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Convergence and Stability of Galerkin Finite Element Method for Hyperbolic Partial Differential Equation with Piecewise Continuous Arguments of Advanced Type

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Abstract. This paper deals with the convergence and stability of Galerkin finite element method for a hyperbolic partial differential equations with piecewise continuous arguments of advanced type. First of all, we obtain the expression of analytic solution by the method of separation variable, then the sufficient conditions for stability are obtained. Semidiscrete and fully discrete schemes are derived by Galerkin finite element method, and their convergence are both analyzed in L^2-norm. Moreover, the stability of the two schemes are investigated. The semidiscrete scheme can achieve unconditionally stability. The sufficient conditions of stability for fully discrete scheme are derived under which the analytic solution is asymptotically stable. Finally, some numerical experiments are presented to illustrate the theoretical results.
Keywords: hyperbolic partial differential equation, piecewise continuous arguments, Galerkin finite element method, convergence, stability.

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1 Introduction

In this paper, we consider the following differential equation with piecewise continuous arguments (EPCA)

$$\begin{aligned}
&u_{tt}(x,t) = a^2 u_{xx}(x,t) + b u_{xx}(x,[t]) + c u_{xx}(x,[t+1]) \text{ in } \Omega \times J,\\
&u(x,0) = v(x),\ u_t(x,0) = w(x) \text{ in } \Omega, \qquad (1.1)\\
&u(x,t) = 0 \text{ on } \partial\Omega \times J,
\end{aligned}$$

where $a, b \in \mathbb{R}$, $\Omega = [0, \pi]$ with smooth boundary $\partial\Omega$, J denotes the time interval $[0, +\infty)$ and $[\cdot]$ denotes the greatest integer function. In the past few decades, EPCA has been applied successfully in economy [2], competition [12], population growth [11] and so on. This class of equations is a hybrid of continuous and discrete dynamical systems, combining the properties of differential and difference equations. The continuity of solution at the points connecting any two continuous intervals implies the recurrence of the value of the solution at those points. Hence, the EPCA is essentially closer to difference equation than differential equation. The studies of these kinds of equations were initially mentioned in [20] and [5]. In the following years, qualitative properties such as the stability and convergence [32, 34], oscillation [4, 8], periodicity [4, 6] of solutions of EPCA have been discussed deeply. Although the numerical study of EPCA starts late, it has gradually become more and more popular since EPCA can hardly be solved by analytical methods or much complicated to deal with. Many numerical methods have been applied to EPCA, such as the θ-methods [26], the Runge-Kutta methods [24, 35], the Euler-Maclaurin methods [18], spectral collocation methods [33] and the linear multistep methods [16] and so on.

However, the literatures mentioned above only focus on the EPCA in case of ordinary differential equations. To the best of our knowledge, there are few publications concerning partial differential equation with piecewise continuous arguments (PEPCA) solved by numerical methods except for [14,15,22,23,25]. Liang et al. investigated PEPCA with the θ-methods [14] and Galerkin finite element method [15], numerical stability was analyzed, respectively. In [23], the θ-methods were also applied to another PEPCA of mixed type and the sufficient conditions for numerical stability were established. In addition, Wang and Wang [25] considered the analytical and numerical stability of PEPCA of alternately retarded and advanced type in the θ-schemes and achieved the corresponding stability conditions. It's worth noticing that published papers mentioned above concerning parabolic PEPCA. Different from them, in this paper, we will investigate a hyperbolic PEPCA of advanced type with homogeneous Dirichlet boundary conditions by Galerkin finite element method. The convergence and stability of numerical schemes are both discussed. For more information on PEPCA, the interested readers can refer to pubications [1,29,30,31] and the references contained therein. As a numerical method for partial differential equation, finite element method is regarded as an improvement of Galerkin method with using finite-dimensional spaces consisting of globally continuous piecewise polynomial functions, which make the application of finite element method more extensive and practical. Specially, Galerkin finite element method for spatial direction and other technique for time direction were proposed to discrete different equations and displayed excellent approximation effects [3, 13, 17]. In addition, Galerkin finite element method is also widely applied in many different fields, including physics [27], medicine [7, 19] and elasticity problems [10]. In our work, we will carry out the numerical approximation scheme for a hyperbolic PEPCA of advanced type by Galerkin finite element method and study its convergence and stability.

The organization of this paper is as follows. In Section 2, some preliminaries are provided. In Section 3, we obtain the expression of analytic solution in vector form and the analytical stability is analyzed. In Sections 4 and 5, the convergence and stability analysis for semidiscrete and fully discrete scheme are discussed and some numerical experiments are presented to verify the theoretical results in Section 6. Finally, we get some conclusions in Section 7.

2 Preliminaries

Definition 1. [28] A function $u(x,t)$ is called a solution of (1.1) if it satisfies the conditions:

(i) $u(x,t)$ is continuous in $\Omega \times J$;

(ii) $\partial^k u/\partial x^k$ and $\partial^k u/\partial t^k$ $(k=1,2)$ exist and are continuous in $\Omega \times J$ with the possible exception of the points (x,n), where one-sided derivatives exist $(n=0,1,2,\ldots)$;

(iii) $u(x,t)$ satisfies $u_{tt}(x,t) = a^2 u_{xx}(x,t) + bu_{xx}(x,[t]) + cu_{xx}(x,[t+1])$ in $\Omega \times J$ with the possible exception of the points (x,n), and conditions $u(x,0)=v(x)$, $u_t(x,0)=w(x)$ in Ω and $u(x,t)=0$ on $\partial\Omega \times J$.

Lemma 1. *[9] The sets of eigenvalues of the matrix S consist of all the eigenvalues of the following family of matrices*

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} & \cdots & S_{1n} \\ S_{21} & S_{22} & S_{23} & \cdots & S_{2n} \\ & \ddots & \ddots & \ddots & \\ S_{(n-1)1} & \cdots & S_{(n-1)(n-2)} & S_{(n-1)(n-1)} & S_{(n-1)n} \\ S_{n1} & \cdots & S_{n(n-2)} & S_{n(n-1)} & S_{nn} \end{pmatrix}.$$

Lemma 2. *[9] The polynomial $x^2 - e_1 x - e_2$ $(e_1, e_2 \in \mathbb{R})$ is Schur polynomial if and only if $|e_1| < 1 - e_2 < 2$.*

3 Analytic solution and stability

Definition 2. If any solution $u(x,t)$ of (1.1) satisfies $\lim_{t\to\infty} u(x,t) = 0$, $x \in \Omega$, the zero solution of (1.1) is asymptotically stable.

Application of the method of separation of variables to look for the solution of (1.1) with the form $u(x,t) = T(t)X(x)$ gives

$$T''(t)X(x) = a^2 T(t)X''(x) + bT([t])X''(x) + cT([t+1])X''(x),$$

so

$$\frac{T''(t)}{a^2T(t) + bT([t]) + cT([t+1])} = \frac{X''(x)}{X(x)} = -P^2,$$

that is

$$X''(x) + P^2 X(x) = 0, \tag{3.1}$$

$$T''(t) + a^2P^2T(t) = -bP^2T([t]) - cP^2T([t+1]). \tag{3.2}$$

(3.1) and the boundary conditions of (1.1) yield $P = i$, $X_i = \sin(xi)\,(i = 1, 2, \ldots)$. So, (3.2) changes into

$$T_i^{''}(t) + a^2 i^2 T_i(t) = -bi^2 T_i([t]) - ci^2 T([t+1]), i = 1, 2, \ldots. \tag{3.3}$$

For convenience, we take $T_i'(t) = V_i(t)$, so, (3.3) gives

$$W_i'(t) = AW_i(t) + BW_i([t]) + CW_i([t+1]), i = 1, 2, \ldots, \tag{3.4}$$

where

$$W_i = \begin{pmatrix} T_i \\ V_i \end{pmatrix}, A = \begin{pmatrix} 0 & 1 \\ -a^2 i^2 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ -bi^2 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 \\ -ci^2 & 0 \end{pmatrix}.$$

On the interval $[n, n+1)$, (3.4) becomes

$$W_i'(t) = AW_i(t) + BC_n + CC_{n+1},\ i = 1, 2, \ldots, \tag{3.5}$$

where $C_n = W_i([t]), C_{n+1} = W_i([t+1])$. From (3.5) we have

$$W_i(t) = \left(A^{-1}BC_n + A^{-1}CC_{n+1} + C_n\right) e^{A(t-n)} - A^{-1}BC_n - A^{-1}CC_{n+1}, i \geq 1,$$

that is

$$W_i(t) = M(t-n)C_n + N(t-n)C_{n+1},\ i = 1, 2, \ldots,$$

where

$$M(t-n) = e^{A(t-n)} + \left(e^{A(t-n)} - I\right)A^{-1}B, \quad N(t-n) = \left(e^{A(t-n)} - I\right)A^{-1}C.$$

Let $t = n+1$, we have $C_{n+1} = M(1)C_n + N(1)C_{n+1}$, where

$$M(1) = e^A + \left(e^A - I\right) A^{-1}B = \begin{pmatrix} (1 + \frac{b}{a^2})\cos(\eta) - \frac{b}{a^2} & \frac{\sin\eta}{\eta} \\ -(1 + \frac{b}{a^2})\eta\sin\eta & \cos\eta \end{pmatrix},$$

$$N(1) = \left(e^A - I\right) A^{-1}C = \begin{pmatrix} -\frac{c}{a^2}(1 - \cos(\eta)) & 0 \\ -\frac{c}{a^2}\eta\sin\eta & 0 \end{pmatrix}, \eta = ai, i = 1, 2, \ldots,$$

so,

$$C_{n+1} = (I - N(1))^{-1} M(1) C_n, \tag{3.6}$$

where

$$(I - N(1))^{-1} M(1) = \begin{pmatrix} -\frac{c\sin^2(\eta) - (a^2+b)\cos(\eta) + b}{a^2 + c(1-\cos(\eta))} & \frac{\sin(\eta)}{\eta} \\ -\frac{(a^2+b)\eta\sin(\eta) + c\eta\sin(\eta)\cos(\eta)}{a^2 + c(1-\cos(\eta))} & \cos(\eta) \end{pmatrix},$$

and we arrive at

$$C_{n+1} = \left((I - N(1))^{-1} M(1)\right)^{n+1} C_0, C_n = \left((I - N(1))^{-1} M(1)\right)^{n} C_0.$$

