Mathematical Modelling and Analysis
Volume 27, Issue 4, 533–546, 2022
https://doi.org/10.3846/mma.2022.15740

Weak Solution for Nonlinear Fractional p(.)-Laplacian Problem with Variable Order via Rothe's Time-Discretization Method

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Received October 7, 2021; revised July 22, 2022; accepted July 22, 2022

Abstract. In this paper, we prove the existence and uniqueness results of weak solutions to a class of nonlinear fractional parabolic p(.)-Laplacian problem with variable order. The main tool used here is the Rothe's method combined with the theory of variable-order fractional Sobolev spaces with variable exponent.

Keywords: fractional $p(.)$-Laplacian, fractional Sobolev space, semi-discretization, Rothe's method, variable exponent, variable order, weak solution.

AMS Subject Classification: 35K55; 34A08; 35R11.

1 Introduction

Let $\Omega \subset \mathbb{R}^d, (d \geq 2)$ be an open bounded domain with a connected Lipschitz boundary $\partial\Omega$ and T be a fixed positive real number. Our aim of this paper is to prove the existence and uniqueness results of weak solutions for the nonlinear fractional parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} + (-\Delta)^{s(.)}_{p(.)} u = f(x,t) \;\; in \;\; Q_T := \Omega \times]0,\ T[, \\ u = 0 \;\; on \quad \Sigma_T := \partial\Omega \times]0,\ T[, \\ u(.,0) = u_0 \;\; in \;\; \Omega, \end{cases} \tag{1.1}$$

where $(-\Delta)^{s(.)}_{p(.)}$ is the fractional $p(x)$-Laplacian operator with variable order which can be defined as

$$(-\Delta)^{s(.)}_{p(.)} u(x) = P \cdot V \cdot \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{d+s(x,y)p(x,y)}} dy, for\ all\ x \in \Omega,$$

and $P.V.$ is a commonly used abbreviation in the principal value sense. $p(.)$ and $s(.)$ are two continuous variable exponents with $s(x,y)p(x,y) < d$ for any $(x,y) \in \overline{\Omega} \times \overline{\Omega}$. f and u_0 are regular data.

The terminology variable-order fractional Laplace operator indicates that $s(.)$ and $p(.)$ are functions and not real numbers. This operator is then a generalization of the fractional Laplacian $(-\Delta)^s$, which corresponds to $p(.) \equiv 2$ and $s(.) \equiv s \in (0,1)$ constant, and of the p -Laplacian $-\Delta_p$, which corresponds to $p(.) \equiv p \in (1,+\infty)$ constant and $s(.) \equiv 1$.

A very interesting area of nonlinear analysis lies in the study of elliptic equations involving fractional operators. Recently, great attention has been focused on these problems, both for pure mathematical research and in view of concrete real-world applications. Indeed, this type of operator arises in a quite natural way in different contexts, such as the description of several physical phenomena, optimization, population dynamics and mathematical finance. The fractional Laplacian operator $(-\Delta)^s$, $0 < s < 1$, also provides a simple model to describe some jump Lévy processes in probability theory (see for example [3, 7, 8, 10, 18] and the references therein).

In last years, a large number of papers are written on fractional Sobolev spaces and nonlocal problems driven by this operator (see for example [7,8,20, 23,24,25] for further details). Specifically, we refer to Di Nezza, Palatucci and Valdinoci [11], for a full introduction to study the fractional Sobolev spaces and the fractional p-Laplacian operators. On the other hand, attention has been paid to the study of partial differential equations involving the $p(x)$-Laplacian operators (see [13,16,19] and the references therein). So, the natural question that arises is to see which result can be obtained if we replace the $p(x)$-Laplacian operator by its fractional version (the fractional $p(x)$-Laplacian operator). Currently, as far as we know, the only results for fractional Sobolev spaces with variable exponents and fractional $p(x)$-Laplacian operator are obtained by [4,5,9,15,27]. In particular, the authors generalized the last operator to fractional case. Then, they introduced an appropriate functional space to study problems in which a fractional variable exponent operator is present. These works are generalized by Reshmi Biswas and Sweta Tiwari in the case of variable order, see [6], they proved interesting properties concerning the spaces of Sobolev with variable order, other works in this direction can be found in [28, 29].

As far as we know, there are only few important contributions concerning the study of nonlinear parabolic problems involving the fractional p-Laplacian operator. One is due to Mazon et al. in [17], where they proved that the problem

$$u_t(t,x)= \int_A \frac{1}{|x-y|^{d+sp}} |u(t,y) - u(t,x)|^{p-2}(u(t,y)-u(t,x))dy \text{ for } x\in\Omega, t>0$$

has a unique strong solution by using the theory of maximal monotone operators.

By using the theory of maximal accretive operators in Banach spaces, J. Giacomoni and S. Tiwari showed in [14] the existence and uniqueness of weak solutions for the parabolic problem involving fractional p -Laplacian,

$$\begin{cases} u_t + (-\Delta)_p^s u + g(x,u) = f(x,u) & \text{in } Q_T := \Omega \times (0,T), \\ u = 0 \quad \text{in } \mathbb{R}^d \backslash \Omega \times (0,T), \\ u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^d, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^d$ (at least $C^2\big)$, $s \in (0,1), 1 < p < \frac{d}{s}$, $u_0 \in L^\infty(\Omega)$ and $f(x,z), g(x,z)$ are Carathéodory functions, locally Lipschitz with respect to z uniformly in x and satisfying some growth conditions.

