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The Conditional Stability and an Iterative Regularization Method for a Fractional Inverse Elliptic Problem of Tricomi-Gellerstedt-Keldysh Type

Sebti Djemoui[a], Mohamed S.E. Meziani[b] and Nadjib Boussetila[c]

[a] *Higher School of industrial Technologies Annaba*
P.O.Box 218, Safsaf, 23000 Annaba, Algeria
[b] *Department of Mathematics, ENSET Skikda*
frères Bouceta, Azzaba, 21001 Skikda, Algeria
[c] *Department of Mathematics, University 8 Mai 1945*
P.O.Box 401, 24000 Guelma, Algeria
E-mail(*corresp.*): s.djemoui@esti-annaba.dz; sebtijd@yahoo.fr
E-mail: mse.meziani@enset-skikda.dz; mmsemath@gmail.com
E-mail: boussetila.nadjib@univ-guelma.dz; n.boussetila@gmail.com

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Abstract. The present paper is devoted to identifying an inaccessible boundary condition for a fractional elliptic problem of Tricomi-Gellerstedt-Keldysh-type. Using the expansion Fourier method, the considered problem can be reformulated as an operator equation of the first kind. To construct a stabilized approximate solution we employ a variant of the iterative method. We also present error estimates between the exact solution and the regularized solution by the a priori and the a posteriori parameter choice rules. Finally, some numerical verifications on the efficiency and accuracy of the proposed algorithm is presented.

Keywords: fractional elliptic equations, Tricomi-Gellerstedt-Keldysh equations, ill-posed problems, inverse problems, a posteriori parameter choice rule, iterative regularization method.

AMS Subject Classification: 35R25; 35R30; 65J22; 34K37.

1 Introduction

The first appearance of mixed partial differential equations and their applications dates back to the beginning of the $20th$ century. This type of equations is found in transonic flux theory and this gives rise to particular boundary value problems. The theory of boundary value problems of mixed type is originated in the fundamental research of Francesco Tricomi [26] and has been applied to describe the theory of plane transonic flow [18, 22]. This idea was exploited and developed by several authors [2, 3, 4].
A very useful model of mixed equations was proposed by Lavrent'ev and Bitsadze [17]. The various boundary value problems (BVP) for mixed type equations are well-known as one of the most important problems in mathematical physics. This type of topic is used to investigate wide applications in transonic gas dynamics (see [16]). Several boundary value problems (BVP) for mixed-type partial differential equations with the Riemann-Liouville fractional differential operator have been studied [8, 11].

Degenerate equations are involved in many physics and mechanics problems, including those involving the theory of boundary layers, the science of diffusion processes, particularly the theory of Brownian motion. The degenerate equations of the second order of elliptic and parabolic form are the subject of the most intense research [21, 27]. Fractional analogues of elliptic equations have been the topic of many articles [19, 28]. The reliance of the formulation of a boundary value problem on the type of elliptic equation degeneration at the boundary was originally noted by M.V. Keldysh.

When the determination of a physical domain boundary is inaccessible, an internal measurement can be considered as additional information to ensure the unique resolvability of the problem. Thus, in recent years, research has intensified on the problems of direct and inverse boundary values for partial differential equations of fractional order. In their works [1,9] the authors studied a class of BVP with integral conditions with both Riemann-Liouville and the Caputo operators. Due to the importance of this equations type, the treatment of problems related to degenerate equations has become one of the principal subject in the theory of partial differential equations.
The present work is a generalization of the results established in [24], where the treated problem is governed by a generalized elliptic equation with a fractional subdiffusion operator.

Recently, in the paper [23] the authors have been developed a theoretical analysis devoted to direct problems governed by Tricomi-Gellerstedt-Keldysh-type fractional elliptical problems. Our objective in this work is to extend this study to the case of inverse problems. More precisely, we are concerned with the identification problem of the missing boundary conditions [23]:

$$D^{2\alpha}u(x,y) - x^{2\beta}Au(x,y) = 0, \quad (x,y) \in \mathbb{R}^+ \times \varOmega, \tag{1.1}$$

with $\frac{1}{2} < \alpha \leq 1,\ \beta > -\alpha$ and $\varOmega \subset \mathbb{R}^N,\ N \geq 1$. This equation is a generalization of several other classical well-known types, we quote for example, Laplace and Tricomi equations [26], it plays a principal role in the mathematical analysis of the transonic flows, as it is of elliptic and hyperbolic type [10]. This work is a new investigation in the field of ill-posed and inverse problems associated with degenerate partial differential equations and their applications.