Therefore, we obtain

$$\begin{aligned} W_i(t) &= M(t-n)C_n + N(t-n)C_{n+1} = M(t-n)\left((I - N(1))^{-1} M(1)\right)^n C_0 \\ &+ N(t-n)\left((I - N(1))^{-1} M(1)\right)^{n+1} C_0 = M(t-[t])\left((I - N(1))^{-1} M(1)\right)^{[t]} C_0 \\ &+ N(t-[t])\left((I - N(1))^{-1} M(1)\right)^{[t+1]} C_0. \end{aligned} \tag{3.7}$$

Then the analytic solution $u(x,t)$ is the first component of the following vector in series form $\hat{u}(x,t)=\sum_{i=1}^{\infty} D_i \sin(xi) W_i(t)$. Let $t=0$, we have

$$\hat{u}(x,0)=\sum_{i=1}^{\infty} D_i \sin(xi) C_0=\sum_{i=1}^{\infty} D_i \sin(xi)\begin{pmatrix} T_i(0) \\ T_i^{'}(0) \end{pmatrix}. \tag{3.8}$$

From (3.8) and the initial conditions of (1.1) we have

$$\sum_{i=1}^{\infty} D_i \sin(xi) T_i(0)=v(x), \quad \sum_{i=1}^{\infty} D_i \sin(xi) T_i^{'}(0)=w(x),$$

so, we get

$$D_i T_i(0)=\frac{\pi}{2}\int_0^{\pi} v(x)\sin(xi)dx, \quad D_i T_i^{'}(0)=\frac{\pi}{2}\int_0^{\pi} w(x)\sin(xi)dx.$$

Thus, $\hat{u}(x,t)=\sum_{i=1}^{\infty}\sin(xi)W_i(t)d_i$, where

$$d_i=\begin{pmatrix} D_i T_i(0) \\ D_i T_i^{'}(0) \end{pmatrix}=\begin{pmatrix} \frac{\pi}{2}\int_0^{\pi} v(x)\sin(xi)dx \\ \frac{\pi}{2}\int_0^{\pi} w(x)\sin(xi)dx \end{pmatrix}.$$

Theorem 1. *The zero solution of Problem* (1.1) *is asymptotically stable if and only if*

$$c>0, b<c, a^2+b+c>0. \tag{3.9}$$

Proof. From (3.6), we know that the zero solution of Problem (1.1) is asymptotically stable if and only if $\max\left|\lambda_{(I-N(1))^{-1}M(1)}\right|<1$. According to Lemma 1, the eigenvalues of $(I-N(1))^{-1}M(1)$ are the roots of the following characteristic equations

$$\lambda^2+\frac{(b+c)(1-\cos(\eta))-2a^2\cos(\eta)}{a^2+c(1-\cos(\eta))}\lambda+\frac{a^2+b(1-\cos(\eta))}{a^2+c(1-\cos(\eta))}=0, \eta=ai,\ i\geq 1.$$

From Lemma 2, we need to verify $1-e_2<2$ and $|e_1|<1-e_2$ under some conditions. Here

$$e_1=-\frac{(b+c)(1-\cos(\eta))-2a^2\cos(\eta)}{a^2+c(1-\cos(\eta))},\quad 1-e_2=\frac{2a^2+(b+c)(1-\cos(\eta))}{a^2+c(1-\cos(\eta))}.$$

If

$$\left(2a^2+(b+c)(1-\cos(\eta))\right)/\left(a^2+c(1-\cos(\eta))\right)<2,$$

we can derive

$$(b-c)(1-\cos(\eta))/\left(a^2+c(1-\cos(\eta))\right)<0.$$

(i) When $a^2+c(1-\cos(\eta))>0$, we obtain $c>0, b-c<0$. From $e_1<1-e_2$ we get $a^2+b+c>0$, while $1+\cos(\eta)>0$ holds obviously from $e_1>-(1-e_2)$.

(ii) When $a^2+c(1-\cos(\eta))<0$, we get the contradiction $1+\cos(\eta)<0$ in the process of deducing $e_1>-(1-e_2)$. Hence, this case does not exist.

The proof is finished. □

4 Semidiscrete scheme for Galerkin FE method

Denote $H^s(\Omega)$ be the Sobolev space on Ω and $\|\cdot\|_s$ is the related norm. We define $H_0^1 = \{\phi \in H^1(\Omega) : \phi = 0 \text{ on } \partial\Omega\}$, which is the completion of $C_0^\infty(\Omega)$ under $L^2(\Omega)$-norm $\|\cdot\|$.

Let r be integer with $r \geq 2$, consider a family of partitions $x_0 < x_1 < \cdots < x_{N_h} (h \in \mathbb{N}^+)$ of Ω into subintervals $I_n = [x_{n-1}, x_n]$, set $h = \max_{1 \leq n \leq N_h} (x_n - x_{n-1})$, and S_h be the piecewise polynomial spline space

$$S_h = \{u_h \in C(\Omega) : u_h|_{I_n} \in P^{(r-1)}(I_n), 1 \leq n \leq N_h\},$$

where $P^{(r-1)}(I_n)$ denotes the space of all (real) polynomials on I_n of degree no more than $r-1$.

In the first step of defining the spatial semi-discrete approximate solution of Problem (1.1), we write it in weak form: Find u: $\bar{J} \to H_0^1$, and apply Green's formula to the second, third and fourth term, we have

$$\begin{aligned} &(u_{tt}(x,t), \phi) + a^2(\nabla u(x,t), \nabla\phi) + b(\nabla u(x,[t]), \nabla\phi) \\ &\qquad + c(\nabla u(x,[t+1]), \nabla\phi) = 0, \quad \forall \phi \in H_0^1, \ \ t > 0, \\ &u(0) = v, \quad u_t(0) = w, \end{aligned} \tag{4.1}$$

where

$$(f,g) = \int_\Omega fg\,dx, (\nabla f, \nabla g) = \int_\Omega \frac{\partial f}{\partial x}\frac{\partial g}{\partial x}\,dx.$$

Then, we give the approximate problem to find $u^h(t) = u^h(\cdot,t) : \bar{J} \to S_h$, belonging to S_h for each t

$$\begin{aligned} &(u^h_{tt}(x,t), \chi) + a^2(\nabla u^h(x,t), \nabla\chi) + b(\nabla u^h(x,[t]), \nabla\chi) \\ &\qquad + c(\nabla u^h(x,[t+1]), \nabla\chi) = 0, \quad \forall \chi \in S_h, \ \ t > 0, \\ &u^h(0) = v^h, \quad u^h_t = w^h, \end{aligned} \tag{4.2}$$

where v^h and w^h are some approximations of v and w in S_h, respectively.

4.1 Convergence analysis

We introduce the Ritz projection $R_h : H_0^1(\Omega) \to S_h$ as the orthogonal projection with respect to the inner product $(\nabla\varphi, \nabla\chi)$, so that

$$(\nabla R_h\varphi, \nabla\chi) = (\nabla\varphi, \nabla\chi), \forall \chi \in S_h, \ for\, \varphi \in H_0^1. \tag{4.3}$$

Lemma 3. *[21] For $\varphi \in H^s \cap H_0^1$, if*

$$\inf_{\chi \in S_h} \{\|\varphi - \chi\| + h\|\nabla(\varphi - \chi)\|\} \leq Ch^s\|\varphi\|_s, \ for\, 1 \leq s \leq r$$

holds, then

$$\|R_h\varphi - \varphi\| + h\|\nabla(R_h\varphi - \varphi)\| \leq Ch^s\|\varphi\|_s, \ for\, \varphi \in H^s \cap H_0^1, 1 \leq s \leq r,$$

where R_h is defined in (4.3)*, C denotes a positive constant.*

Theorem 2. *Let u and u^h be the solutions of (4.1) and (4.2), respectively, for $t\in[n,n+1)(n\in\mathbb{Z})$, if $\|v^h-v\|\le Ch^r\|v\|_r$ and $\|w^h-w\|\le Ch^r\|w\|_r$, then*

$$\|u^h(t)-u(t)\|\le C(t)h^r\Big\{\|v\|_r+\|w\|_r+\int_0^t\|u_s(s)\|ds+\int_0^t\Big(\int_0^t\|u_{ss}(s)\|_r^2ds\Big)^{\frac{1}{2}}ds\Big\},$$

where $C(t)$ is a function with respect to t.

Proof. Let

$$u^h(t)-u(t)=(u^h(t)-R_hu(t))+(R_hu(t)-u(t))\triangleq\theta(t)+\rho(t). \tag{4.4}$$

For $t\in[n,n+1)$, according to Lemma 3 we get

$$\begin{aligned}\|\rho(t)\|=\|R_hu(t)-u(t)\|\le Ch^r\|u(t)\|_r&=Ch^r\Big\|u(0)+\int_0^t u_sds\Big\|_r\\&\le Ch^r\Big(\|v\|_r+\int_0^t\|u_s\|_rds\Big).\end{aligned} \tag{4.5}$$

In addition, from $(R_h(u))_{tt}=R_hu_{tt}$ (see [21]) and (4.3) we have

$$\begin{aligned}&(\theta_{tt}(t),\chi)+a^2(\nabla\theta(t),\nabla\chi)+b(\nabla\theta([t]),\nabla\chi)+c(\nabla\theta([t+1]),\nabla\chi)\\&\quad=(u^h_{tt}(t),\chi)-((R_hu)_{tt}(t),\chi)+a^2(\nabla u^h(t),\nabla\chi)-a^2(\nabla R_hu(t),\nabla\chi)\\&\qquad+b(\nabla u^h([t]),\nabla\chi)-b(\nabla R_hu^h([t]),\nabla\chi)+c(\nabla u^h([t+1]),\nabla\chi)\\&\qquad-c(\nabla R_hu^h([t+1]),\nabla\chi)=-((R_hu)_{tt}(t),\chi)-a^2(\nabla R_hu(t),\nabla\chi)\\&\qquad-b(\nabla R_hu([t]),\nabla\chi)-c(\nabla R_hu([t+1]),\nabla\chi)\\&\quad=-(R_hu_{tt}(t),\chi)-a^2(\nabla u(t),\nabla\chi)-b(\nabla u([t],\nabla\chi)-c(\nabla u([t+1]),\nabla\chi)\\&\quad=(u_{tt}(t)-R_hu_{tt}(t),\chi)=-(\rho_{tt}(t),\chi).\end{aligned} \tag{4.6}$$