In addition, by SOLA method, B. Abdellaoui et al. studied in [2] the existence of weak solutions of the nonlinear fractional p-Laplacian problem with Dirichlet boundary condition

$$\begin{cases} u_t + \left(-\Delta_p^s\right) u = f(x,t) & \text{in } \Omega_T \equiv \Omega \times (0,T), \\ u = 0 & \text{in } \left(\mathbb{R}^d \backslash \Omega\right) \times (0,T), \\ u(x,0) = u(x) & \text{in } \Omega, \end{cases}$$

with $(f, u_0) \in L^1\left(\Omega_T\right) \times L^1(\Omega)$.

Motivated by these works, and by using the Rothe's method, we study the existence and uniqueness question of weak solutions to the non-linear parabolic problem (1.1), we apply here a time discretization of the problem (1.1) by Euler forward scheme and we show existence, uniqueness and stability of weak solutions to the discretized problem. After, we will construct from the weak solution of the discretized problem a sequence that we show converging to a weak solution of the nonlinear parabolic problem (1.1). We recall that The Rothe's method was introduced by E. Rothe in 1930 and it has been used and developed by many authors, e.g., P.P. Mosolov, K. Rektorys in linear and quasilinear parabolic problems. This method has been used by several authors while studying time discretization of nonlinear parabolic problems, we refer to the works [12, 21, 22] for some details. The advantage of our method is that we cannot only obtain the existence and uniqueness of weak solutions to the problem (1.1), but also compute the numerical approximations.

2 Preliminaries and notations

In this section, we will recall some notations and definitions and we will state some results which will be used in this work.
Let Ω be a smooth bounded open set in $\mathbb{R}^d$, we consider the set

$$C_+(\bar{\Omega}) = \{q \in C(\bar{\Omega}) : 1 < q^- < q(x) < q^+ < \infty \text{ for all } x \in \bar{\Omega}\},$$

where

$$q^- = \inf_{x \in \overline{\Omega}} q(x) \text{ and } q^+ = \sup_{x \in \overline{\Omega}} q(x).$$

For any $q \in C_+(\bar{\Omega})$, we define the variable exponent Lebesgue space as

$$L^{q(.)}(\Omega) = \Big\{ u : \text{function } u : \Omega \to \mathbb{R} \text{ is measurable with } \int_\Omega |u(x)|^{q(x)} dx < \infty \Big\},$$

which is endowed with the so-called Luxemburg norm

$$\|u\|_{q(\cdot)} = \inf \left\{ \gamma > 0 : \int_\Omega |u(x)/\gamma|^{q(x)}\, dx \leq 1 \right\}.$$

$\left(L^{q(\cdot)}(\Omega), \|\cdot\|_{q(\cdot)}\right)$ is a separable reflexive Banach space see, for example [16]. Let $p : \overline{\Omega} \times \overline{\Omega} \longrightarrow (1, +\infty)$ and $s : \overline{\Omega} \times \overline{\Omega} \longrightarrow (0, 1)$ be two continuous functions such that

$$1 < p^- = \min_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) \leqslant p(x,y) \leqslant p^+ = \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) < +\infty, \tag{2.1}$$

$$0 < s^- = \min_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} s(x,y) \leqslant s(x,y) < s^+ = \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} s(x,y) < 1, \tag{2.2}$$

$$0 < s^- < s^+ < 1 < p^- \leqslant p^+. \tag{2.3}$$

We set $\overline{p}(x) = p(x,x)$ and $\overline{s}(x) = s(x,x)$ for all $x \in \overline{\Omega}$.

We assume that p and s are symmetric, that is

$$p(x,y) = p(y,x), \quad s(x,y) = s(y,x) \;\; \forall (x,y) \in \overline{\Omega} \times \overline{\Omega}. \tag{2.4}$$

The variable-order fractional Sobolev space with variable exponent via the Gagliardo approach is defined by

$$X = W^{s(\cdot),p(\cdot)}(\Omega) = \Bigg\{ u \in L^{\bar{p}(x)}(\Omega) : \\ \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{\gamma^{p(x,y)} |x-y|^{d+s(x,y)p(x,y)}} dxdy < \infty \text{ for some } \gamma > 0 \Bigg\},$$

with the norm $\|u\|_X = \|u\|_{\bar{p}(x)} + [u]_{s(\cdot),p(\cdot)}$, where

$$[u]_{s(\cdot),p(\cdot)} = \inf \left\{ \gamma > 0 : \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{\gamma^{p(x,y)} |x-y|^{d+s(x,y)p(x,y)}} dxdy < 1 \right\}$$

is a Gagliardo seminorm with variable-order and variable exponent. The space X is a separable reflexive Banach space, see [6]. Next we define the subspace X_0 of X as

