}_t P^{\mathbf{N},\mathbf{N}'}_{\mathbf{M},\mathbf{M}'}Z(x,t))\|_{L^2([0,1)\times[0,1))}\\
&+\Big\|\alpha_3\Big(\frac{\partial Z(x,t)}{\partial x}-\frac{\partial P^{\mathbf{N},\mathbf{N}'}_{\mathbf{M},\mathbf{M}'}Z(x,t)}{\partial x}\Big)\Big\|_{L^2([0,1)\times[0,1))}\\
&+\Big\|\alpha_4\Big(\frac{\partial^2 Z(x,t)}{\partial x^2}-\frac{\partial^2 P^{\mathbf{N},\mathbf{N}'}_{\mathbf{M},\mathbf{M}'}Z(x,t)}{\partial x^2}\Big)\Big\|_{L^2([0,1)\times[0,1))}\\
\leq\ & \alpha_1\frac{\mathbb{R}^{\eta,\eta_1}_{M,M'}}{\Gamma(M\eta+1)\Gamma(M'\eta_1)\sqrt{(2M\eta+1)(2M'\eta_1-1)}}\\
&+\alpha_2\int_0^1\frac{\mathbb{M}\mathbb{R}^{\eta,\eta_1}_{M,M'}}{\Gamma(M\eta+1)\Gamma(M'\eta_1-\mu+1)\sqrt{(2M\eta+1)(2M'\eta_1-2\mu+1)}}d\mu\\
&+\alpha_3\frac{\mathbb{R}^{\eta,\eta_1}_{M,M'}}{\Gamma(M\eta)\Gamma(M'\eta_1+1)\sqrt{(2M\eta-1)(2M'\eta_1+1)}}\\
&+\alpha_4\frac{\mathbb{R}^{\eta,\eta_1}_{M,M'}}{\Gamma(M\eta-1)\Gamma(M'\eta_1+1)\sqrt{(2M\eta-3)(2M'\eta_1+1)}}\\
\leq\ & \alpha_1\frac{\mathbb{R}^{\eta,\eta_1}_{M,M'}}{\Gamma(M\eta+1)\Gamma(M'\eta_1)\sqrt{(2M\eta+1)(2M'\eta_1-1)}}\\
&+\alpha_2\frac{\mathbb{M}\mathbb{R}^{\eta,\eta_1}_{M,M'}}{\Gamma(M\eta+1)\Gamma(M'\eta_1-\mu+1)\sqrt{(2M\eta+1)(2M'\eta_1-2\mu+1)}}\\
&+\alpha_3\frac{\mathbb{R}^{\eta,\eta_1}_{M,M'}}{\Gamma(M\eta)\Gamma(M'\eta_1+1)\sqrt{(2M\eta-1)(2M'\eta_1+1)}}\\
&+\alpha_4\frac{\mathbb{R}^{\eta,\eta_1}_{M,M'}}{\Gamma(M\eta-1)\Gamma(M'\eta_1+1)\sqrt{(2M\eta-3)(2M'\eta_1+1)}},
\end{aligned}
$$

where $\|\rho(\mu)\|\leq\mathbb{M}$. On the other hand, since $P^{\mathbf{N},\mathbf{N}'}_{\mathbf{M},\mathbf{M}'}Z(x,t)$ is the best approximation of $Z(x,t)$, then we have:

$$
\begin{aligned}
\|Z(x,t)-P^{\mathbf{N},\mathbf{N}'}_{\mathbf{M},\mathbf{M}'}Z(x,t)\|_2^2 &\leq \|Z(x,t)-\mathbb{H}^{\eta,\eta_1}_{M,M'}(x,t)\|_2^2\\
&=\int_0^1\int_0^1\mathbb{W}^{(\eta)}(t)\mathbb{W}^{(\eta)}(x)\Big|Z(x,t)-\mathbb{H}^{\eta,\eta_1}_{M,M'}(x,t)\Big|^2 dtdx.
\end{aligned}
$$

Therefore, by using Equation (4.1), we obtain:

$$
\|Z(x,t)-P^{\mathbf{N},\mathbf{N}'}_{\mathbf{M},\mathbf{M}'}Z(x,t)\|_2^2\leq\frac{(r^{\eta,\eta_1}_{M,M'})^2}{\eta^2\Gamma^2(M\eta+1)\Gamma^2(M'\eta_1+1)}.
$$