Take $\chi=a^2\theta_t(t)+b\theta_t([t])+c\theta_t([t+1])$, from (4.6) we derive

$$\begin{aligned}&(\theta_{tt}(t),a^2\theta_t(t)+b\theta_t([t])+c\theta_t([t+1]))+\frac{1}{2}\frac{d}{dt}\big\|a^2\nabla\theta(t)+b\nabla\theta([t])+c\theta([t+1])\big\|^2\\&\quad=-(\rho_{tt}(t),a^2\theta_t(t)+b\theta_t([t])+c\theta_t([t+1])),\end{aligned}$$

then,

$$\begin{aligned}&\frac{a^2}{2}\frac{d}{dt}\|\theta_t(t)\|^2+b\frac{d}{dt}(\theta_t(t),\theta_t(n))+c\frac{d}{dt}(\theta_t(t),\theta_t(n+1))\\&\quad\le\|\rho_{tt}(t)\|\|a^2\theta_t(t)+b\theta_t(n)+c\theta_t(n+1)\|.\end{aligned} \tag{4.7}$$

Integrating (4.7) from n to t, we get

$$\begin{aligned}&\frac{a^2}{2}\|\theta_t(t)\|^2-\frac{a^2}{2}\|\theta_t(n)\|^2+b(\theta_t(t),\theta_t(n))-b\|\theta_t(n)\|^2\\&\quad+c(\theta_t(t),\theta_t(n+1))-c(\theta_t(n),\theta_t(n+1))\\&\le\int_n^t\|\rho_{ss}(s)\|\|a^2\theta_s(s)+b\theta_s(n)+c\theta_s(n+1)\|ds.\end{aligned}$$

By using Schwarz inequality and Cauchy inequality we have

$$b(\theta_t(t),\theta_t(n)) \le |b|\|\theta_t(t)\|\|\theta_t(n)\| \le \frac{|b|\varepsilon_1}{2}\|\theta_t(t)\|^2 + \frac{|b|}{2\varepsilon_1}\|\theta_t(n)\|^2,$$

$$c(\theta_t(t),\theta_t(n+1)) \le |c|\|\theta_t(t)\|\|\theta_t(n+1)\| \le \frac{|c|\varepsilon_2}{2}\|\theta_t(t)\|^2 + \frac{|c|}{2\varepsilon_2}\|\theta_t(n+1)\|^2,$$

$$c(\theta_t(n),\theta_t(n+1)) \le |c|\|\theta_t(n)\|\|\theta_t(n+1)\| \le \frac{|c|\varepsilon_3}{2}\|\theta_t(n)\|^2 + \frac{|c|}{2\varepsilon_3}\|\theta_t(n+1)\|^2,$$

$$\|\rho_{ss}(s)\|\|a^2\theta_s(s)\| \le \frac{a^2\varepsilon_4}{2}\|\rho_{ss}(s)\|^2 + \frac{a^2}{2\varepsilon_4}\|\theta_s(s)\|^2,$$

$$\|\rho_{ss}(s)\|\|b\theta_s(n)\| \le \frac{|b|\varepsilon_5}{2}\|\rho_{ss}(s)\|^2 + \frac{|b|}{2\varepsilon_5}\|\theta_s(n)\|^2,$$

$$\|\rho_{ss}(s)\|\|c\theta_s(n+1)\| \le \frac{|c|\varepsilon_6}{2}\|\rho_{ss}(s)\|^2 + \frac{|c|}{2\varepsilon_6}\|\theta_s(n+1)\|^2, \varepsilon_i > 0, i = 1,2,\ldots,6,$$

so,

$$\begin{aligned}
&\frac{a^2}{2}\|\theta_t(t)\|^2 - \frac{a^2}{2}\|\theta_t(n)\|^2 - \frac{|b|\varepsilon_1}{2}\|\theta_t(t)\|^2 - \frac{|b|}{2\varepsilon_1}\|\theta_t(n)\|^2 - b\|\theta_t(n)\|^2 \\
&\quad - \frac{|c|\varepsilon_2}{2}\|\theta_t(t)\|^2 - \frac{|c|}{2\varepsilon_2}\|\theta_t(n+1)\|^2 - \frac{|c|\varepsilon_3}{2}\|\theta_t(n)\|^2 - \frac{|c|}{2\varepsilon_3}\|\theta_t(n+1)\|^2 \\
&\le \frac{a^2\varepsilon_4}{2}\int_n^t \|\rho_{ss}(s)\|^2 ds + \frac{a^2}{2\varepsilon_4}\int_n^t \|\theta_s(s)\|^2 ds + \frac{|b|\varepsilon_5}{2}\int_n^t \|\rho_{ss}(s)\|^2 ds \\
&\quad + \frac{|b|}{2\varepsilon_5}\int_n^t \|\theta_s(n)\|^2 ds + \frac{|c|\varepsilon_6}{2}\int_n^t \|\rho_{ss}(s)\|^2 ds + \frac{|c|}{2\varepsilon_6}\int_n^t \|\theta_s(n+1)\|^2 ds,
\end{aligned}$$

further,

$$\begin{aligned}
&\left(\frac{a^2 - |b|\varepsilon_1 - |c|\varepsilon_2}{2}\right)\|\theta_t(t)\|^2 \le \left(\frac{|c|}{2\varepsilon_2} + \frac{|c|}{2\varepsilon_3} + \frac{|c|}{2\varepsilon_6}(t-n)\right)\|\theta_t(n+1)\|^2 \\
&\quad + \left(\frac{a^2}{2} + \frac{|b|}{2\varepsilon_1} + |b| + \frac{|c|\varepsilon_3}{2} + \frac{|b|}{2\varepsilon_5}(t-n)\right)\|\theta_t(n)\|^2 \\
&\quad + \frac{a^2\varepsilon_4 + |b|\varepsilon_5 + |c|\varepsilon_6}{2}\int_n^t \|\rho_{ss}(s)\|^2 ds + \frac{a^2}{2\varepsilon_4}\int_n^t \|\theta_s(s)\|^2 ds.
\end{aligned}$$

So, we can take $\varepsilon_1, \varepsilon_2$ suitably to make $a^2 - |b|\varepsilon_1 - |c|\varepsilon_2 > 0$ hold, and let $\tilde{a} = (a^2 - |b|\varepsilon_1 - |c|\varepsilon_2)/2$, then

$$\begin{aligned}
\|\theta_t(t)\|^2 \le &\left(\frac{|c|}{2\tilde{a}\varepsilon_2} + \frac{|c|}{2\tilde{a}\varepsilon_3} + \frac{|c|}{2\tilde{a}\varepsilon_6}(t-n)\right)\|\theta_t(n+1)\|^2 \\
&+ \left(\frac{a^2}{2\tilde{a}} + \frac{|b|}{2\tilde{a}\varepsilon_1} + \frac{|b|}{\tilde{a}} + \frac{|c|\varepsilon_3}{2\tilde{a}} + \frac{|b|}{2\tilde{a}\varepsilon_5}(t-n)\right)\|\theta_t(n)\|^2 \\
&+ \frac{a^2\varepsilon_4 + |b|\varepsilon_5 + |c|\varepsilon_6}{2\tilde{a}}\int_n^t \|\rho_{ss}(s)\|^2 ds + \frac{a^2}{2\tilde{a}\varepsilon_4}\int_n^t \|\theta_s(s)\|^2 ds.
\end{aligned}$$

Gronwall inequality implies that

$$\begin{aligned}\|\theta_t(t)\|^2 \le& \left(\frac{|c|}{2\tilde{a}\varepsilon_2}+\frac{|c|}{2\tilde{a}\varepsilon_3}+\frac{|c|}{2\tilde{a}\varepsilon_6}(t-n)\right)e^{\frac{a^2}{2\tilde{a}\varepsilon_4}(t-n)}\|\theta_t(n+1)\|^2\\ &+\left(\frac{a^2}{2\tilde{a}}+\frac{|b|}{2\tilde{a}\varepsilon_1}+\frac{|b|}{\tilde{a}}+\frac{|c|\varepsilon_3}{2\tilde{a}}+\frac{|b|}{2\tilde{a}\varepsilon_5}(t-n)\right)e^{\frac{a^2}{2\tilde{a}\varepsilon_4}(t-n)}\|\theta_t(n)\|^2\\ &+\frac{a^2\varepsilon_4+|b|\varepsilon_5+|c|\varepsilon_6}{2\tilde{a}}e^{\frac{a^2}{2\tilde{a}\varepsilon_4}(t-n)}\int_n^t\|\rho_{ss}(s)\|^2ds.\end{aligned}$$

Let $t=n+1$ and denote

$$\begin{aligned}&\tilde{b}=\left(\frac{|c|}{2\tilde{a}\varepsilon_2}+\frac{|c|}{2\tilde{a}\varepsilon_3}+\frac{|c|}{2\tilde{a}\varepsilon_6}\right)e^{\frac{a^2}{2\tilde{a}\varepsilon_4}},\quad \tilde{d}=\left(\frac{a^2\varepsilon_4+|b|\varepsilon_5+|c|\varepsilon_6}{2\tilde{a}}\right)e^{\frac{a^2}{2\tilde{a}\varepsilon_4}},\\ &\tilde{c}=\left(\frac{a^2}{2\tilde{a}}+\frac{|b|}{2\tilde{a}\varepsilon_1}+\frac{|b|}{\tilde{a}}+\frac{|c|\varepsilon_3}{2\tilde{a}}+\frac{|b|}{2\tilde{a}\varepsilon_5}\right)e^{\frac{a^2}{2\tilde{a}\varepsilon_4}},\end{aligned}$$

then we obtain

$$(1-\tilde{b})\|\theta_t(n+1)\|^2\le \tilde{c}\|\theta_t(n)\|^2+\tilde{d}\int_n^{n+1}\|\rho_{ss}(s)\|^2ds.$$