$$X_0 = X_0^{s(.),p(.)}(\Omega) := \{ u \in X : u = 0 \text{ a.e.in } \Omega^c \}$$

endowed by the norm $\|u\|_{X_0} := [u]_{s(\cdot),p(\cdot)}$. The space X_0 is a separable reflexive Banach space, see [6]. We define the convex modular function $\varrho_{p(\cdot)}^{s(\cdot)} : X_0 \to \mathbb{R}$ by

$$\varrho_{p(\cdot)}^{s(\cdot)}(u) = \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{d+s(x,y)p(x,y)}} dxdy,$$

whose associated norm define by

$$\|u\| = \|u\|_{\rho_{p(.)}^{s(.)}} = \inf \left\{ \gamma > 0 : \varrho_{p(\cdot)}^{s(\cdot)} \{u/\gamma\} \leq 1 \right\},$$

which is equivalent to the norm $\|\cdot\|_{X_0}$.

Proposition 1. *[6] Let $u \in X_0$ and $\{u_n\} \subset X_0$, then*

1. $\|u\|_{X_0} < 1(\text{ resp. } = 1, > 1) \Longleftrightarrow \rho^{s(\cdot)}_{p(\cdot)}(u) < 1(\text{ resp. } = 1, > 1)$,
2. $\|u\|_{X_0} < 1 \Rightarrow \|u\|^{p^+}_{X_0} \leq \rho^{s(\cdot)}_{p(\cdot)}(u) \leq \|u\|^{p^-}_{X_0}$,
3. $\|u\|_{X_0} > 1 \Rightarrow \|u\|^{p^-}_{X_0} \leq \rho^{s(\cdot)}_{p(\cdot)}(u) \leq \|u\|^{p^+}_{X_0}$,
4. $\lim_{n\to\infty} \|u_n\|_{X_0} = 0(\infty) \Longleftrightarrow \lim_{n\to\infty} \rho^{s(\cdot)}_{p(\cdot)}(u_n) = 0(\infty)$,
5. $\lim_{n\to\infty} \|u_n - u\|_{X_0} = 0 \Longleftrightarrow \lim_{n\to\infty} \rho^{s(\cdot)}_{p(\cdot)}(u_n - u) = 0$.

Theorem 1. *[6] Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain and let $p(.)$ and $s(.)$ be two continuous variable exponents satisfying (2.1)–(2.4) with $s(.)p(.) < d$. Assume that $r : \overline{\Omega} \longrightarrow (1, +\infty)$ is a continuous variable exponent such that*

$$p^*_{s(.)}(x) = \frac{d\overline{p}(x)}{d - \overline{s}(x)\overline{p}(x)} > r(x) \geqslant r^- = \min_{x\in\overline{\Omega}} r(x) > 1 \quad \text{for all } x \in \overline{\Omega}.$$

Then, there exists a positive constant $C = C(d, s, p, r, \Omega)$ such that, for any $u \in X_0$

$$\|u\|_{L^{r(x)}(\Omega)} \leqslant C \|u\|_{X_0}.$$

Thus, X_0 is continuously embedded in $L^{r(x)}(\Omega)$ for any $r \in (1, p^*_{s(.)})$. Moreover, this embedding is compact.

Let $q' \in C_+(\overline{\Omega})$ be the conjugate exponent of q, that is, $\frac{1}{q(x)} + \frac{1}{q'(x)} = 1$ for all $x \in \overline{\Omega}$, then we have the following Hölder's inequality:

Lemma 1. *[13]**(Hölder's inequality)**. If $u \in L^{q(x)}(\Omega)$ and $v \in L^{q'(x)}(\Omega)$, then*

$$\left|\int_\Omega uv dx\right| \leqslant \left(\frac{1}{q^-} + \frac{1}{q'^-}\right) \|u\|_{L^{q(x)}(\Omega)} \|v\|_{L^{q'(.)}(\Omega)} \leqslant 2 \|u\|_{L^{q(x)}(\Omega)} \|v\|_{L^{q'(x)}(\Omega)}.$$

Let X be a Banach space and let $T > 0$. For $1 \leq p \leq \infty$, the space $L^p(0, T; X)$ consists of all measurable functions $u : [0, T] \to X$ such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|^p_X \, dt\right)^{\frac{1}{p}} < \infty \quad \text{if } 1 \leq p < \infty,$$
$$\|u\|_{L^\infty(0,T;X)} = \text{ess-sup}_{t\in[0,T]} \|u(t)\|_X < \infty.$$

The space $C(0, T; X)$ is a space of all continuous functions $u : [0, T] \to X$ such that

$$\|u\|_{C(0,T;X)} = \max_{t\in[0,T]} \|u(t)\|_X < \infty.$$

The spaces $L^p(0, T; X)$ and $C(0, T; X)$ equipped with the norms from the above definitions are the Banach spaces.