We note here that the problem in [24] is a special case of the present equation (1.1) ($\beta = 0$). The formal solution of our problem is expressed by the three parameters Mittag-Leffler function (Kilbas-Saigo function), which makes the computations rather complicated. The important contribution in our study is the choice of the regularization parameter with the a priori and a posteriori rules.

The outline of this paper is arranged as follows. In Section 2, we present some preliminary results and interesting estimates which will be used in the paper. Section 3 is devoted to describing the statement of the problem. In Section 4, ill-posedness and conditional stability for the fractional elliptic inverse problem of Tricomi-Gellerstedt-Keldysh-type are provided. In Section 5, we will recover the unknown data by the iterative method of regularization proposed by Kozlov-Maz'ya and one will establish some results of convergence under the a priori and a posteriori parameter choice rule. Some numerical results are illustrated in Section 6. Finally, we give a brief conclusion in Section 7.

2 Preliminaries

In this section, we start by the definition of the three-parameter Mittag-Leffler function (Kilbas-Saigo function) and some of its properties. The Mittag-Leffler function with three parameters was introduced by Kilbas and Saigo [12] with the convergent series representation

$$E_{\alpha,m,l}(z) = \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{\Gamma(1+\alpha((k-1)m+l))}{\Gamma(1+\alpha((k-1)m+l+1))} \right) z^n, \quad z \in \mathbb{C},$$

where $\alpha, m > 0$ and $l > \frac{-1}{\alpha}$. For $\alpha \in (0,1), m > 0$ and $x \geq 0$, we have the following inequality (see [6])

$$\frac{1}{1+\Gamma(1-\alpha)x} \leq E_{\alpha,m,m-1}(-x) \leq \frac{1}{1+\frac{\Gamma(1+(m-1)\alpha)}{\Gamma(1+m\alpha)}x}. \tag{2.1}$$

The Kilbas-Saigo function is one of the most important mechanisms that are applied to solve different types of integral and partial differential equations of fractional order.

For $x = \lambda L^{\alpha+\beta}$ and $m = 1 + \frac{\beta}{\alpha}$, we can deduce the following estimate

$$\frac{\eta_1}{\lambda} \leq E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\lambda L^{\alpha+\beta}) \leq \frac{\eta_2}{\lambda}, \quad \lambda \geq \lambda_1 > 0, \tag{2.2}$$

where $\eta_1 = (\lambda_1^{-1} + \Gamma(1-\alpha)L^{\alpha+\beta})^{-1}$, $\eta_2 = 1/(C_{\alpha,1+\frac{\beta}{\alpha}} L^{\alpha+\beta})$ and $C_{\alpha,m} = \Gamma(1+(m-1)\alpha)/\Gamma(1+m\alpha)$. From (2.2), we can deduce that

$$\left(E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\lambda L^{\alpha+\beta})\right)^{-\theta} \leq (\lambda/\eta_1)^{\theta}, \quad \theta > 0. \tag{2.3}$$

Remark 1. ([5]). As for the classical case $m = 1$, these bounds are optimal because of the asymptotic behaviors

$$
\begin{aligned}
&1 - E_{\alpha,m,m-1}(-x) \sim x\Gamma(1+\alpha(m-1))/\Gamma(1+\alpha m), \;\; as\, x \to 0, \\
&E_{\alpha,m,m-1}(-x) \sim 1/\big(\Gamma(1-\alpha)x\big), \;\; as \;\; x \to \infty,
\end{aligned}
\tag{2.4}
$$

where $\alpha \in (0, 1)$, $x \geq 0$.

Now, we will recall some well-known facts about non-expansive operators.