Similarly, we take $\tilde{a},\varepsilon_2,\varepsilon_3,\varepsilon_6$ suitably to make $1-\tilde{b}>0$ hold. Therefore,

$$\begin{aligned}\|\theta_t(n+1)\|^2 \le& \frac{\tilde{c}}{1-\tilde{b}}\|\theta_t(n)\|^2+\frac{\tilde{d}}{1-\tilde{b}}\int_n^{n+1}\|\rho_{ss}(s)\|^2ds\le\frac{\tilde{c}}{1-\tilde{b}}\\ &\times\left(\frac{\tilde{c}}{1-\tilde{b}}\|\theta_t(n-1)\|^2+\frac{\tilde{d}}{1-\tilde{b}}\int_{n-1}^{n}\|\rho_{ss}(s)\|^2ds\right)+\frac{\tilde{d}}{1-\tilde{b}}\int_n^{n+1}\|\rho_{ss}(s)\|^2ds\\ =&\left(\frac{\tilde{c}}{1-\tilde{b}}\right)^2\|\theta_t(n-1)\|^2+\frac{\tilde{c}\tilde{d}}{(1-\tilde{b})^2}\int_{n-1}^{n}\|\rho_{ss}(s)\|^2ds+\frac{\tilde{d}}{1-\tilde{b}}\int_n^{n+1}\|\rho_{ss}(s)\|^2ds\\ \le&\left(\frac{\tilde{c}}{1-\tilde{b}}\right)^{n+1}\|\theta_t(0)\|^2+\sum_{j=1}^{n+1}\frac{\tilde{c}^{n-j+1}\tilde{d}}{(1-\tilde{b})^{n-j+2}}\int_{j-1}^{j}\|\rho_{ss}(s)\|^2ds,\end{aligned}$$

here

$$\begin{aligned}&\|\theta_t(0)\|=\|w^h-R_hw\|\le\|w^h-w\|+\|R_hw-w\|\le\|w^h-w\|+Ch^r\|w\|_r,\\ &\|\rho_{tt}(t)\|=\|R_hu_{tt}-u_{tt}\|\le Ch^r\|u_{tt}\|_r,\end{aligned}$$

so,

$$\|\theta_t(n+1)\|^2\le\left(\frac{\tilde{c}}{1-\tilde{b}}\right)^{n+1}\|w^h-w\|^2+Ch^{2r}\sum_{j=1}^{n+1}\frac{\tilde{c}^{n-j+1}\tilde{d}}{(1-\tilde{b})^{n-j+2}}\int_{j-1}^{j}\|u_{ss}(s)\|^2ds,$$

$$\|\theta_t(t)\|^2\le C(t)\|w^h-w\|^2+C(t)h^{2r}\|w\|_r^2+C(t)h^{2r}\int_0^t\|u_{ss}(s)\|_r^2ds,$$

where $C(t)$ is a function with $t \in [n, n+1)$. Therefore, if $\|w^h - w\| \le Ch^r\|w\|_r$, then,

$$\|\theta_t(t)\| \le C(t)\|w^h - w\| + C(t)h^r\|w\|_r + C(t)h^r \left(\int_0^t \|u_{ss}(s)\|_r^2 ds\right)^{\frac{1}{2}}$$
$$\le C(t)h^r\Big(\|w\|_r + \left(\int_0^t \|u_{ss}(s)\|_r^2 ds\right)^{\frac{1}{2}}\Big).$$

Due to

$$\|\theta(t)\| = \|\theta(0) + \int_0^t \theta_s(s)ds\| \le \|\theta(0)\| + \int_0^t \|\theta_s(s)\|ds,$$
$$\|\theta(0)\| = \|v^h - R_h v\| \le \|v^h - v\| + \|R_h v - v\| \le \|v^h - v\| + Ch^r\|v\|_r, \quad (4.8)$$

we can get

$$\|\theta(t)\| \le C(t)h^r(\|v\|_r + \|w\|_r) + C(t)h^r \int_0^t \left(\int_0^t \|u_{ss}(s)\|_r^2 ds\right)^{\frac{1}{2}} ds.$$

Hence,

$$\|u^h - u\| \le \|\theta(t)\| + \|\rho(t)\|$$
$$\le C(t)h^r\Big(\|v\|_r + \|w\|_r + \int_0^t \|u_s(s)\|ds + \int_0^t \left(\int_0^t \|u_{ss}(s)\|_r^2 ds\right)^{\frac{1}{2}} ds\Big).$$

When $h \to 0$, we have $u^h \to u$. This completes the proof. □

4.2 Stability analysis

Considering the basis $\{\Phi_j\}_1^{N_h}$ of S_h, for any $u^h \in S_h$, we have

$$u^h(x,t) = \sum_{j=1}^{N_h} \beta_j(t)\Phi_j(x),$$

where $\beta_j(t)$ is undetermined coefficient, such that

$$\sum_{j=1}^{N_h} \beta_j^{''}(t)(\Phi_j, \Phi_k) + \sum_{j=1}^{N_h} \left(a^2\beta_j(t) + b\beta_j([t]) + c\beta_j([t+1])\right)(\nabla\Phi_j, \nabla\Phi_k) = 0,$$
$$\beta_k(0) = \alpha_k, \quad \beta_k^{'}(0) = \gamma_k, \quad k = 1, 2, \dots, N_h, \quad (4.9)$$

where α_k and γ_k are the components of the given initial approximation v^h and w^h, respectively. (4.9) can be expressed as the following matrix form

$$A_1\beta^{''}(t) + a^2A_2\beta(t) + bA_2\beta([t]) + cA_2\beta([t+1]) = 0, t > 0,$$
$$\beta(0) = \alpha, \quad \beta^{'}(0) = \gamma, \quad (4.10)$$

where $A_1 = (a_{jk})$ is the mass matrix with elements $a_{jk} = (\Phi_j, \Phi_k)$, $A_2 = (b_{jk})$ is the stiffness matrix with $b_{jk} = (\nabla\Phi_j, \nabla\Phi_k), \beta(t)$ is the vector of unknown β_j, and $\alpha = (\alpha_k), \gamma = (\gamma_k)$. The dimension of all these items equals $N_h = \dim(S_h)$.

Since A_1 and A_2 are Gram matrices, positive definite and invertible ([21]), so the equation in (4.10) can be written as

$$\beta^{''}(t) + a^2 A_1^{-1} A_2 \beta(t) + b A_1^{-1} A_2 \beta([t]) + c A_1^{-1} A_2 \beta([t+1]) = 0.$$

DEFINITION 3. If any solution $u^h(x,t)$ of (4.2) satisfies

$$\lim_{t\to\infty} u^h(x,t) = 0, x \in \Omega,$$

then the zero solution of (4.2) is asymptotically stable.

Here, we introduce $\mu(t) = \beta^{'}(t)$, together with (4.2), we have

$$Z^{'}(t) = B_1 Z(t) + B_2 Z([t]) + B_3 Z([t+1]),$$

where

$$Z(t) = \begin{pmatrix} \beta(t) \\ \mu(t) \end{pmatrix}, \qquad B_1 = \begin{pmatrix} O & I \\ -a^2 A_1^{-1} A_2 & O \end{pmatrix},$$
$$B_2 = \begin{pmatrix} O & O \\ -b A_1^{-1} A_2 & O \end{pmatrix}, \quad B_3 = \begin{pmatrix} O & O \\ -c A_1^{-1} A_2 & O \end{pmatrix}.$$

From (3.4) we obtain

$$Z(t) = M_1(t-n) Z_n + N_1(t-n) Z_{n+1},\ t \in [n, n+1), \tag{4.11}$$

where

$$M_1(t-n) = e^{B_1(t-n)} + \left(e^{B_1(t-n)} - I\right) B_1^{-1} B_2,$$
$$N_1(t-n) = \left(e^{B_1(t-n)} - I\right) B_1^{-1} B_3, \quad Z_n = Z(n), \ \ Z_{n+1} = Z(n+1),$$

and let $t = n+1$, (4.11) can be written as

$$Z_{n+1} = (I - N_1(1))^{-1} M_1(1) Z_n. \tag{4.12}$$

Theorem 3. *Under the condition* (3.9)*, the zero solution of* (4.2) *is asymptotically stable.*

Proof. From (4.12), we know that the zero solution of (4.2) is asymptotically stable if and only if $\max |\lambda_{(I-N_1(1))^{-1}M_1(1)}| < 1$. Since A_2 is a positive definite matrix, there exists an invertible matrix G, such that $A_2 = G^T G$. Moreover,

$$G A_1^{-1} A_2 G^{-1} = G A_1^{-1} G^T G G^{-1} = G A_1^{-1} G^T, \tag{4.13}$$

which implies that $A_1^{-1}A_2$ is positive definite. Therefore, we can find a positive definite matrix K such that $A_1^{-1}A_2 = K^2$. We compute that

$$\begin{aligned}&(I - N_1(1))^{-1} M_1(1)\\ &= \begin{pmatrix} -\frac{c\sin^2(aK)-(a^2+b)\cos(aK)+bI}{a^2+c(1-\cos(aK))} & (aK)^{-1}\sin(aK) \\ -\frac{(a^2+b)(aK)\sin(aK)+c(aK)\sin(aK)\cos(aK)}{a^2+c(1-\cos(aK))} & \cos(aK) \end{pmatrix}.\end{aligned}$$

From Lemma 1, we can conclude that the eigenvalues of $(I - N_1(1))^{-1} M_1(1)$ are the roots of the following characteristic equation

$$\lambda^2 + \frac{(b+c)(1-\cos(\tilde{\lambda})) - 2a^2\cos(\tilde{\lambda})}{a^2 + c(1-\cos(\tilde{\lambda}))}\lambda + \frac{a^2 + b(1-\cos(\tilde{\lambda}))}{a^2 + c(1-\cos(\tilde{\lambda}))} = 0, \quad (4.14)$$

where $\tilde{\lambda}$ is the eigenvalue of aK. Under the condition (3.9), it is obvious that coefficients of (4.14) satisfy the requirements of Lemma 2. Therefore, we can get $\max|\lambda_{(I-N_1(1))^{-1}M_1(1)}| < 1$. The proof is complete. □

5 Fully discrete scheme for Galerkin FE method

Let $p = 1/m, m \geq 1$ be the time step size, $\{t_n\}$ be the uniform partition of $[0, +\infty)$ with $t_n = np, n = 0, 1, 2, \ldots$, U^n be the approximation in S_h of $u(t)$ at t_n and denote $\partial_{tt}U^n = (U^{n+1} - 2U^n + U^{n-1})/p^2$, then Galerkin finite element method to (1.1) reads

$$(\partial_{tt}U^n, \chi) + a^2(\nabla U^n, \nabla\chi) + b(\nabla U^{n,p}, \nabla\chi) + c(\nabla U^{n+1,p}, \nabla\chi) = 0, \forall\chi \in S_h, \quad (5.1)$$

where $U^{n,p}$ and $U^{n+1,p}$ denote a given approximation to $u(x, [t_n])$ and $u(x, [t_{n+1}])$, respectively $(n = 1, 2, 3, \ldots)$.