Thus, we find that if $M,M'\to\infty$, we have $\|Z(x,t)-P^{\mathbf{N},\mathbf{N}'}_{\mathbf{M},\mathbf{M}'}Z(x,t)\|_2\to 0$.
□

5 Numerical results

In this section, we apply the proposed method for solving two test equations, which are given by Equations (1.1) and (1.2), to show the accuracy and capability of the proposed method. To evaluate and measure the accuracy of the proposed method, the absolute error(AE) and L_∞ error are measured based on the following formulae:

$$AE = \left|Z(x,t) - P^{\mathbf{N},\mathbf{N}'}_{\mathbf{M},\mathbf{M}'} Z(x,t)\right|, \quad L_\infty = \max_{x,t\in[0,1)} \left|Z(x,t) - P^{\mathbf{N},\mathbf{N}'}_{\mathbf{M},\mathbf{M}'} Z(x,t)\right|,$$

in which $Z(x,t)$ denotes the exact solution and $P^{\mathbf{N},\mathbf{N}'}_{\mathbf{M},\mathbf{M}'} Z(x,t)$ is the approximate solution. All numerical simulations of the proposed method are obtained by applying Matlab software.

Example 1. Consider the following distributed order mobile-immobile advection - dispersion equation:

$$Z_t(x,t)+\int_0^1 \Gamma(3.5-\mu)\,{}^C\mathfrak{D}^\mu_t Z(x,t)d\mu = -Z_x(x,t)+Z_{xx}(x,t)+h_1(x,t)+h_2(x,t),$$

where

$$\begin{aligned} h_1(x,t) &= t^{\frac{5}{2}}(x-1)^2 + 2t^{\frac{5}{2}}x(x-1) + \frac{5}{2}t^{\frac{3}{2}}x(x-1)^2, \\ h_2(x,t) &= \frac{15\sqrt{\pi}t^{\frac{5}{3}}(x-1)^2 x(t-1)}{8\log t} - 2t(3x-2), \end{aligned}$$

subject to the initial and boundary conditions:

$$\begin{aligned} &Z(x,0) = 0, \quad x\in[0,1], \\ &Z(0,t) = 0, \quad Z(1,t) = 0, \quad t\in[0,1]. \end{aligned}$$

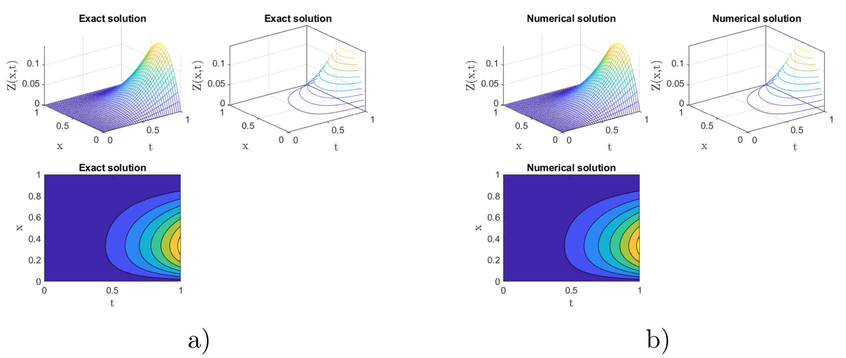
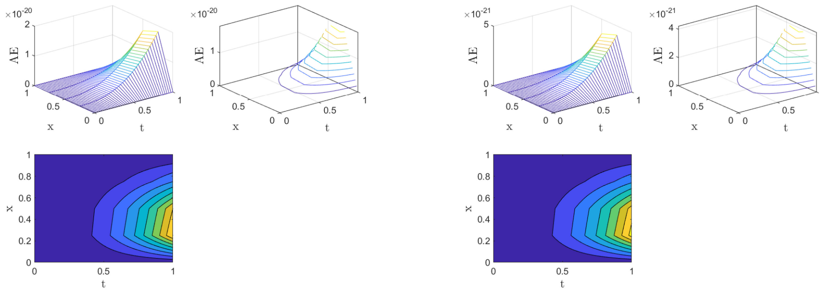


Figure 1. The exact solution (a) and approximate solution (b) with $m = m' = 1$, $M = M' = 4$ and $\eta = \eta_1 = \frac{1}{2}$ for Example 1.