Lemma 2. *[1] For $\xi,\ \eta \in \mathbb{R}^d$ and $1<p<\infty$, we have*

$$\frac{1}{p}|\xi|^p - \frac{1}{p}|\eta|^p \le |\xi|^{p-2}\xi(\xi-\eta).$$

Lemma 3. *[26] For any $x, y \in \mathbb{R}^d$, we have*

$$\begin{cases} |x-y|^p \le c_p\left(|x|^{p-2}x - |y|^{p-2}y\right)\cdot(x-y) & \text{for } p \ge 2,\\ |x-y|^p \le C_p\left[\left(|x|^{p-2}x-|y|^{p-2}y\right)\cdot(x-y)\right]^{\frac{p}{2}}\left(|x|^p+|y|^p\right)^{\frac{2-p}{2}} & \text{for } 1<p<2, \end{cases}$$

where $c_p = (\frac{1}{2})^{-p}$ and $C_p = \frac{1}{p-1}$.

Remark 1. Hereinafter k, τ, T are strictly positive real numbers, N is a strictly positive natural number and $C(X), C_i(X)$ $(i \in \mathbb{N})$ are positive constants depending only on X.

3 Main result

In this section, we give the notion of weak solutions for the nonlinear fractional parabolic problem (1.1) and we state the main result of this paper, Firstly, we assume that

$$f \in L^\infty(Q_T) \text{ and } u_0 \in L^\infty(\Omega) \cap X_0. \tag{3.1}$$

Definition 1. A measurable function $u : Q_T \to \mathbb{R}$ is a weak solution to the nonlinear fractional parabolic problem (1.1) in Q_T if $u(.,0) = u_0$ in Ω, $u \in C(0,T;L^2(\Omega)) \cap L^{p^-}(0,T;X_0)$, $\frac{\partial u}{\partial t} \in L^2(Q_T)$, and for all $\varphi \in C^1(Q_T)$, we have

$$\begin{aligned} \int_0^T \int_\Omega \frac{\partial u}{\partial t}\varphi \, dx\, dt + \int_0^T \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{d+s(x,y)p(x,y)}} \\ \times \Big(\varphi(x)-\varphi(y)\Big)\, dx\, dy\, dt = \int_0^T \int_\Omega f\varphi \, dx\, dt. \end{aligned} \tag{3.2}$$

Now, we state our main result of this work.

Theorem 2. *Let $p(.)$ and $s(.)$ be two continuous variable exponents satisfying* (2.1)–(2.4) *with $s(x,y)p(x,y) < d$ for all $(x,y) \in \overline{\Omega} \times \overline{\Omega}$. If hypothesis* (3.1) *holds, then, the problem* (1.1) *has a unique weak solution.*

4 Proof of the main result

The proof of our main result is divided into three steps, in the first one, using Euler forward scheme, we discretize the problem (1.1) and we study the existence and uniqueness questions of weak solutions to the discretized problems. In the second step, we give some stability results for the discrete entropy solutions. Finally and by Rothe's function, we construct a sequence of functions that we show that this sequence converges to a weak solution of the nonlinear fractional parabolic problem (1.1). We finish this step by proving the uniqueness result of weak solutions.

Step 1. The semi-discrete problem.
By Euler forward scheme, we discretize the problem (1.1), we obtain the following problems

$$\begin{cases} U_n + \tau\,(-\Delta)^{s(.)}_{p(.)}\,U_n = \tau f_n + U_{n-1} \text{ in } \Omega, \\ U_n = 0 \text{ on } \partial\Omega, \\ U_0 = u_0 \text{ in } \Omega, \end{cases} \tag{4.1}$$

where $N\tau = T,\ 0 < \tau < 1,\ 1 \le n \le N,\ t_n = n\tau,\ f_n(.) = \frac{1}{\tau}\int_{t_{n-1}}^{t_n} f(s,.)ds$ in Ω.

DEFINITION 2. A weak solution for the discretized problems (4.1) is a sequence $(U_n)_{0\le n\le N}$ such that $U_0 = u_0$ and U_n is defined by induction as a weak solution of the problem

$$\begin{cases} u + \tau\,(-\Delta)^{s(.)}_{p(.)}\,u = \tau f_n + U_{n-1} \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

i.e., $U_n \in X_0$ and $\forall\, \varphi \in X_0,\ \forall\, \tau > 0$, we have

$$\int_\Omega U_n\varphi\, dx + \tau\int_\Omega\int_\Omega \frac{|U_n(x)-U_n(y)|^{p(x,y)-2}(U_n(x)-U_n(y))}{|x-y|^{d+s(x,y)p(x,y)}} \times \Big(\varphi(x)-\varphi(y)\Big)\, dx\, dy = \int_\Omega (\tau f_n + U_{n-1})\varphi\, dx. \tag{4.2}$$

Lemma 4. *Let hypothesis* (3.1) *be satisfied. If* $(U_n)_{0\le n\le N}$ *is a weak solution of the discretized problem* (4.1)*, then for all* $n = 1,\dots,N$*, we have* $U_n \in L^\infty(\Omega)$*.*