Let $n = km + l, k = 0, 1, 2, \ldots, l = 1, 2, \ldots, m$, then $U^{n,p}$ and $U^{n+1,p}$ can be written as U^{km} and $U^{(k+1)m}$ according to Definition 1. Therefore, (5.1) turns into

$$(\partial_{tt}U^{km+l}, \chi) + a^2(\nabla U^{km+l}, \nabla\chi) + b(\nabla U^{km}, \nabla\chi) + c(\nabla U^{(k+1)m}, \nabla\chi) = 0, \forall\chi \in S_h,$$

that is

$$\begin{aligned}(U^{km+l+1}, \chi) =& 2(U^{km+l}, \chi) - a^2p^2(\nabla U^{km+l}, \nabla\chi) - (U^{km+l-1}, \chi)\\ &- bp^2(\nabla U^{km}, \nabla\chi) - cp^2(\nabla U^{(k+1)m}, \nabla\chi).\end{aligned} \quad (5.2)$$

Similar to the semidiscrete case, (5.2) can be written as

$$A_1\beta^{km+l+1} = (2A_1 - a^2p^2A_2)\beta^{km+l} - A_1\beta^{km+l-1} - bp^2A_2\beta^{km} - cp^2A_2\beta^{(k+1)m}, \quad (5.3)$$

where A_1, A_2 are defined in (4.10).

5.1 Convergence analysis

Theorem 4. *Let* U^n *and* u *be the solutions of* (5.1) *and* (4.1), *respectively. If* $\|v^h - v\| \le Ch^r\|v\|_r$, *then*

$$\|U^n - u(t_n)\| \le Ch^r\Big(\|v\|_r + \frac{p}{6}\sum_{i=0}^{n}\int_{t_i}^{t_{i+2}}\|u_{ttt}\|_r dt + \sum_{i=0}^{n}\|u(t_i)\|_r + \int_0^{t_n}\|u_s\|ds\Big)$$
$$+ Cp\Big(\sum_{i=0}^{n}\frac{1}{6}\int_{t_i}^{t_{i+2}}\|u_{tttt}\|dt\Big),$$

where C *is independent of* h *and* p.

Proof. In analogy with (4.4), take $n = km + l$ then we have

$$\begin{aligned} U^{km+l} - u(t_{km+l}) &= (U^{km+l} - R_h u(t_{km+l})) + (R_h u(t_{km+l}) - u(t_{km+l})) \\ &\triangleq \theta^{km+l} + \rho^{km+l}, \end{aligned}$$

and $\rho^{km+l} = \rho(t_{km+l})$ is bounded as claimed in (4.5). Then, a calculation similar to (4.6) yields

$$(\partial_{tt}\theta^{km+l}, \chi) + (a^2\nabla\theta^{km+l} + b\nabla\theta^{km} + c\nabla\theta^{(k+1)m}, \nabla\chi) = -(q^{km+l}, \chi),\ \chi \in S_h, \tag{5.4}$$

where

$$\begin{aligned} q^{km+l} &= R_h\partial_{tt}u(t_{km+l}) - u_{tt}(t_{km+l}) \\ &= (R_h - I)\partial_{tt}u(t_{km+l}) + (\partial_{tt}u(t_{km+l}) - u_{tt}(t_{km+l})) \triangleq q_1^{km+l} + q_2^{km+l}. \end{aligned}$$

Let $\chi = a^2(\theta^{km+l+1} - \theta^{km+l}) + b\theta^{km} + c\theta^{(k+1)m}$, by the Schwarz inequality and Cauchy inequality:

$$\begin{aligned} (a^2(\theta^{km+l+1} - \theta^{km+l}), \theta^{km+l} - \theta^{km+l-1}) &\le \frac{a^2 r_1}{2}\|\theta^{km+l+1} - \theta^{km+l}\|^2 \\ &\quad + \frac{a^2}{2r_1}\|\theta^{km+l} - \theta^{km+l-1}\|^2, \\ (\theta^{km+l+1} - \theta^{km+l}, b\theta^{km}) &\le \frac{|b|r_2}{2}\|\theta^{km+l+1} - \theta^{km+l}\|^2 + \frac{|b|}{2r_2}\|\theta^{km}\|^2, \\ (\theta^{km+l} - \theta^{km+l-1}, b\theta^{km}) &\le \frac{|b|r_3}{2}\|\theta^{km+l} - \theta^{km+l-1}\|^2 + \frac{|b|}{2r_3}\|\theta^{km}\|^2, \\ (\theta^{km+l+1} - \theta^{km+l}, c\theta^{(k+1)m}) &\le \frac{|c|r_4}{2}\|\theta^{km+l+1} - \theta^{km+l}\|^2 + \frac{|c|}{2r_4}\|\theta^{(k+1)m}\|^2, \\ (a^2\nabla(\theta^{km+l+1} - \theta^{km+l}), a^2\nabla\theta^{km+l}) &\le \frac{a^4 r_5}{2}\|\nabla(\theta^{km+l+1} - \theta^{km+l})\|^2 \\ &\quad + \frac{a^4}{2r_5}\|\nabla\theta^{km+l}\|^2, \end{aligned}$$

$$(a^2\nabla\theta^{km+l}, b\nabla\theta^{km}) \le \frac{a^2|b|r_6}{2}\|\nabla\theta^{km+l}\|^2 + \frac{a^2|b|}{2r_6}\|\nabla\theta^{km}\|^2,$$
$$(a^2\nabla\theta^{km+l}, c\nabla\theta^{(k+1)m}) \le \frac{a^2|c|r_7}{2}\|\nabla\theta^{km+l}\|^2 + \frac{a^2|c|}{2r_7}\|\nabla\theta^{(k+1)m}\|^2,$$
$$(a^2\nabla(\theta^{km+l+1} - \theta^{km+l}), b\nabla\theta^{km}) \le \frac{a^2|b|r_8}{2}\|\nabla(\theta^{km+l+1} - \theta^{km+l})\|^2$$
$$+ \frac{a^2|b|}{2r_8}\|\nabla\theta^{km}\|^2, \quad (b\nabla\theta^{km}, c\nabla\theta^{(k+1)m}) \le \frac{|b||c|r_9}{2}\|\nabla\theta^{km}\|^2$$
$$+ \frac{|b||c|}{2r_9}\|\nabla\theta^{(k+1)m}\|^2, \quad (a^2\nabla(\theta^{km+l+1} - \theta^{km+l}), c\nabla\theta^{(k+1)m}) \le \frac{a^2|c|r_{10}}{2}$$
$$\times \|\nabla(\theta^{km+l+1} - \theta^{km+l})\|^2 + \frac{a^2|c|}{2r_{10}}\|\nabla\theta^{(k+1)m}\|^2,$$
$$a^2\|q^{km+l}\|\|\theta^{km+l+1} - \theta^{km+l}\| \le \frac{a^2r_{11}}{2}\|\theta^{km+l+1} - \theta^{km+l}\|^2 + \frac{a^2}{2r_{11}}\|q^{km+l}\|^2,$$
$$|b|\|q^{km+l}\|\|\theta^{km}| \le \frac{|b|r_{12}}{2}\|\theta^{km}\|^2\frac{|b|}{2r_{12}}\|q^{km+l}\|^2,$$
$$|c|\|q^{km+l}\|\|\theta^{(k+1)m}| \le \frac{|c|r_{13}}{2}\|\theta^{(k+1)m}\|^2\frac{|c|}{2r_{13}}\|q^{km+l}\|^2, r_i>0, i=1,2,\ldots,13,$$

we have

$$\begin{aligned}
a^2\|\theta^{km+l+1} - \theta^{km+l}\|^2 &\le \frac{a^2r_1}{2}\|\theta^{km+l+1} - \theta^{km+l}\|^2 + \frac{a^2}{2r_1}\|\theta^{km+l} - \theta^{km+l-1}\|^2 \\
&+ \frac{|b|r_2}{2}\|\theta^{km+l+1} - \theta^{km+l}\|^2 + \frac{|b|}{2r_2}\|\theta^{km}\|^2 + \frac{|b|}{2r_3}\|\theta^{km}\|^2 + \frac{|c|}{2r_4}\|\theta^{(k+1)m}\|^2 \\
&+ \frac{|b|r_3}{2}\|\theta^{km+l} - \theta^{km+l-1}\|^2 + \frac{|c|r_4}{2}\|\theta^{km+l} - \theta^{km+l-1}\|^2 + \frac{a^4p^2}{2r_5}\|\nabla\theta^{km+l}\|^2 \\
&+ \frac{a^4p^2r_5}{2}\|\nabla(\theta^{km+l+1} - \theta^{km+l})\|^2 + \frac{a^2|b|p^2r_6}{2}\|\nabla\theta^{km+l}\|^2 + \frac{a^2|b|p^2}{2r_6}\|\nabla\theta^{km}\|^2 \\
&+ \frac{a^2|c|p^2r_7}{2}\|\nabla\theta^{km+l}\|^2 + \frac{a^2|c|p^2}{2r_7}\|\nabla\theta^{(k+1)m}\|^2 + \frac{a^2|b|p^2r_8}{2} \\
&\times \|\nabla(\theta^{km+l+1} - \theta^{km+l})\|^2 + \frac{a^2|b|p^2}{2r_8}\|\nabla\theta^{km}\|^2 - b^2p^2\|\nabla\theta^{km}\|^2 \\
&+ |b||c|p^2r_9\|\nabla\theta^{km}\|^2 + \frac{|b||c|p^2}{r_9}\|\nabla\theta^{(k+1)m}\|^2 + \frac{a^2|c|p^2r_{10}}{2}\|\nabla(\theta^{km+l+1} - \theta^{km+l})\|^2 \\
&+ \frac{a^2|c|p^2}{2r_{10}}\|\nabla\theta^{(k+1)m}\|^2 - c^2p^2\|\nabla\theta^{(k+1)m}\|^2 \\
&+ \frac{a^2p^2r_{11}}{2}\|\theta^{km+l+1} - \theta^{km+l}\|^2 + \frac{a^2p^2}{2r_{11}}\|q^{km+l}\|^2 + \frac{|b|p^2r_{12}}{2}\|\theta^{km}\|^2 \\
&+ \frac{|b|p^2}{2r_{12}}\|q^{km+l}\|^2 + \frac{|c|p^2r_{13}}{2}\|\theta^{(k+1)m}\|^2 + \frac{|c|p^2}{2r_{13}}\|q^{km+l}\|^2,
\end{aligned}$$