The analytical solution of this problem is $Z(x,t) = t^{\frac{5}{2}}(x-1)^2 x$. We solve this example by the proposed method for various values of m, m', η, η_1, M and M'.

Figure 1 reports the graphs of the exact and approximate solutions for this example with $m = m' = 1$, $M = M' = 4$ and $\eta = \eta_1 = \frac{1}{2}$.



a) b)

Figure 2. The AE functions of applying the proposed method for $M = M' = 32$, $\eta = \eta_1 = \frac{1}{2}$ (a), $M = M' = 64$ and $\eta = \eta_1 = \frac{1}{2}$ (b) in Example 1.

Figure 2 is plotted to demonstrate the graphical behavior of the AE functions with $m = m' = 1$ and $\eta = \eta_1 = \frac{1}{2}$ for $M = M' = 32$ and $M = M' = 64$.

Graphs of the AE functions at several values of M and M' reported in Figure 3. The plots of the approximate solution behavior and exact solution of this example at $t = 0.5$ with $m = m' = 1$ and $\eta = \eta_1 = \frac{1}{2}$ for different values of M and M' are illustrated in Figure 4.

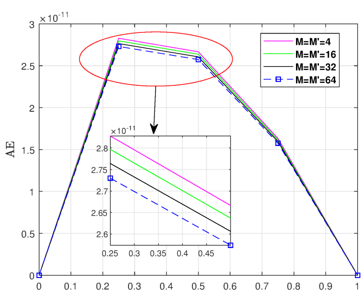


Figure 3. The AE functions of applying the proposed method with $m = m' = 1$, $\eta = \eta_1 = \frac{1}{2}$ and different values of $M = M'$ in Example 1.

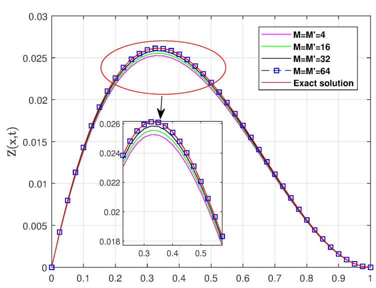


Figure 4. The comparison between the exact solution and the approximate solutions with $m = m' = 1$, $\eta = \eta_1 = \frac{1}{2}$ and different values of $M = M'$ for Example 1.

Table 1 shows the absolute error and CPU time for the numerical solutions by applying the proposed method with $m = m' = 1$, $\eta = 1$ and $M = M' = 4$ for various choices of η_1.

Table 2 shows the L_∞ error by applying the proposed method with $m =$

Table 1. The comparison of the absolute error (AE) obtained by our numerical method at some selected points with $m = m' = 1$, $M = M' = 64$, $\eta = 1$ and different values for η_1 in Example 1.

(x, t)	*AE*			
η_1	$\eta_1 = \frac{1}{4}$	$\eta_1 = \frac{1}{3}$	$\eta_1 = \frac{1}{2}$	$\eta_1 = \frac{2}{3}$
$(0.1, 0.1)$	1.9197×10^{-24}	2.5597×10^{-24}	3.8395×10^{-24}	5.1194×10^{-24}
$(0.2, 0.2)$	1.7161×10^{-23}	2.2882×10^{-23}	3.4323×10^{-23}	4.5764×10^{-23}
$(0.3, 0.3)$	5.4311×10^{-23}	7.2415×10^{-23}	1.0862×10^{-22}	1.4483×10^{-22}
$(0.4, 0.4)$	1.0921×10^{-22}	1.4562×10^{-22}	2.18430×10^{-22}	2.9124×10^{-22}
$(0.5, 0.5)$	1.6561×10^{-22}	2.2082×10^{-22}	3.3123×10^{-22}	4.4164×10^{-22}
$(0.6, 0.6)$	2.0064×10^{-22}	2.6752×10^{-22}	4.0128×10^{-22}	5.3504×10^{-22}
$(0.7, 0.7)$	1.9357×10^{-22}	2.5810×10^{-22}	3.8715×10^{-22}	5.1620×10^{-22}
$(0.8, 0.8)$	1.3729×10^{-22}	1.8305×10^{-22}	2.7458×10^{-22}	3.6611×10^{-22}
$(0.9, 0.9)$	5.1834×10^{-23}	6.9112×10^{-23}	1.0366×10^{-22}	1.3822×10^{-22}
CPU times	6.261	6.260	6.307	6.292

$m' = 1$ and $M = M' = 64$ for different values of η and η_1. The obtained numerical results of figures and tables confirm the high accuracy of the proposed method. Also, we see that the approximate solutions have a good accuracy.