Proof. Let $k > 0$ and $1 \le n \le N$, we take $\varphi = |U_n|^k U_n$ as test function in the Equation (4.2) we obtain

$$\int_\Omega |U_n|^{k+2}dx + \tau\int_\Omega\int_\Omega \frac{|U_n(x)-U_n(y)|^{p(x,y)-2}(U_n(x)-U_n(y))}{|x-y|^{d+s(x,y)p(x,y)}} \times \Big(|U_n|^k U_n(x) - |U_n|^k U_n(y)\Big)\, dx\, dy = \int_\Omega (\tau f_n + U_{n-1})|U_n|^k U_n dx.$$

Using Hölder's inequality and the hypothesis (3.1), we obtain

$$\|U_n\|^{k+2}_{k+2} \le \tau C_1 \|U_n\|^{k+1}_{k+1} + \|U_{n-1}\|_{k+2}\, \|U_n\|^{k+1}_{k+2}.$$

Since $\|U_n|_{k+1} \le C_2\, \|U_n\|_{k+2}$, it follows that

$$\|U_n\|_{k+2} \le \tau C_3 + \|U_{n-1}\|_{k+2}\,,$$

and, by induction, we deduce that

$$\|U_n\|_{k+2} \le TC_4 + \|U_0\|_\infty\,.$$

Taking the limit as $k \to \infty$, we deduce the desired result. □

Theorem 3. *Let $p(.)$ and $s(.)$ be two continuous variable exponents satisfying (2.1)–(2.4) with $s(x,y)p(x,y) < d$ for all $(x,y) \in \overline{\Omega} \times \overline{\Omega}$. If hypothesis (3.1) holds, then, the problem (4.1) has a unique weak solution $(U_n)_{0 \leq n \leq N}$ and for all $n = 1, ..., N$, $U_n \in L^\infty(\Omega) \cap X_0$.*

Proof. For $n = 1$, we pose $u = U_1$, we rewrite the problem (4.1) as

$$\begin{cases} u + \tau \, (-\Delta)^{s(.)}_{p(.)} \, u = \tau f_1 + U_0 \text{ in } \ \Omega, \\ u = 0 \ \text{ on } \ \partial\Omega. \end{cases} \tag{4.3}$$

Existence part. For any $u \in X_0$, define the following functional

$$\begin{aligned} \mathcal{F}(u) :=& \frac{1}{2} \int_\Omega u^2(x) dx \int_\Omega + \tau \int_\Omega \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{d+s(x,y)p(x,y)} p(x,y)} dxdy \\ & - \tau \int_\Omega f_1(x) u(x) dx - \int_\Omega U_0 u(x) dx. \end{aligned}$$

Note that the functional $\mathcal{F}(u)$ is bounded and strictly convex (this holds since for any x and y the function $t \longmapsto t^{p(x,y)}$ is strictly convex). Now our goal is to prove that $\mathcal{F}(u)$ has a unique minimizer in X_0. This minimizer shall provide the unique weak solution to the problem (4.3). For that, let $u \in X_0$ with $\|u\|_{X_0} \geqslant 1$, by using Proposition 1 and Theorem 1, we derive that

$$\begin{aligned} \mathcal{F}(u) \geq & \frac{\tau}{p_+} \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{d+s(x,y)p(x,y)}} dxdy - \tau \left\| f_1 \right\|_{L^{(\overline{p}(x))'}(\Omega)} \left\| u \right\|_{L^{\overline{p}(x)}(\Omega)} \\ & - \left\| U_0 \right\|_{L^{(\overline{p}(x))'}(\Omega)} \left\| u \right\|_{L^{\overline{p}(x)}(\Omega)} \geq \frac{\tau}{p_+} \left\| u \right\|^{p_-}_{X_0} - C_5 \left\| u \right\|_{X_0} - C_6 \left\| u \right\|_{X_0}, \end{aligned}$$

which implies that $\mathcal{F}(u)$ is coercive on X_0. Then, there is a unique minimizer u of $\mathcal{F}(u)$. It remains to show that when u is a minimizer to $\mathcal{F}(u)$ then it is a weak solution to the problem (4.3). Indeed, given a $\varphi \in X_0$ we compute

$$\begin{aligned} 0 = & \frac{d}{dt} \mathcal{F}(u + t\varphi) \Big|_{t=0} = \int_\Omega \frac{d}{dt} \frac{(u(x) + t\varphi(x))^2}{2} dx \Big|_{t=0} \\ & + \tau \int_\Omega \int_\Omega \frac{d}{dt} \frac{|u(x) - u(y) + t(\varphi(x) - \varphi(y))|^{p(x,y)}}{p(x,y)|x-y|^{d+s(x,y)p(x,y)}} dxdy \Big|_{t=0} \\ & - \tau \int_\Omega \frac{d}{dt} f_1(x)(u(x) + t\varphi(x)) dx \Big|_{t=0} - \int_\Omega \frac{d}{dt} U_0(x)(u(x) + t\varphi(x)) dx \Big|_{t=0} \\ = & \int_\Omega u(x)\varphi(x) dx + \tau \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))}{|x-y|^{d+s(x,y)p(x,y)}} \\ & \times (\varphi(x) - \varphi(y)) dxdy - \tau \int_\Omega f_1(x)\varphi(x) dx - \int_\Omega U_0(x)\varphi(x) dx. \end{aligned}$$

As u is a minimizer of $\mathcal{F}$. Thus, we deduce that u is a weak solution to the problem (4.3). The proof of the converse (that every weak solution is a minimizer of $\mathcal{F}$) is standard and I leave the details to the reader.