that is

$$\left(a^2 - \frac{a^2r_1}{2} - \frac{|b|r_2}{2} - \frac{|c|r_4}{2} - \frac{a^2p^2r_{11}}{2}\right)\|\theta^{km+l+1} - \theta^{km+l}\|^2$$

$$
\begin{aligned}
&\leq \left(\frac{a^2}{2r_1}+\frac{|b|r_3}{2}\right)\|\theta^{km+l}-\theta^{km+l-1}\|^2+\left(\frac{|b|}{2r_2}+\frac{|b|}{2r_3}+\frac{|b|p^2r_{12}}{2}\right)\|\theta^{km}\|^2\\
&\quad+\left(\frac{|c|}{2r_4}+\frac{|c|p^2r_{13}}{2}\right)\|\theta^{(k+1)m}\|^2+\left(\frac{a^4p^2}{2r_5}+\frac{a^2|b|p^2r_6}{2}+\frac{a^2|c|p^2r_7}{2}\right)\\
&\quad\times\|\nabla\theta^{km+l}\|^2+\left(\frac{a^4p^2r_5}{2}+\frac{a^2|b|p^2r_8}{2}+\frac{a^2|c|p^2r_{10}}{2}\right)\|\nabla(\theta^{km+l+1}-\theta^{km+l})\|^2\\
&\quad+\left(\frac{a^2|c|p^2}{2r_7}+\frac{|b||c|p^2}{r_9}+\frac{a^2|c|p^2}{2r_{10}}-c^2p^2\right)\|\nabla\theta^{(k+1)m}\|^2\\
&\quad+\left(\frac{a^2|b|p^2}{2r_6}+\frac{a^2|b|p^2}{2r_8}-b^2p^2+|b||c|p^2r_9\right)\|\nabla\theta^{km}\|^2\\
&\quad+\left(\frac{a^2p^2}{2r_{11}}+\frac{|b|p^2}{2r_{12}}+\frac{|c|p^2}{2r_{13}}\right)\|q^{km+l}\|^2. \qquad (5.5)
\end{aligned}
$$

From (5.5), it's necessary to estimate $\|\nabla(\theta^{km+l+1}-\theta^{km+l})\|^2$, $\|\nabla\theta^{km+l}\|^2$, $\|\nabla\theta^{(k+1)m}\|^2$ and $\|\nabla\theta^{km}\|^2$. Let $\chi=\theta^{km+l}$, $\chi=b\theta^{km}$ and $\chi=c\theta^{(k+1)m}$, then we have

$$
\begin{aligned}
a^2\|\nabla\theta^{km+l}\|^2 &\leq \|q^{km+l}\|\|\theta^{km+l}\|+|b|\|\nabla\theta^{km}\|\|\nabla\theta^{km+l}\|\\
&\quad+|c|\|\nabla\theta^{(k+1)m}\|\|\nabla\theta^{km+l}\|+\|\partial_{tt}\theta^{km+l}\|\|\theta^{km+l}\|, \qquad (5.6)\\
b^2\|\nabla\theta^{km}\|^2 &\leq |b|\|q^{km+l}\|\|\theta^{km}\|+a^2|b|\|\nabla\theta^{km+l}\|\|\nabla\theta^{km}\|\\
&\quad+|b||c|\|\nabla\theta^{(k+1)m}\|\|\nabla\theta^{km}\|+|b|\|\partial_{tt}\theta^{km+l}\|\|\theta^{km}\|, \qquad (5.7)\\
c^2\|\nabla\theta^{(k+1)m}\|^2 &\leq |c|\|q^{km+l}\|\|\theta^{(k+1)m}\|+a^2|c|\|\nabla\theta^{km+l}\|\|\nabla\theta^{(k+1)m}\|\\
&\quad+|b||c|\|\nabla\theta^{km}\|\|\nabla\theta^{(k+1)m}\|+|c|\|\partial_{tt}\theta^{km+l}\|\|\theta^{(k+1)m}\|. \qquad (5.8)
\end{aligned}
$$

Similar to (5.5), a calculation on (5.6), (5.7) and (5.8) with Poincaré inequality

$$
\begin{aligned}
&\|\theta^{km+l}\|\leq C_2\|\nabla\theta^{km+l}\|,\ \ C_2=C_2(\Omega),\\
&\|\theta^{(k+1)m}\|\leq \widehat{C_2}\|\nabla\theta^{(k+1)m}\|,\ \ \widehat{C_2}=\widehat{C_2}(\Omega),\quad \|\theta^{km}\|\leq\widetilde{C_2}\|\nabla\theta^{km}\|,
\end{aligned}
$$

which implies that the value range of $\|\nabla\theta^{km+l}\|^2,\|\nabla\theta^{km}\|^2$ and $\|\nabla\theta^{(k+1)m}\|^2$ relate to items

$$
\|q^{km+l}\|,\ \ \|\theta^{km+l+1}-\theta^{km+l}\|,\ \ \|\theta^{km+l}-\theta^{km+l-1}\|.
$$

Moreover, subtracting $a^2(\nabla\theta^{km+l+1},\nabla\chi)$ on both sides of (5.4) and taking $\chi=\theta^{km+l+1}-\theta^{km+l}$ and $\chi=\theta^{km+l+1}+\theta^{km+l}$, respectively, we can derive

$$
\begin{aligned}
&a^2\|\nabla(\theta^{km+l+1}-\theta^{km+l})\|^2\leq\|\partial_{tt}\theta^{km+l}\|\|\theta^{km+l+1}-\theta^{km+l}\|\\
&\quad+a^2\|\nabla\theta^{km+l+1}\|\|\nabla(\theta^{km+l+1}-\theta^{km+l})\|+|b|\|\nabla\theta^{km}\|\|\nabla(\theta^{km+l+1}-\theta^{km+l})\|\\
&\quad+|c|\|\nabla\theta^{(k+1)m}\|\|\nabla(\theta^{km+l+1}-\theta^{km+l})+\|q^{km+l}\|\|\theta^{km+l+1}-\theta^{km+l}\|,\\
&a^2\|\nabla\theta^{km+l+1}\|^2-a^2\|\nabla\theta^{km+l}\|^2\leq\|q^{km+l}\|\|\theta^{km+l+1}+\theta^{km+l}\|\\
&\quad+\|\partial_{tt}\theta^{km+l}\|\|\theta^{km+l+1}+\theta^{km+l}\|+a^2\|\nabla\theta^{km+l+1}\|\|\nabla(\theta^{km+l+1}+\theta^{km+l})\|
\end{aligned}
$$

$$+|b|\|\nabla\theta^{km}\|\|\nabla(\theta^{km+l+1}+\theta^{km+l})\|+|c|\|\nabla\theta^{(k+1)m}\|\|\nabla(\theta^{km+l+1}+\theta^{km+l})\|. \tag{5.9}$$

Similar to the above process we can also obtain the value range of item $\|\nabla(\theta^{km+l+1}-\theta^{km+l})\|^2$ relate to items

$$\|q^{km+l}\|, \quad \|\theta^{km+l+1}-\theta^{km+l}\|, \quad \|\theta^{km+l}-\theta^{km+l-1}\|.$$

Based on the discussion about $\|\nabla(\theta^{km+l+1}-\theta^{km+l})\|^2$, $\|\nabla\theta^{km+l}\|^2$, $\|\nabla\theta^{(k+1)m}\|^2$ and $\|\nabla\theta^{km}\|^2$, (5.5) turns into

$$\begin{aligned}&\|\theta^{km+l+1}-\theta^{km+l}\|^2 \le s_1\|\theta^{km+l}-\theta^{km+l-1}\|^2+s_2\|q^{km+l}\|^2+s_2\|q^{km+l}\|^2\\ &\le s_1(s_1\|\theta^{km+l-1}-\theta^{km+l-2}\|^2+s_2\|q^{km+l-1}\|^2)=s_1^2\|\theta^{km+l-1}-\theta^{km+l-2}\|^2\\ &\quad+s_1s_2\|q^{km+l-1}\|^2+s_2\|q^{km+l}\|^2\le s_1^l\|\theta^{km+1}-\theta^{km}\|^2\\ &\quad+\sum_{j=1}^{l}s_1^{l-j}s_2\|q^{km+j}\|^2\le s_1^l\|\theta^{km+1}\|^2+s_1^l\|\theta^{km}\|^2+\sum_{j=1}^{l}s_1^{l-j}s_2\|q^{km+j}\|^2,\end{aligned}$$

where s_1, s_2 are positive numbers and determined by inequalities (5.5)–(5.9). Hence, we have

$$\begin{aligned}\|\theta^{km+l+1}\| &\le \|\theta^{km+l}\|+s_1^l\|\theta^{km+1}\|+s_1^l\|\theta^{km}\|+\sum_{j=1}^{l}s_1^{l-j}s_2\|q^{km+j}\|^2|\\ &\le \|\theta^{km+1}\|+ls_1^l\|\theta^{km+1}\|+ls_1^l\|\theta^{km}\|+l\sum_{j=1}^{l}s_1^{l-j}s_2\|q^{km+j}\|^2\\ &= \left(1+ls_1^l\right)\|\theta^{km+1}\|+ls_1^l\|\theta^{km}\|+l\sum_{j=1}^{l}s_1^{l-j}s_2\|q^{km+j}\|^2\\ &\le \left(1+ls_1^l\right)^2\|\theta^{km}\|+l(1+ls_1^l)s_1^l\|\theta^{km}\|+l\left(1+ls_1^l\right)\sum_{j=1}^{l}s_1^{l-j}s_2\|q^{km+j}\|^2\\ &\quad+ls_1^l\|\theta^{km}\|+l\sum_{j=1}^{l}s_1^{l-j}s_2\|q^{km+j}\|^2\\ &= \left(\left(1+ls_1^l\right)^2+l(2+ls_1^l)s_1^l\right)\|\theta^{km}\|+l\left(2+ls_1^l\right)\sum_{j=1}^{l}s_1^{l-j}s_2\|q^{km+j}\|^2.\end{aligned}$$