DEFINITION 1. [15] A linear bounded operator $B : H \to H$ is called non-expansive if $\|B\| \leq 1$.

The convergence theorem below makes it possible to solve the following equation in the case of a non-expansive operator

$$
(I - B)f = g. \tag{2.5}
$$

Theorem 1. *(see [15]). Let B be a non-expansive, selfadjoint positive operator on H. Let g be such that (2.5) it has a solution. If 1 is not an eigenvalue of B then the successive approximations*

$$
f_{n+1} = Bf_n + g, \quad n \geq 0
$$

converge to a solution to (2.5) for any initial data $f_0 \in H$. Moreover, $B^n f \to 0$ for every $f \in H$ as $n \to \infty$. In other words, $\widetilde{f} - f_n = B^n(f_0 - \widetilde{f}) \to 0$, $n \to \infty$, where $\widetilde{f}$ is a solution to (2.5).

Theorem 2. *(Generalized Picard Theorem)[A. I. Prilepko (2000), p. 502]. Let us assume that H is a Hilbert space and S is a positive self-adjoint, unbounded linear operator on H, and $\Theta : \sigma(A) \longrightarrow \mathbb{R}$ is a continuous function not identically equal to zero, such that*

$$
\Theta(S) = \int_0^{+\infty} \Theta(\lambda) dE_\lambda \in \mathcal{L}(H),
$$

where $\{E_\lambda, \; \lambda \geq> 0\}$ is the spectral resolution of the identity associated to S.

Let $Z(\Theta) = \{\lambda \in \sigma(A) : \Theta(\lambda) = 0\}$ the set of zeroes of the characteristic function $\Theta(\lambda)$ supposed to be either is empty or contains isolated point only. Then, the equation $\Theta(S)u = v$ is correctly solvable if and only if

1. *$Z(\Theta) \cap \sigma(A) = \emptyset$ (uniqueness condition).*

2. *$\int_0^{+\infty} \frac{1}{|\Theta(\lambda)|^2} d\|E_\lambda v\|^2 < +\infty$ (existence condition).*

3 Statement of the problem

For $\frac{1}{2} < \alpha \leq 1$ and $\beta > -\alpha$, we consider the problem of finding the function $\varphi \in H$ in the following fractional elliptic system equations

$$\begin{cases} D_x^{2\alpha} u(x,y) - x^{2\beta} A u(x,y) = 0, & (x,y) \in (0,\infty) \times \Omega, \\ u(x,y) = 0, & (x,y) \in (0,\infty) \times \partial\Omega, \\ u(0,y) = \varphi(y), & y \in \Omega, \\ \lim\limits_{x\to\infty} u(x,y) = 0, & y \in \Omega, \end{cases}$$

where $A : D(A) \subset H \to H$ denotes a positive, linear densely defined self-adjoint operator with compact resolvent, Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$ and H is a separable Hilbert space endowed with the inner product $(.,.)$ and the norm $\|.\|$. The pair (e_n, λ_n) represents the eigenvalues and orthonormal eigenfunctions respectively associated with the operator A, such that

$$\begin{aligned} &Ae_n = \lambda_n e_n, \; n \in \mathbb{N}^*, \\ &0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ..., \quad \lim_{n\to\infty} \lambda_n = \infty, \end{aligned}$$

$\forall f \in H$, $f = \sum_{n=1}^{\infty} f_n e_n$, $f_n = (f, e_n)$. Here, $D_x^{2\alpha} = \partial_{0+,x}^{\alpha} \partial_{0+,x}^{\alpha}$, and $\partial_{0+,x}^{\alpha}$ is the Caputo fractional derivatives of order α defined by

$$\partial_{0+,x}^{\alpha} u(x,y) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-s)^{-\alpha} \partial_s u(s,y) ds.$$

3.1 Solution of the direct problem

Let us consider the following problem

$$\begin{cases} D_x^{2\alpha} u(x,y) - x^{2\beta} A u(x,y) = 0, & (x,y) \in (0,\infty) \times \Omega, \\ u(x,y) = 0, & (x,y) \in (0,\infty) \times \partial\Omega, \\ u(0,y) = \varphi(y), & y \in \Omega, \\ \lim\limits_{x\to\infty} u(x,y) = 0, & y \in \Omega. \end{cases} \tag{3.1}$$