Uniqueness part. Let u and v be two weak solutions of the problem (4.3). For the solution u, we take $\varphi = u - v$ as test function and for the solution v we take $\varphi = v - u$ as test function in the equality (4.2), then we have

$$\int_\Omega u(u-v)dx + \tau \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{d+s(x,y)p(x,y)}} \times \Big(u(x)-u(y)-\big(v(x)-v(y)\big)\Big)\,dx\,dy = \int_\Omega (\tau f_1 + U_0)(u-v)dx,$$

and

$$\int_\Omega v(v-u)dx + \tau \int_\Omega \int_\Omega \frac{|v(x)-v(y)|^{p(x,y)-2}(v(x)-v(y))}{|x-y|^{d+s(x,y)p(x,y)}} \times \Big(v(x)-v(y)-\big(u(x)-u(y)\big)\Big)\,dx\,dy = \int_\Omega (\tau f_1 + U_0)(v-u)dx.$$

By summing up the two above equalities, we get

$$\int_\Omega (u-v)^2 dx \ + \tau \Big\langle (-\Delta)^{s(.)}_{p(.)}\, u - (-\Delta)^{s(.)}_{p(.)}\, v, u-v \Big\rangle = 0.$$

Using Lemma 3, we deduce that

$$\Big\langle (-\Delta)^{s(.)}_{p(.)}\, u - (-\Delta)^{s(.)}_{p(.)}\, v, u-v \Big\rangle \geqslant 0.$$

Therefore, $u = v$ a.e in Ω. By induction, using the same argument above, we prove that the problem (4.1) has a unique weak solution $(U_n)_{0\le n\le N}$ such that $n = 1, ..., N$, $U_n \in L^\infty(\Omega) \cap X_0$. □

Step 2. Stability results.
In this section, we give some a priori estimates for the discrete weak solution $(U_n)_{1\le n\le N}$ which will be used to derive the convergence results for the Euler forward scheme.

Theorem 4. *Let hypothesis* (3.1) *be satisfied. Then, there exists a positive constant* $C(u_0, f)$ *depending on the data but not on* N *such that for all* $n = 1, \dots, N$, *we have*

$$(a)\ \|U_n\|_2^2 \le C(u_0,f), \quad (b)\ \sum_{i=1}^n \|U_i - U_{i-1}\|_2^2 \le C(u_0,f),$$

$$(c)\ \tau \sum_{i=1}^n \|U_i\|_{X_0}^{p^-} \le C(u_0,f), \quad (d)\ \sum_{i=1}^n \|U_i - U_{i-1}\|_{L^1(\Omega)} \le C(u_0,f).$$

Proof. **For (a) and (b).** Let $1 \le i \le N$, we take $\varphi = U_i$ as test function in the Equation (4.2) we obtain

$$\int_\Omega (U_i - U_{i-1})U_i dx + \tau \int_\Omega \int_\Omega \frac{|U_i(x)-U_i(y)|^{p(x,y)}}{|x-y|^{d+s(x,y)p(x,y)}} dxdy = \int_\Omega (\tau f_i + U_{i-1})U_i dx. \tag{4.4}$$

With the aid of the identity $2a(a-b) = a^2 - b^2 + (a-b)^2$, from (4.4) we obtain

$$\frac{1}{2}\|U_i\|_2^2 - \frac{1}{2}\|U_{i-1}\|_2^2 + \|U_i - U_{i-1}\|_2^2 + \tau\int_\Omega\int_\Omega \frac{|U_i(x)-U_i(y)|^{p(x,y)}}{|x-y|^{d+s(x,y)p(x,y)}}dxdy \le \tau C_7\|U_i\|_2. \tag{4.5}$$

Now, summing (4.5) from $i=1$ to n and using the Lemma 4, we get

$$\frac{\|U_n\|_2^2 - \|U_0\|_2^2}{2} + \sum_{i=1}^n \|U_i - U_{i-1}\|_2^2 + \tau\sum_{i=1}^n \int_\Omega\int_\Omega \frac{|U_i(x)-U_i(y)|^{p(x,y)}}{|x-y|^{d+s(x,y)p(x,y)}}dxdy \le C_8. \tag{4.6}$$

Hence, the stability result (a) and (b) are then proved.

For (c). We pose $m_0 = \{i \in 1;2;...;N : \|U_i\|_{X_0} \le 1\}$, we have

$$\tau\sum_{i=1}^n \|U_i\|_{X_0}^{p^-} \le \tau\sum_{i\in m_0}\|U_i\|_{X_0}^{p^-} + \tau\sum_{i\notin m_0}\|U_i\|_{X_0}^{p^-} \le T + \tau C_9\sum_{i\notin m_0}\varrho_{p(\cdot)}^{s(\cdot)}(U_i).$$

And, by the inequality (4.6), we deduce the stability result (c).