Table 2. The comparison of the L_∞ error, L_2 error and *RMS* obtained by our numerical method with $m = m' = 1$ and $M = M' = 64$ for different values for η_1 and η for Example 1.

η, η_1	$\eta = 1, \eta_1 = \frac{1}{4}$	$\eta = 1$, $\eta_1 = \frac{1}{2}$	$\eta = 1$, $\eta_1 = \frac{2}{3}$	$\eta = 1 = \eta_1 = 1$
L_∞	1.0539×10^{-21}	2.1079×10^{-21}	2.8106×10^{-21}	4.2159×10^{-21}
L_2	5.1462×10^{-20}	7.2593×10^{-20}	8.3824×10^{-20}	1.0263×10^{-20}
RMS	5.1341×10^{-21}	7.2471×10^{-21}	8.3703×10^{-21}	1.0151×10^{-21}

Example 2. Let us consider the following distributed order mobile-immobile advection-dispersion equation:

$$Z_t(x,t)+\int_0^1 \Gamma(3-\mu)^C\mathfrak{D}_t^\mu Z(x,t)d\mu = -Z_x(x,t) + Z_{xx}(x,t) + h_1(x,t)+h_2(x,t),$$

where

$$\begin{aligned} h_1(x,t) &= -2t(x-1)\cos(x) - t^2\cos(x) + t^2(x-1)\sin(x), \\ h_2(x,t) &= -2t^2\sin(x) - (x-1)t^2\cos(x) - \frac{(x-1)\cos(x)2t(t-1)}{\log t}, \end{aligned}$$

with the initial and boundary conditions:

$$\begin{aligned} &Z(x,0) = 0, \ x \in [0,1], \\ &Z(0,t) = t^2, \ Z(1,t) = 0, \ t \in [0,1]. \end{aligned}$$

The exact solution of this example is $Z(x,t) = -t^2(x-1)\cos(x)$. We use the proposed method to solve this problem for various values of m, m', η, η_1, M and M'.

Figure 5 reports the graphs of the exact and approximate solutions obtained by applying the proposed method for this example with $m = m' = 1$, $M = M' = 4$ and $\eta = \eta_1 = \frac{1}{2}$.

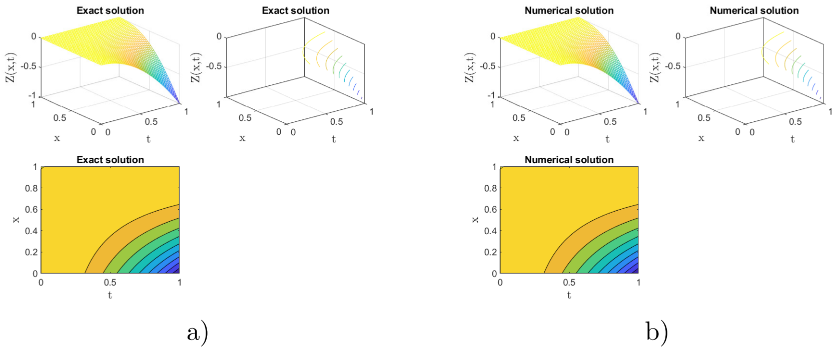


Figure 5. The exact solution (a) and approximate solution (b) with $m = m' = 1$, $M = M' = 4$ and $\eta = \eta_1 = \frac{1}{2}$ for Example 2.

Plots of absolute error functions of approximate solutions for $M = M' = 32$, $M = M' = 64$ and $\eta = \eta_1 = \frac{1}{2}$ has been reported in Figure 6.