Denote

$$\begin{aligned}H_1&=\left(1+(m-1)s_1^{m-1}\right)^2+(m-1)(2+(m-1)s_1^{m-1})s_1^{m-1},\\ H_2&=(m-1)\left(2+(m-1)s_1^{m-1}\right),\end{aligned}$$

then,

$$\|\theta^{(k-1)m}\|\le H_1\|\theta^{km}\|+H_2\sum_{j=1}^{m-1}s_1^{m-1-j}s_2\|q^{km+j}\|\le H_1^2\|\theta^{(k-1)m}\|$$

$$+ H_1 H_2 \sum_{j=1}^{m-1} s_1^{m-1-j} s_2 \|q^{(k-1)m+j}\| + H_2 \sum_{j=1}^{m-1} s_1^{m-1-j} s_2 \|q^{km+j}\|$$

$$\le H_1^{k+1} \|\theta^{(0)}\| + H_1^k H_2 \sum_{j=1}^{m-1} s_1^{m-1-j} s_2 \|q^j\| + \cdots + H_2 \sum_{j=1}^{m-1} s_1^{m-1-j} s_2 \|q^{km+j}\|.$$

Here, $\theta^{(0)} = \theta(0)$ is bounded as desired in (4.8). We write

$$\begin{aligned} q_1^i &= (R_h - I)\partial_{tt} u(t_i) = (R_h - I) p^{-2} (u(t_{i+1}) - 2u(t_i) + u(t_{i-1})) \\ &= \frac{1}{6p^2} \Big(\int_{t_{i-1}}^{t_i} (t - t_{i-1})^3 (R_h - I) u_{ttt} dt - \int_{t_i}^{t_{i+1}} (t - t_{i+1})^3 (R_h - I) u_{ttt} dt \\ &\quad + 6p^2 (R_h - I) u(t_i) \Big), \end{aligned}$$

and obtain

$$\begin{aligned} \|q_1^i\| &\le \frac{1}{6p^2} \Big(p^3 \int_{t_{i-1}}^{t_i} \|(R_h - I) u_{ttt}\| dt + p^3 \int_{t_i}^{t_{i+1}} \|(R_h - I) u_{ttt}\| dt + 6p^2 \|(R_h - I) u(t_i)\| \Big) \\ &\le \frac{1}{6} \Big(p \int_{t_{i-1}}^{t_i} C h^r \|u_{ttt}\|_r dt + p \int_{t_i}^{t_{i+1}} C h^r \|u_{ttt}\|_r dt + 6 C h^r \|u(t_i)\|_r \Big) \\ &= C h^r \Big(\frac{p}{6} \int_{t_{i-1}}^{t_{i+1}} \|u_{ttt}\|_r dt + \|u(t_i)\|_r \Big). \end{aligned}$$

Further,

$$\begin{aligned} q_2^i &= \partial_{tt} u(t_i) - u_{tt}(t_i) = \frac{1}{p^2} (u(t_{i+1}) - 2u(t_i) + u(t_{i-1}) - p^2 u_{tt}(t_i)) \\ &= \frac{1}{6p^2} \Big(\int_{t_{i-1}}^{t_i} (t - t_{i-1})^3 u_{tttt} dt - \int_{t_i}^{t_{i+1}} (t - t_{i+1})^3 u_{tttt} dt \Big), \end{aligned}$$

so that,

$$\begin{aligned} \|q_2^i\| &= \frac{1}{6p^2} \Big\| \int_{t_{i-1}}^{t_i} (t - t_{i-1})^3 u_{tttt} dt - \int_{t_i}^{t_{i+1}} (t - t_{i+1})^3 u_{tttt} dt \Big\| \\ &\le \frac{1}{6p^2} \Big(p^3 \int_{t_{i-1}}^{t_i} \|u_{tttt}\| dt + p^3 \int_{t_i}^{t_{i+1}} \|u_{tttt}\| dt \Big) \le \frac{p}{6} \int_{t_{i-1}}^{t_{i+1}} \|u_{tttt}\| dt. \end{aligned}$$

Thus, we have

$$\begin{aligned} \|U^n - u(t_n)\| &\le \|\rho^n\| + \|\theta^n\| \\ &\le C h^r \left(\|v\|_r + \frac{p}{6} \sum_{i=0}^{n} \int_{t_i}^{t_{i+2}} \|u_{ttt}\|_r dt + \sum_{i=0}^{n} \|u(t_i)\|_r + \int_0^{t_n} \|u_s\| ds \right) \\ &\quad + C p \left(\sum_{i=0}^{n} \frac{1}{6} \int_{t_i}^{t_{i+2}} \|u_{tttt}\| dt \right). \end{aligned}$$

So, $U^n \to u(t_n)$ as $h \to 0$ and $p \to 0$. The proof is complete. □

5.2 Stability analysis

Definition 4. If any solution U^n of (5.1) satisfies

$$\lim_{n\to\infty} U^n = 0, x \in \Omega,$$

then the zero solution of (5.1) is asymptotically stable.

From (5.3) we obtain

$$\begin{aligned}\beta^{km+l+1} &= (2I - a^2p^2A_1^{-1}A_2)\beta^{km+l} - \beta^{km+l-1} - bp^2A_1^{-1}A_2\beta^{km} \\ &\quad - cp^2A_1^{-1}A_2\beta^{(k+1)m}.\end{aligned}$$

For convenience, we introduce $\bar{u}^{km+l+1} = \beta^{km+l}$, so

$$z^{km+l+1} = R_1z^{km+l} + R_2z^{km} + R_3z^{(k+1)m},$$

where $z^{km+l} = (\beta^{km+l}, \bar{u}^{km+l})^T$ and

$$R_1 = \begin{pmatrix} 2I - a^2p^2A_1^{-1}A_2 & -I \\ I & O \end{pmatrix}, \quad R_2 = \begin{pmatrix} -bp^2A_1^{-1}A_2 & O \\ O & O \end{pmatrix},$$
$$R_3 = \begin{pmatrix} -cp^2A_1^{-1}A_2 & O \\ O & O \end{pmatrix}.$$

Therefore, we derive that

$$\begin{aligned} z^{km+l+1} &= R_1z^{km+l} + R_2z^{km} + R_3z^{(k+1)m} \\ &= R_1^2z^{km+l-1} + (R_1+I)R_2z^{km} + (R_1+I)R_3z^{(k+1)m} \\ &= \left(R_1^{l+1} + (R_1^{l+1} - I)(R_1 - I)^{-1}R_2\right)z^{km} + (R_1^{l+1} - I)(R_1 - I)^{-1}R_3z^{(k+1)m}, \end{aligned}$$

that is,

$$z^{(k+1)m} = Mz^{km}, \tag{5.10}$$

where

$$M = \frac{I + (R_1^m - I)(I + (R_1 - I)^{-1}R_2)}{I - (R_1^m - I)(R_1 - I)^{-1}R_3}.$$

From Lemma 1 we know that the eigenvalues of matrix $I + (R_1 - I)^{-1}R_2$ are $\xi_1 = 1$, $\xi_2 = 1 + b/a^2$, and the eigenvalues of matrix $(R_1 - I)^{-1}R_3$ are $\eta_1 = 0$, $\eta_2 = c/a^2$. So, we have the following result.

Theorem 5. *Under the condition*

$$p^2 < \min 2/(a^2\lambda_{A_1^{-1}A_2}), \tag{5.11}$$

the eigenvalue of R_1 *satisfies* $|\lambda_{R_1}| < 1$.

Proof. From (4.13), we know that $\lambda_{A_1^{-1}A_2} > 0$. By Lemma 1, we obtain the eigenvalue of R_1 consist of the roots of the following equation

$$\lambda^2 - (2 - a^2p^2\lambda_{A_1^{-1}A_2})\lambda + 1 = 0. \tag{5.12}$$

It's obvious to notice that neither $\lambda = 0$ nor $\lambda = 1$ is the root of (5.12). Thus, we need to verify eigenvalue of R_1 satisfying $0 < |\lambda| < 1$ under some conditions. Firstly, we denote the function

$$f(\lambda) = \lambda^2 - (2 - a^2 p^2 \lambda_{A_1^{-1}A_2})\lambda + 1.$$

(i) When m is even, we can obtain $p^2 < \min 4/(a^2 \lambda_{A_1^{-1}A_2})$ from (5.11) as well as $0 < (2 - a^2 p^2 \lambda_{A_1^{-1}A_2})/2 < 1$ holds, which means the root of (5.12) exists in the interval $(0, 1)$ under $f(0) = 1 > 0, f(1) = a^2 p^2 \lambda_{A_1^{-1}A_2} > 0$. In addition, inequality $p^2 < \min 4/(a^2 \lambda_{A_1^{-1}A_2})$ guarantees $-1 < (2 - a^2 p^2 \lambda_{A_1^{-1}A_2})/2 < 0$ and $f(-1) > 0$ so that the root of (5.12) also exists in the interval $(-1, 0)$. Therefore, we get $0 < |\lambda| < 1$.

(ii) When m is odd, by (5.11), $0 < (2 - a^2 p^2 \lambda_{A_1^{-1}A_2})/2 < 1$ holds so that the root of (5.12) is in the interval $(0, 1)$. Hence, we obtain $0 < \lambda < 1$.