For (d). Let $k>0$, we define the following function

$$T_k(m) := \begin{cases} m, & \text{if } |m| \le k,\\ k\ sign\ (m), & \text{if } |m| > k,\end{cases} \quad \text{where } sign\ (m) := \begin{cases} 1, & \text{if } m>0,\\ 0, & \text{if } m=0,\\ -1, & \text{if } m<0.\end{cases}$$

Taking $\varphi = T_\tau(U_i - U_{i-1})$ in the Equation (4.2) and dividing this equation by τ, we obtain by applying Lemma 2 that

$$\int_\Omega (U_i - U_{i-1})\frac{T_\tau(U_i - U_{i-1})}{\tau}dx + \int_{\Omega_\tau^i(y)}\int_{\Omega_\tau^i(x)}\left(\frac{|U_i(x)-U_i(y)|^{p(x,y)}}{p(x,y)|x-y|^{d+s(x,y)p(x,y)}} - \frac{|U_{i-1}(x)-U_{i-1}(y)|^{p(x,y)}}{p(x,y)|x-y|^{d+s(x,y)p(x,y)}}\right)dxdy \le \tau\|f_i\|_{L^1(\Omega)}, \tag{4.7}$$

where $\Omega_\tau^i(z) = \{|U_i(z) - U_{i-1}(z)| \le \tau\}$. Summing the inequality (4.7) from $i=1$ to n, we get

$$\sum_{i=1}^n\int_\Omega (U_i - U_{i-1})\frac{T_\tau(U_i - U_{i-1})}{\tau}dx \le \frac{1}{p^-}\int_\Omega\int_\Omega \frac{|U_0(x)-U_0(y)|^{p(x,y)}}{|x-y|^{d+s(x,y)p(x,y)}}dxdy + C_{10}.$$

Then, letting τ approach to 0 in the above inequality, and using the fact that

$$\lim_{k\to 0} m(x)\frac{T_k(m(x))}{k} = |m(x)|,$$

we deduce the stability result (d). □

Step 3. Weak solution of the continuous problem.

Let us introduce the following piecewise linear extension (called Rothe function)

$$\begin{cases} u_N(0) := u_0,\\ u_N(t) := U_{n-1} + (U_n - U_{n-1})\frac{(t-t_{n-1})}{\tau}, \ \forall\, t \text{ in } \ ]t_{n-1},\ t_n],\ n = 1,\ldots,N \text{ in } \Omega.\end{cases}$$

And the following piecewise constant function

$$\begin{cases} \overline{u}_N(0) := u_0, \\ \overline{u}_N(t) := U_n \ \forall\ t \text{ in } \ ]t_{n-1},\ t_n], \quad n = 1, ..., N \text{ in } \ \Omega. \end{cases}$$

We have by Theorem 3 that for any $N \in \mathbb{N}$, the weak solution $(U_n)_{1 \le n \le N}$ of problems (4.1) is unique, thus, the two sequences $(u_N)_{N \in \mathbb{N}}$ and $(\overline{u}_N)_{N \in \mathbb{N}}$ are uniquely defined.

Lemma 5. *Let hypothesis* (3.1) *be satisfied. Then, there exists a positive constant* $C(T, u_0, f)$ *independent of* N *such that for all* $N \in \mathbb{N}$, *we have*

$$(1)\ \ \|\overline{u}_N - u_N\|^2_{L^2(Q_T)} \le \frac{1}{N} C(T, u_0, f), \quad (2)\ \ \|\overline{u}_N\|_{L^\infty(0,T,L^2(\Omega))} \le C(T, u_0, f),$$

$$(3)\ \ \|u_N\|_{L^\infty(0,T,L^2(\Omega))} \le C(T, u_0, f), \quad (4)\ \ \|\overline{u}_N\|_{L^{p^-}(0,T,X_0)} \le C(T, u_0, f),$$

$$(5)\ \ \left\| \frac{\partial u_N}{\partial t} \right\|^2_{L^2(Q_T)} \le C(T, u_0, f).$$

Proof. **For (1).** We have

$$\begin{aligned} \|\overline{u}_N - u_N\|^2_{L^2(Q_T)} &= \int_0^T \int_\Omega |\overline{u}_N - u_N|^2 dxdt \\ &\le \sum_{i=1}^{i=N} \int_{t^{n-1}}^{t^n} \int_\Omega |U_n - U_{n-1}|^2 \left(\frac{t^n - t}{\tau} \right)^2 dxdt = \frac{\tau}{3} \sum_{i=1}^{i=N} \|U_n - U_{n-1}\|_2^2. \end{aligned}$$

From **(b)** of Theorem 4, we get

$$\|\overline{u}_N - u_N\|^2_{L^2(Q_T)} \le \frac{1}{N} C(T, u_0, f).$$

And in the same manner, we show the results (2), (3) and (4).