Let us now note that:

$$\|\varphi\|^2 = \int_\Omega |\varphi(y)|^2 dy, \qquad \|u(x,.)\|^2 = \int_\Omega |u(x,y)|^2 dy.$$

Theorem 3. *([23]). Let $\varphi \in H$. Then the problem* (3.1) *admits a unique generalized solution given by*

$$u(x,y) = \sum_{n=1}^{\infty} \varphi_n E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}} (-\sqrt{\lambda_n} x^{\alpha+\beta}) e_n(y).$$

In addition, the following estimates are satisfied

$$\begin{aligned} &\|u(x,.)\|_{C(\mathbb{R}_+, L^2(\Omega))} \leq \|\varphi\|_{L^2(\Omega)}, \\ &\sup_{x>0} \left\| x^{-2\beta} D_x^{2\alpha} u(x,.) \right\|_{L^2(\Omega)} \leq \|\varphi\|_H, \quad \sup_{x>0} \||Au(x,.)\|_{L^2(\Omega)} \leq \|\varphi\|_H. \end{aligned}$$

From the Theorem 3, we can write

$$u(x,y)=\sum_{n=1}^{\infty}\varphi_n E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\sqrt{\lambda_n}x^{\alpha+\beta})e_n(y)=R_{\alpha,\beta}(x)\varphi(y), \tag{3.2}$$

where $R_{\alpha,\beta}(x)=E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\sqrt{A}x^{\alpha+\beta})$ is a strongly continuous function defined via the spectral diagonalization of A and $E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(z)$ is the Kilbas-Saigo function.

4 Ill-posedness of the inverse problem

We consider the problem of finding the function $u(0,y)=\varphi(y)\in H$ in the following elliptic fractional equation

$$\begin{cases} D_x^{2\alpha}u(x,y)-x^{2\beta}Au(x,y)=0, & (x,y)\in[0,\infty[\times\Omega,\\ u(x,y)=0, & (x,y)\in[0,\infty[\times\partial\Omega,\\ \lim\limits_{x\to\infty}u(x,y)=0, & y\in\Omega, \end{cases} \tag{4.1}$$

under the internal condition at $x=L>0$, i.e.,

$$u(x=L,y)=\psi(y). \tag{4.2}$$

Making use of the supplementary condition (4.2), we have

$$u(x=L,y)=R_{\alpha,\beta}(L)\varphi(y)=\psi(y),$$

which implies

$$\varphi(y)=R_{\alpha,\beta}^{-1}(L)\psi(y).$$

So,

$$\varphi_n=\frac{\psi_n}{E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\sqrt{\lambda_n}L^{\alpha+\beta})},\quad \varphi=\sum_{n=1}^{\infty}\frac{\psi_n}{E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\sqrt{\lambda_n}L^{\alpha+\beta})}e_n.$$

Replacing φ by its value in (3.2), we get

$$u(x,y)=\sum_{n=1}^{\infty}\frac{E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\sqrt{\lambda_n}x^{\alpha+\beta})}{E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\sqrt{\lambda_n}L^{\alpha+\beta})}\psi_n e_n=R_{\alpha,\beta}(x)R_{\alpha,\beta}^{-1}(L)\psi(y). \tag{4.3}$$

Using (2.1), with $m=1+\frac{\beta}{\alpha}$, we derive the following

$$\frac{1}{1+T_\alpha x}\le E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-x)\le\frac{1}{1+F_{\alpha,\beta}x},\quad x>0, \tag{4.4}$$

where $T_\alpha=\Gamma(1-\alpha)$ and $F_{\alpha,\beta}=\frac{\Gamma(1+\beta)}{\Gamma(1+\alpha+\beta)}$. Taking into account (4.4) and (4.3), then, for some $\lambda_n\ge\lambda_1$ and $x\ge\varepsilon>0$, it results