The proof is finished. □

Theorem 6. *Under the conditions* (3.9) *and* (5.11), *if*

$$-a^2 < b < a^2, 0 < c < a^2, a^2 + c - b > 0 \tag{5.13}$$

hold, then the zero solution of (5.1) *is asymptotically stable.*

Proof. From (5.10) we know that the zero solution of (5.1) is asymptotically stable if and only if the eigenvalue of matrix M satisfies

$$|\lambda_M| < 1, \tag{5.14}$$

that is,

$$\left| \frac{1 + (\lambda_{R_1}^m - 1)\xi_i}{1 - (\lambda_{R_1}^m - 1)\eta_i} \right| < 1, i = 1, 2. \tag{5.15}$$

From (5.15) we have

$$(1 + (\lambda_{R_1}^m - 1)\xi_i)^2 < (1 - (\lambda_{R_1}^m - 1)\eta_i)^2, i = 1, 2,$$

so,

$$(1 - \lambda_{R_1}^m)(\xi_i + \eta_i)\left(2 - (1 - \lambda_{R_1}^m)(\xi_i - \eta_i)\right) > 0. \tag{5.16}$$

Under the condition (3.9), if (5.13) holds, we get

$$\xi_i + \eta_i > 0, \xi_i - \eta_i < 2, i = 1, 2.$$

When m is even, in view of (5.11), we can get $\lambda_{R_1} \neq 0$ and $-1 < \lambda_{R_1} < 1$. Then together with (3.9) and (5.13), we know that (5.16) is satisfied. So, (5.14) holds. Hence, the zero solution of (5.1) is asymptotically stable. When m is odd, a similar analysis can be obtained. This completes the proof. □

6 Numerical experiments

In this section, some experiments are provided for verifying the theoretical results. Consider the following problem

$$\begin{aligned}&u_{tt}(x,t) = 144u_{xx}(x,t) - 2u_{xx}(x,[t]) + u_{xx}(x,[t+1]), [0,\pi]\times[0,+\infty),\\ &u(x,0) = \sin(x),\ u_t(x,0) = 0,\ x \in [0,\pi], \\ &u(0,t) = u(\pi,t) = 0,\ t\in[0,+\infty).\end{aligned} \tag{6.1}$$

It is not difficult to verify that the coefficients a, b and c satisfy the conditions (3.9) and (5.13). The analytic solution of (6.1) is the first component of

$$\hat{u}(x,t) = \sin(x)W_1(t)d_1, \tag{6.2}$$

where $d_1 = (1,0)^T$, $W_1(t)$ is defined in (3.7). When t is an integer, (6.2) gives

$$\hat{u}(x,t) = \sin(x)\left((I-N(1))^{-1}M(1)\right)^t d_1,$$

where

$$(I-N(1))^{-1}M(1) = \begin{pmatrix} -\frac{c\sin^2(a)-(a^2+b)\cos(a)+b}{a^2+c(1-\cos(a))} & \frac{\sin(a)}{a} \\ -\frac{(a^2+b)a\sin(a)+ca\sin(a)\cos(a)}{a^2+c(1-\cos(a))} & \cos(a) \end{pmatrix}.$$

Moreover, we consider piecewise linear functions as the bases of S_h

$$\Phi_{i-1}(x) = \begin{cases} (x_i - x)/(x_i - x_{i-1}), & x\in[x_{i-1},x_i], \\ 0, & \text{elsewhere}, \end{cases}$$

$$\Phi_{i}(x) = \begin{cases} (x - x_{i-1})/(x_i - x_{i-1}), & x\in[x_{i-1},x_i], \\ 0, & \text{elsewhere}, \end{cases}$$

where $i = 1, 2, \cdots, N_h$, That is, we obtain the convergence order of semidiscrete case $O(h^2)$ and fully discrete case $O(h^2+p)$.

In semidiscrete scheme and fully discrete scheme, the order of convergence is defined as

$$order = \frac{\log(AE_*(h_i)/AE_*(h_{i+1}))}{\log(h_i/h_{i+1})},$$

where $AE_*(h_i)$ is the error calculated in L^∞ norm and L^2 norm by the following formulas when taking step-size h_i and $*$ represents 2-norm or ∞-norm:

$$L^\infty = \|u-U\|_{L^\infty} = \max_{0\le i\le N}|u(x_i,t)-U(x_i,t)|,$$

$$L^2 = \|u-U\|_{L^2} = \left(\int_\Omega (u-U)^2dx\right)^{1/2} \approx \left(h\sum_{i=1}^{N-1}(u(x_i,t)-U(x_i,t))^2\right)^{1/2}.$$

We take step-size $h = \pi/N_h$ and obtain numerical results in 2-norm and infinite norm at $t = 7$, which are shown in Table 1.

Table 1. Semidiscrete error estimations of Problem (6.1) at $t = 7$.

h	AE_2	$rate$	AE_∞	$rate$
$\pi/8$	3.5988e-1	-	2.8714e-1	-
$\pi/16$	1.1944e-2	1.5912	9.5298e-2	1.5912
$\pi/24$	5.6868e-2	1.8301	4.5374e-2	1.8301
$\pi/32$	3.3821e-2	1.8064	2.6985e-2	1.8203
$\pi/64$	1.1058e-2	1.6128	8.8823e-3	1.6128

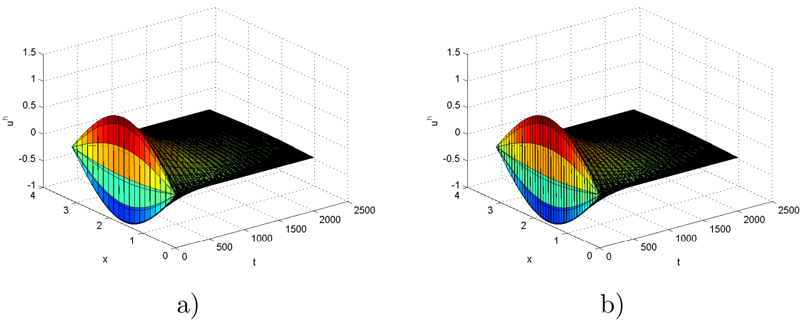


Figure 1. The semidiscrete numerical solution of Problem (6.1) with a) $N_h = 20$, b) $N_h = 40$.

From this table, we can see that the semidiscrete scheme is convergent with second order. Thus, these numerical results validate the theoretical error estimates in Theorem 2.

From Figure 1 we see that the numerical solution of (6.1) is asymptotically stable under the condition (3.9), which is consistent with Theorem 3.

Furthermore, in order to obtain the second-order for fully discrete case, we take $p = 1/N_h^2$ and good convergence is shown in Table 2 at $t = 2$ in different norms when h and p decreasing simultaneously. So, Theorem 4 is verified.

Table 2. Fully discrete error estimations of Problem (6.1) at $t = 2$.

h	p	AE_2	rate	AE_∞	rate
1/8	1/64	3.1037e-1	-	2.4764e-1	-
1/16	1/256	7.3422e-2	2.0797	5.8582e-2	2.0797
1/32	1/1024	1.8017e-2	2.0269	1.4375e-2	2.0269
1/64	1/4096	4.4820e-3	2.0071	3.5761e-3	2.0071
1/128	1/16384	1.1183e-3	2.0028	8.9224e-4	2.0029

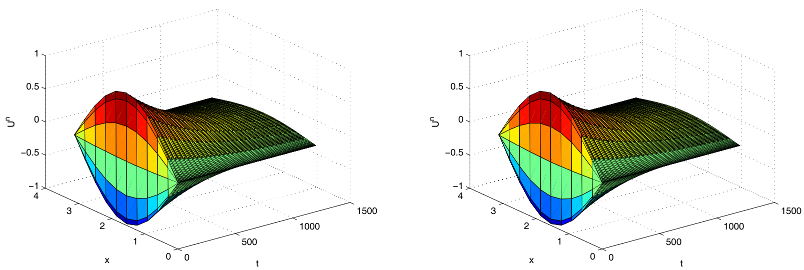
Figures 2–3 are presented to illustrate the stability of numerical solution under fully discrete case by different time steps. These figures are in accordance with Theorem 6. Some detailed analysis are as follows.

Let $m = 100$ and $m = 110$, then we have

$$p^2 = 1e{-}4 < \min \frac{2}{a^2 \lambda_{A_1^{-1} A_2}} = 1.0349e{-}4$$

$$p^2 = 8.2645e{-}5 < \min \frac{2}{a^2 \lambda_{A_1^{-1} A_2}} = 1.0349e{-}4,$$

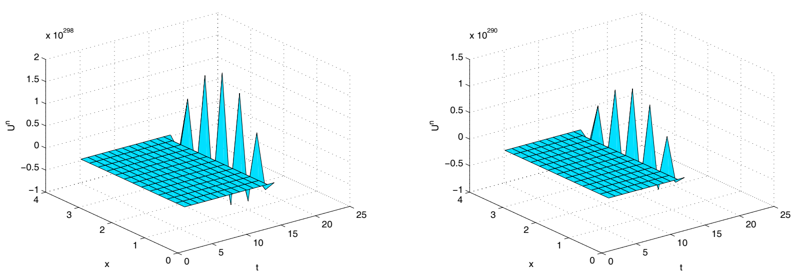
respectively, so the condition (5.11) is satisfied and the numerical solution of Problem (6.1) is asymptotically stable according to Theorem 6, see Figure 2.



a) b)

Figure 2. The fully discrete numerical solution of Problem (6.1) with a) $N_h = 10, m = 100$, b)$N_h = 10, m = 110$.

However, when $m = 17$ and $m = 20$, the condition (5.11) is not satisfied. As we observe from Figure 3 the numerical solution of Problem (6.1) is unstable, which is also in accordance with Theorem 6.



a) b)

Figure 3. The fully discrete numerical solution of Problem (6.1) with a) $N_h = 12, m = 17$, b) $N_h = 12, m = 20$.

7 Conclusions

This paper deals with the numerical approximation of semidiscrete scheme and fully discrete scheme of hyperbolic PEPCA of advanced type by Galerkin finite element method (for spatial derivatives) and finite difference scheme (for time derivative). Rigorous theoretical analysis for convergence and stability of the two numerical schemes are presented. The results show that the semidiscrete scheme can achieve unconditionally stability and some sufficient conditions are put to guarantee the asymptotical stability of numerical solution for fully discrete scheme. In the future study, we will focus on high dimension problem and nonlinear problem.

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