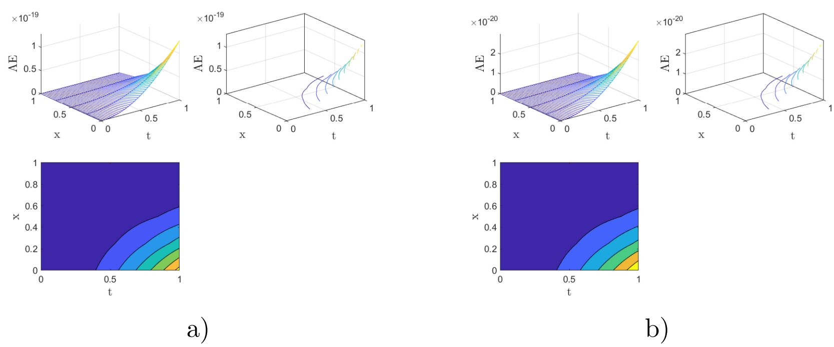


Figure 6. The AE functions of applying the proposed method with $M = M' = 32$, $\eta = \eta_1 = \frac{1}{2}$ (a) and $M = M' = 64$ and $\eta = \eta_1 = \frac{1}{2}$ (b) for Example 2.

Graphs of absolute error with $m = m' = 1$ and $\eta = \eta_1 = \frac{1}{2}$ and different values of $M = M'$ are depicted in Figure 7. Plots of the approximate solution behavior and exact solution of this example at $t = 0.5$ with $m = m' = 1$ and $\eta = \eta_1 = \frac{1}{2}$ for different values of M and M' are illustrated in Figure 8.

Absolute error and CPU time for the numerical solutions by applying the proposed method with $m = m' = 1$, $M = M' = 64$ and $\eta = 1$ for various choices η_1 are presented in Table 3. Also, the L_∞ errors obtained by our numerical method with $m = m' = 1$ and $M = M' = 64$ for different values for

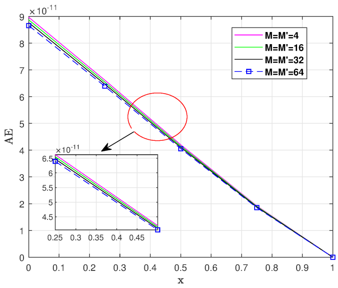


Figure 7. The graph of the AE functions of applying the proposed method with $m = m' = 1$, $\eta = \eta_1 = \frac{1}{2}$ and different values of $M = M'$ for Example 2.

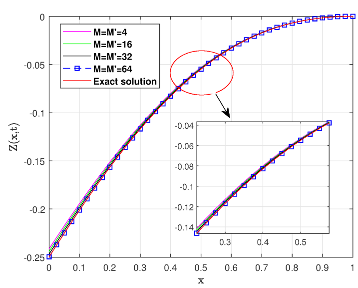


Figure 8. The comparison between the exact and the approximate solutions with $m = m' = 1$, $\eta = \eta_1 = \frac{1}{2}$ and different values of $M = M'$ for Example 2.

η_1 and η are presented in Table 4.

Table 3. The comparison of the absolute error (AE) obtained by our numerical method at some selected points with $m = m' = 1$, $M = M' = 64$, $\eta = 1$ and different values of η_1 for Example 2.

(x,t)	AE			
η_1	$\eta_1 = \frac{1}{4}$	$\eta_1 = \frac{1}{3}$	$\eta_1 = \frac{1}{2}$	$\eta_1 = \frac{2}{3}$
$(0.1, 0.1)$	6.0406×10^{-23}	8.0541×10^{-23}	1.2081×10^{-22}	1.6108×10^{-22}
$(0.2, 0.2)$	1.8804×10^{-22}	2.5072×10^{-22}	3.7609×10^{-22}	5.0145×10^{-22}
$(0.3, 0.3)$	3.1576×10^{-22}	4.2102×10^{-22}	6.3153×10^{-22}	8.4204×10^{-22}
$(0.4, 0.4)$	3.9763×10^{-22}	5.3017×10^{-22}	7.9526×10^{-22}	1.0603×10^{-21}
$(0.5, 0.5)$	4.1109×10^{-22}	5.4812×10^{-22}	8.2218×10^{-22}	1.0962×10^{-21}
$(0.6, 0.6)$	3.5630×10^{-22}	4.7507×10^{-22}	7.1261×10^{-22}	9.5015×10^{-22}
$(0.7, 0.7)$	2.5280×10^{-22}	3.3706×10^{-22}	5.0560×10^{-22}	6.7413×10^{-22}
$(0.8, 0.8)$	1.3367×10^{-22}	1.7823×10^{-22}	2.6735×10^{-22}	3.5647×10^{-22}
$(0.9, 0.9)$	3.7737×10^{-23}	5.0316×10^{-23}	7.5475×10^{-23}	1.0063×10^{-22}
CPU times	5.197	5.229	5.329	5.198