$$\begin{aligned} \frac{1}{1+T_\alpha L^{\alpha+\beta}\sqrt{\lambda_n}} &\le E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-L^{\alpha+\beta}\sqrt{\lambda_n})\le\frac{1}{1+F_{\alpha,\beta}L^{\alpha+\beta}\sqrt{\lambda_n}},\\ \frac{1}{1+T_\alpha x^{\alpha+\beta}\sqrt{\lambda_n}} &\le E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-x^{\alpha+\beta}\sqrt{\lambda_n})\le\frac{1}{1+F_{\alpha,\beta}x^{\alpha+\beta}\sqrt{\lambda_n}}, \end{aligned} \tag{4.5}$$

$$1+F_{\alpha,\beta}L^{\alpha+\beta}\sqrt{\lambda_n}\le\frac{1}{E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-L^{\alpha+\beta}\sqrt{\lambda_n})}\le 1+T_\alpha L^{\alpha+\beta}\sqrt{\lambda_n}. \tag{4.6}$$

From (4.5) and (4.6), we deduce that

$$\frac{1+F_{\alpha,\beta}L^{\alpha+\beta}\sqrt{\lambda_n}}{1+T_\alpha x^{\alpha+\beta}\sqrt{\lambda_n}} \leq \frac{E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-x^{\alpha+\beta}\sqrt{\lambda_n})}{E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-L^{\alpha+\beta}\sqrt{\lambda_n})} \leq \frac{1+T_\alpha L^{\alpha+\beta}\sqrt{\lambda_n}}{1+F_{\alpha,\beta}x^{\alpha+\beta}\sqrt{\lambda_n}}.$$

In this step, we want to study the stability of the solution u for all $x \in [0,\infty[$.

$$\begin{aligned}\left\|R_{\alpha,\beta}(x)R^{-1}_{\alpha,\beta}(L)\right\| &= \sup_{\lambda_n\geq\lambda_1}\left|\frac{1+T_\alpha L^{\alpha+\beta}\sqrt{\lambda_n}}{1+F_{\alpha,\beta}x^{\alpha+\beta}\sqrt{\lambda_n}}\right| = \frac{T_\alpha L^{\alpha+\beta}\sqrt{\lambda_n}}{F_{\alpha,\beta}x^{\alpha+\beta}\sqrt{\lambda_n}} \\ &= \Gamma(1-\alpha)\Gamma(1+\alpha+\beta)\left(\Gamma(1+\beta)\right)^{-1}(L/x)^{\alpha+\beta} = C(\alpha,\beta)\,(L/x)^{\alpha+\beta}, \quad (4.7)\end{aligned}$$

where $C(\alpha,\beta) = \Gamma(1-\alpha)\Gamma(1+\alpha+\beta)\left(\Gamma(1+\beta)\right)^{-1}$. From (4.7) and according to the values of the variable x, we can distinguish the following three cases.

- If $x \in [L,\infty[$, we have $\left\|R_{\alpha,\beta}(x)R^{-1}_{\alpha,\beta}(L)\right\| \leq 1$, then $\|u(x,.)\| \leq \|\psi\|$, which leads to the stability of u.
- For $\epsilon > 0$ and $x \in [\epsilon, L]$, we can get, $\|u(x,.)\| \leq C(\alpha,\beta)\left(\frac{L}{\epsilon}\right)^{\alpha+\beta}\|\psi\|$, and this also signified the stability of the solution u.
- If $x \in [0,\epsilon[$, we have $\lim\limits_{x\to 0} u(x,y) = \sum\limits_{n=1}^{\infty} \frac{E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\sqrt{\lambda_n}x^{\alpha+\beta})}{E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\sqrt{\lambda_n}L^{\alpha+\beta})}\psi_n e_n(y)$, and $\frac{1}{E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\sqrt{\lambda_n}L^{\alpha+\beta})} \longrightarrow +\infty$ as $n \longrightarrow \infty$, which shows the ill-posedness nature of the problem.

In conclusion, the reconstruction of the solution $u(x,y)$ for all $x > 0$ from $u(x = L, y)$ is stable, however, for $x = 0$ the reconstruction of the solution $u(0,y)$ from $u(x = L,y)$ is the only case of instability.