For (5). In the weak formulation (4.2) we take $\varphi = U_n - U_{n-1}$ and summing this equality from $i = 1$ to N, we get by applying Lemma 2 and hypothesis (3.1) that

$$\begin{aligned} &\sum_{i=1}^{i=N} \int_\Omega \frac{(U_i - U_{i-1})^2}{\tau} dx + \sum_{i=1}^{i=N} \int_\Omega \int_\Omega \left(\frac{|U_i(x) - U_i(y)|^{p(x,y)}}{p(x,y)|x-y|^{d+s(x,y)p(x,y)}} \right. \\ &\qquad \left. - \frac{|U_{i-1}(x) - U_{i-1}(y)|^{p(x,y)}}{p(x,y)|x-y|^{d+s(x,y)p(x,y)}} \right) dxdy \le C_{11} \sum_{i=1}^{i=N} \|U_i - U_{i-1}\|_{L^1(\Omega)} . \end{aligned}$$

This implies that

$$\left\| \frac{\partial u_N}{\partial t} \right\|^2_{L^2(Q_T)} \le \frac{1}{p^-} \int_\Omega \int_\Omega \frac{|U_0(x) - U_0(y)|^{p(x,y)}}{|x-y|^{d+s(x,y)p(x,y)}} dxdy + C_{12} \sum_{i=1}^{i=N} \|U_i - U_{i-1}\|_{L^1(\Omega)} .$$

Then, we apply the result **(d)** of Theorem 4 and hypothesis (3.1) we obtain the result (5). □

Now, using the two results (2) and (3) of Lemma 5, the sequences $(u_N)_{N\in\mathbb{N}}$ and $(\overline{u}_N)_{N\in\mathbb{N}}$ are uniformly bounded in $L^\infty(0,T,L^2(\Omega))$, therefore, there exist two elements u and v in $L^\infty(0,T,L^2(\Omega))$ such that

$$\overline{u}_N \to u \text{ weakly in } L^\infty(0,T,L^2(\Omega)), \;\; u_N \to v \text{ weakly in } L^\infty(0,T,L^2(\Omega)).$$

And from the result (1) of Lemma 5, it follows that $u \equiv v$. Furthermore, by Lemma 5 we have that

$$\frac{\partial u_N}{\partial t} \to \frac{\partial u}{\partial t} \text{ in } L^2(Q_T), \;\; \overline{u}_N \to u \text{ in } L^{p^-}(0,T,X_0),$$
$$\frac{|\overline{u}_N(x)-\overline{u}_N(y)|^{p(x,y)}}{|x-y|^{\frac{d+s(x,y)p(x,y)}{p'(x,y)}}} \to \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{\frac{d+s(x,y)p(x,y)}{p'(x,y)}}} \text{ weakly in} \left(L^{p'(x,y)}(\Omega\times\Omega\times]0;T[)\right)^d.$$

On the other hand, we have by Lemma 5 and Aubin-Simon's compactness result that $u_N \to u$ in $C(0,T,L^2(\Omega))$. Now, we prove that the limit function u is a weak solution of problem (1.1). Firstly, we have $u_N(0) = U_0 = u_0$ for all $N \in \mathbb{N}$, then $u(0,.) = u_0$. Secondly, let $\varphi \in C^1(Q_T)$, we rewrite (3.2) in the forms of

$$\int_0^T \int_\Omega \frac{\partial u_N}{\partial t}\varphi \, dx\, dt + \int_0^T \int_\Omega \int_\Omega \frac{|\overline{u}_N(x)-\overline{u}_N(y)|^{p(x,y)-2}(\overline{u}_N(x)-\overline{u}_N(y))}{|x-y|^{d+s(x,y)p(x,y)}}$$
$$\times \Big(\varphi(x)-\varphi(y)\Big)\, dx\, dy\, dt = \int_0^T \int_\Omega f_N\varphi \, dx, \tag{4.8}$$

where $f_N(t,x) = f_n(x), \forall\, t \in]t^{n-1},\; t^n],\; n = 1,\dots,N$. Taking limits as $N \to \infty$ in (4.8) and using the above results, we deduce that u is a weak solution of the nonlinear fractional parabolic problem (1.1).

Uniqueness part. Let u and v be two weak solutions of the problem (1.1). For the solution u, we take $\varphi = u - v$ as test function and for the solution v we take $\varphi = v - u$ as test function in the Equation (3.2), then we have

$$\int_0^T \int_\Omega \frac{\partial u}{\partial t}(u-v)dx\; dt + \int_0^T \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{d+s(x,y)p(x,y)}}$$
$$\times \Big(u(x)-u(y)-\big(v(x)-v(y)\big)\Big)\, dx\, dy\, dt = \int_0^T \int_\Omega f(u-v)dx\, dt,$$
$$\int_0^T \int_\Omega \frac{\partial v}{\partial t}(v-u)dx\; dt + \int_0^T \int_\Omega \int_\Omega \frac{|v(x)-v(y)|^{p(x,y)-2}(v(x)-v(y))}{|x-y|^{d+s(x,y)p(x,y)}}$$
$$\times \Big(v(x)-v(y)-\big(u(x)-u(y)\big)\Big)\, dx\, dy\, dt = \int_0^T \int_\Omega f(v-u)dx\, dt.$$

By summing up the two above equalities, and using the same arguments in the proof of uniqueness part of Theorem 3, we conclude that $u = v$ a.e in Ω.

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