The figures and tables show that the proposed method provides approximate solutions with acceptable accuracy. Also, we see that the approximate solutions have a good accuracy. In addition to, from Tables we conclude that proposed method is appropriate for this example and that by increasing the number of the M and M' the accuracy is further improved.

Table 4. The comparison of the L_∞ error obtained by our numerical method with $m = m' = 1$ and $M = M' = 64$ for different values for η_1 and η for Example 2.

η, η_1	$\eta = 1, \eta_1 = \frac{1}{4}$	$\eta = 1$, $\eta_1 = \frac{1}{2}$	$\eta = 1$, $\eta_1 = \frac{2}{3}$	$\eta = 1 = \eta_1 = 1$
L_∞	1.6443×10^{-21}	1.4989×10^{-20}	1.9987×10^{-20}	2.9979×10^{-20}

6 Conclusions

This paper proposed an accurate and efficient composite collocation method based on the Chelyshkov wavelets. Using the fractional integrals of Chelyshkov wavelets, this collocation method was applied to reduce the solution of the distributed-order fractional mobile-immobile advection-dispersion problem to a system of algebraic equations in order to calculate the unknown function. Two numerical examples have provided to show the validity and applicability of the proposed method. The numerical results illustrate that the proposed method is effective and simple with allowing a more conclusive analysis of the distributed-order fractional mobile-immobile advection-dispersion problem.

Acknowledgements

This project has been supported by a research grant of the University of Tabriz (number 822). Authors are very grateful to reviewers for carefully reading this paper and for their comments and suggestions which have improved the paper.

References

[1] A.A. Alikhanov. Numerical methods of solutions of boundary value problems for the multi-term variable-distributed order diffusion equation. *Appl. Math. Comput.*, **268**:12–22, 2015. https://doi.org/10.1016/j.amc.2015.06.045.

[2] S. Bazm and A. Hosseini. Numerical solution of nonlinear integral equations using alternative Legendre polynomials. *J. Appl. Math. Comput.*, **56**(1):25–51, 2018. https://doi.org/10.1007/s12190-016-1060-5.

[3] M. Caputo. *Elasticitàe Dissipazione.* Zanichelli, 1969.

[4] M. Caputo. Mean fractional-order-derivatives differential equations and filters. *Annal. Del. Universitá di Ferrara.*, **41**(1):73–84, 1995. https://doi.org/10.1007/BF02826009.

[5] M. Dehghan and M. Abbaszadeh. A Legendre spectral element method (SEM) based on the modified bases for solving neutral delay distributed-order fractional damped diffusion-wave equation. *Math. Methods. Appl. Sci.*, **41**(9):3476–3494, 2018. https://doi.org/10.1002/mma.4839.

[6] K. Diethelm and N.J. Ford. Numerical analysis for distributed-order differential equations. *J. Comput. Appl. Math.*, **225**(1):96–104, 2009. https://doi.org/10.1016/j.cam.2008.07.018.

[7] S.S. Ezz-Eldien. New quadrature approach based on operational matrix for solving a class of fractional variational problems. *J. Comput. phys.*, **317**:362–381, 2016. https://doi.org/10.1016/j.jcp.2016.04.045.

[8] N.J. Ford and M.L. Morgado. Distributed order equations as boundary value problems. *Comput. Math. Appl.*, **64**(10):2973–2981, 2012. https://doi.org/10.1016/j.camwa.2012.01.053.