On the basis of (e_n) we introduce the Hilbert scale $H^s, s \in \mathbb{R}$ induced by A as follows:

$$H^s = \mathcal{D}(A^s) = \Big\{h \in H : \|h\|^2_{H^s} = \sum_{n=1}^{\infty}\lambda_n^{2s}|(h,e_n)|^2 < +\infty\Big\}.$$

Let $0 < \theta_1 < \theta_2$ and let $0 < \theta_3 < \theta_4$. Then we have the following topological inclusions: $H^{\theta_2} \subset H^{\theta_1} \subset H^0 = H \subset H^{-\theta_3} \subset H^{-\theta_4}$.

Remark 2. For $s > 0$, the Hilbert space H^{-s} is the topological dual space of H^s, that is, $H^{-s} = (H^s)'$.

Corollary 1. The problem (4.1)–(4.2) admits a unique solution if and only if $\psi \in H^{\frac{1}{2}}$, i.e.,

$$\sum_{n=1}^{\infty}\lambda_n|\psi_n|^2 < \infty. \quad (4.8)$$

Proof. From (2.2) and by virtue of the generalized Picard theorem, we deduce that the inverse problem (4.1)–(4.2) is correctly solvable if and only if $\psi \in H^{\frac{1}{2}}$. □

Theorem 4. *Assuming that ψ satisfies the existence condition* (4.8)*, then the problem* (4.1) *has a unique strong generalized solution given by* (4.3)*. In addition, we have the following dependence estimates*

$$\sup_{x\geq L}\|u(x,.)\|\leq\|\psi\|,\qquad \sup_{\epsilon\leq x\leq L}\|u(x,.)\|\leq C(\alpha,\beta)\,(L/\epsilon)^{\alpha+\beta}\,\|\psi\|.$$

Next, a conditional stability of the ill-posed problem is given in the following theorem.

Theorem 5. *For $\theta>0$, if $\varphi\in H^{\frac{\theta}{4}}$, then we have*

$$\|\varphi\|\leq(1/\eta_1)^{\frac{\theta}{\theta+2}}\,\|\varphi\|_{H^{\frac{\theta}{4}}}^{\frac{2}{\theta+2}}\,\|\psi\|^{\frac{\theta}{\theta+2}}.$$

Proof. We have

$$\begin{aligned}
\|\varphi\|^2 &= \sum_{n=1}^{\infty}\left(\frac{|\psi_n|}{E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-L^{\alpha+\beta}\sqrt{\lambda_n})}\right)^2\\
&= \sum_{n=1}^{\infty}\left(\frac{|\psi_n|^{\frac{4}{\theta+2}}}{E^2_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-L^{\alpha+\beta}\sqrt{\lambda_n})}\right)\left(|\psi_n|^{\frac{2\theta}{\theta+2}}\right).
\end{aligned}$$

Using Hölder's inequality and (4.6), we get

$$\begin{aligned}
\|\varphi\|^2 &\leq \left(\sum_{n=1}^{\infty}\frac{|\psi_n|^2}{E^{\theta+2}_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-L^{\alpha+\beta}\sqrt{\lambda_n})}\right)^{\frac{2}{\theta+2}}\left(\sum_{n=1}^{\infty}|\psi_n|^2\right)^{\frac{\theta}{\theta+2}}\\
&= \left(\sum_{n=1}^{\infty}|\varphi_n|^2\frac{1}{E^{\theta}_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-L^{\alpha+\beta}\sqrt{\lambda_n})}\right)^{\frac{2}{\theta+2}}\left(\|\psi\|^2\right)^{\frac{\theta}{\theta+2}}\\
&\leq \left(\sum_{n=1}^{\infty}|\varphi_n|^2\left(\frac{1}{\eta_1}\right)^{\theta}\lambda_n^{\frac{\theta}{2}}\right)^{\frac{2}{\theta+2}}\|\psi\|^{\frac{2\theta}{\theta+2}}\leq\left(\frac{1}{\eta_1}\right)^{\frac{2\theta}{\theta+2}}\|\varphi\|_{H^{\frac{\theta}{4}}}^{\frac{4}{\theta+2}}\|\psi\|^{\frac{2\theta}{\theta+2}}.
\end{aligned}$$