[9] G.H. Gao, H.W. Sun and Z.Z. Sun. Some high-order difference schemes for the distributed-order differential equations. *J. Comput. Phys.*, **298**:337–359, 2015. https://doi.org/10.1016/j.jcp.2015.05.047.

[10] M. Hamid, O.M. Foong, M. Usman, I. Khan and W. Wang. A new operational matrices-based spectral method for multi-order fractional problems. *Symmetry*, **12**(9):1471, 2020. https://doi.org/10.3390/sym12091471.

[11] M. Hamid, M. Usman, R.U. Haq and Z. Tian. A spectral approach to analyze the nonlinear oscillatory fractional-order differential equations. *Chaos. Solitons. Fractals.*, **146**:110921, 2021. https://doi.org/10.1016/j.chaos.2021.110921.

[12] M. Hamid, M. Usman, R.U. Haq, Z. Tian and W. Wang. Linearized stable spectral method to analyze two-dimensional nonlinear evolutionary and reaction-diffusion models. *Numer. Methods. Partial. Differ. Equ.*, **38**(2):243–261, 2020. https://doi.org/10.1002/num.22659.

[13] M. Hamid, M. Usman, R.U. Haq and W. Wang. A Chelyshkov polynomial based algorithm to analyze the transport dynamics and anomalous diffusion in fractional model. *Physica. A.*, **551**:124227, 2020. https://doi.org/10.1016/j.physa.2020.124227.

[14] M. Hamid, M. Usman, W. Wang and Z. Tian. A stable computational approach to analyze semi-relativistic behavior of fractional evolutionary problems. *Numer. Methods. Partial. Differ. Equ.*, **38**(2):122–136, 2020. https://doi.org/10.1002/num.22617.

[15] M. Hamid, M. Usman, W. Wang and Z. Tian. Hybrid fully spectral linearized scheme for time-fractional evolutionary equations. *Math. Methods. Appl. Sci.*, **44**(5):3890–3912, 2021. https://doi.org/10.1002/mma.6996.

[16] M. Hamid, M. Usman, T. Zubair, R.U. Haq and W. Wang. Innovative operational matrices based computational scheme for fractional diffusion problems with the Riesz derivative. *Eur. Phys. J. Plus.*, **134**(10):1–17, 2019. https://doi.org/10.1140/epjp/i2019-12871-y.

[17] L. Hormander. *The analysis of linear partial differential operators*. 1990.

[18] Z. Jiao, Y.Q. Chen and I. Podlubny. *Distributed-Order Dynamic Systems: Stability, Simulation, Applications and Perspectives*. Springer London, 2012.

[19] Y. Luchko. Boundary value problems for the generalized time-fractional diffusion equation of distributed order. *Fract. Calc. Appl. Anal.*, **12**(4):409–422, 2009.

[20] L. Moradi, F. Mohammadi and D. Baleanu. A direct numerical solution of time-delay fractional optimal control problems by using Chelyshkov wavelets. *J. Vib. Control.*, **25**(2):310–324, 2019. https://doi.org/10.1177/1077546318777338.

[21] M. Pourbabaee and A. Saadatmandi. A novel Legendre operational matrix for distributed order fractional differential equations. *Appl. Math. Comput.*, **361**:215–231, 2019. https://doi.org/10.1016/j.amc.2019.05.030.

[22] P. Rahimkhani, Y. Ordokhani and E. Babolian. Fractional-order Bernoulli wavelets and their applications. *Appl. Math. Model.*, **40**(17-18):8087–8107, 2016. https://doi.org/10.1016/j.apm.2016.04.026.

[23] T. Srokowski. Lévy flights in nonhomogeneous media: Distributed-order fractional equation approach. *Phys. Rev. E.*, **78**(3):031135, 2008. https://doi.org/10.1103/PhysRevE.78.031135.

[24] H. Ye, F. Liu and V. Anh. Compact difference scheme for distributed-order time-fractional diffusion-wave equation on bounded domains. *J. Comput. Phys.*, **298**:652–660, 2015. https://doi.org/10.1016/j.jcp.2015.06.025.

[25] S. Yuzbasi. Numerical solutions of fractional Riccati type differential equations by means of the Bernstein polynomials. *Appl. Math. Comput.*, **219**(11):6328–6343, 2013. https://doi.org/10.1016/j.amc.2012.12.006.