□

5 Iterative procedure and error estimates

For regularizing the problem (4.1), we propose an iterative regularization procedure based on the Kozlov-Maz'ya approach with the help of an extra measurement at an internal point given by (4.2). In [13, 14], Kozlov and Maz'ya proposed an alternating iterative method to solve boundary value problems for general strongly elliptic and formally self-adjoint systems. After that, this method has attracted considerable attention of a lot of mathematicians and the idea has been successfully used for solving various classes of ill-posed (elliptic, parabolic, biparabolic, hyperbolic and fractional evolution) equations; see, for example, [20, 29, 30].

The principle of the alternative iterative method consists in replacing the given problem which is ill-posed by a series of problems which are all of a well-posed nature. First, we start by letting $\varphi_0(y) \in H$ be arbitrary; the initial approximation $u_0(x,y)$ is the solution to the direct well-posed problem

$$\begin{cases} D_x^{2\alpha} u_0(x,y) - x^{2\beta} A u_0(x,y) = 0, & (x,y) \in]0,\infty[\times \Omega, \\ u_0(x,y) = 0, & (x,y) \in]0,\infty[\times \partial\Omega, \\ u_0(0,y) = \varphi_0(y), & \\ \lim\limits_{x\to\infty} u_0(x,y) = 0, & y \in \Omega. \end{cases}$$

Once the pair of solutions (φ_k, u_k) has been constructed, we can adopt the following recurring relation

$$\varphi_{k+1}(y) = \varphi_k(y) - \omega(u_k(x=L,y) - \psi(y)) = (I - \omega K(L))\varphi_k(y) + \omega\psi(y), \quad (5.1)$$

where ω is such that

$$0 < \omega < \omega^* = 1/\|K(L)\|, \quad K(L) = \left(E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-L^{\alpha+\beta}\sqrt{A})\right), \quad (5.2)$$

$$\|K(L)\| = \sup_{n\geq 1}\left(E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-L^{\alpha+\beta}\sqrt{\lambda_n})\right) \leq \left(E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-L^{\alpha+\beta}\sqrt{\lambda_1})\right) < 1.$$

The following problem enables us to define the required solution $u_{k+1}(x,y)$:

$$\begin{cases} D_x^{2\alpha} u_{k+1}(x,y) - x^{2\beta} A u_{k+1}(x,y) = 0, & (x,y) \in]0,\infty[\times \Omega, \\ u_{k+1}(x,y) = 0, & (x,y) \in]0,\infty[\times \partial\Omega, \\ u_{k+1}(0,y) = \varphi_{k+1}(y), & \\ \lim\limits_{x\to\infty} u_{k+1}(x,y) = 0. & \end{cases}$$

By using relation (5.1) and by carrying out a finite number of iterations, we obtain

$$\varphi_{k+1}(y) = (I - \omega K(L))^{k+1}\varphi_0(y) + \omega\sum_{j=0}^{k}(I - \omega K(L))^j\psi(y). \quad (5.3)$$

Proposition 1. *The operator $\Theta = (I - \omega K(L))$ has the following properties*
1. Θ is self-adjoint, 2. Θ is nonexpansive,
3. 1 is not an eigenvalue of Θ, 4. $\left\|\sum\limits_{i=0}^{k-1}(I-\omega K(L))^i\right\| \leq k, \quad k = 1,2,\ldots.$

Then, it follows from Theorem 1 that the sequence of functions $(\varphi_n)_{n\geq 1}$ converges and $(I - \omega K(L))^n \to 0$ as $n \to \infty$ for every $\varphi \in H$.

Lemma 1. *([7]) For $0 < \lambda < 1$, define*

$$p_k(\lambda) = \sum_{i=0}^{k-1}(1-\lambda)^i, \quad r_k(\lambda) = 1 - p_k(\lambda) = (1-\lambda)^k.$$

Then, for $0 \leq \mu \leq 1$, there hold

$$p_k(\lambda)\lambda^u \leq k^{1-\mu}, \quad 0 \leq \mu \leq 1, \quad r_k(\lambda)^k\lambda^\mu \leq \tau_\mu(k+1)^{-\mu},$$

where

$$\tau_\mu = \begin{cases} 1, & 0 \le \mu \le 1, \\ \mu^\mu, & \mu > 1. \end{cases}$$

Theorem 6. *Let $u(x,y)$ be a solution to the inverse problem (4.1). Let $\varphi_0(y)$ be an arbitrary initial data for the iterative procedure proposed above and let $u_k(x,y)$ be the $k-th$ approximate solution. Then,*

$$\sup_{x\in[0,\infty[} \|u(x,.) - u_k(x,.)\| \to 0 \ \text{ as } \ k \to \infty.$$

Proof. From (5.3), we have

$$\varphi_k(y) = (I - \omega K(L))^k \varphi_0(y) + (I - (I - \omega K(L))^k)(K(L))^{-1}\psi(y).$$

It follows that

$$\varphi_k(y) = (I - \omega K(L))^k (\varphi_0(y) - \varphi(y)) + \varphi(y),$$

which implies that

$$u_k(x,y) - u(x,y) = K(x)(\varphi_k(y) - \varphi(y)) = K(x)(I - \omega K(L))^k(\varphi_0(y) - \varphi(y)).$$

Passing to the supremum with respect to $x \in [0,\infty[$ and using (5.2), we get

$$\|u_k(x,.) - u(x,.)\| \le \left\|(I - \omega K(L))^k(\varphi_0 - \varphi)\right\|.$$

By virtue of the Theorem 1, we get $\|u_k(x,.) - u(x,.)\| \to 0$ as $k \to \infty$. □

Theorem 7. *Let $\varphi_0(y)$ be an arbitrary element for the iterative procedure suggested above and $u_k(x,y)$ be the k-th approximate solution. If $(\varphi_0(y) - u(0,y)) \in H^{\frac{\theta}{4}}, \theta > 0$, then the rate of convergence of the method is given by*

$$\|u_k(x,.) - u(x,.)\| \le C_\theta E/(k+1)^{\frac{\theta}{2}}. \tag{5.4}$$

Proof. We get

$$\begin{aligned}
\|u_k(x,.) - u(x,.)\|^2 &= \left\|K(x)(I - \omega K(L))^k(\varphi_0 - \varphi)\right\|^2 \\
&\le \left\|K(x)\|^2\|(I - \omega K(L))^k(\varphi_0 - \varphi)\right\|^2 \le \left\|(I - \omega K(L))^k(\varphi_0 - \varphi)\right\|^2 \\
&\le \sum_{n=1}^{\infty} (1 - \mu_n))^{2k} \left|(\varphi_0 - u(0), e_n)\right|^2,
\end{aligned}$$

where $\mu_n = \omega E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\sqrt{\lambda_n}L^{\alpha+\beta})$, then, we obtain

$$\begin{aligned}
\|u_k(x,.) - u(x,.)\|^2 \quad &\le \sum_{n=1}^{\infty} (1 - \mu_n)^{2k} (\mu_n)^\theta (\mu_n)^{-\theta} \left|(\varphi_0 - u(0), e_n)\right|^2 \\
&= \sum_{n=1}^{\infty} \left((1 - \mu_n)^k \mu_n^{\frac{\theta}{2}}\right)^2 (\mu_n)^{-\theta} \left|(\varphi_0 - u(0), e_n)\right|^2.
\end{aligned}$$

Using (2.3), we get

$$\|u_k(x,.)-u(x,.)\|^2 \le (1/(\omega\eta_1))^\theta \sum_{n=1}^{\infty}\left((1-\mu_n)^k \mu_n^{\frac{\theta}{2}}\right)^2 \lambda_n^\theta |(\varphi_0-u(0),e_n)|^2$$
$$\le (1/(\omega\eta_1))^\theta \left(\tau_{\frac{\theta}{2}}/(1+k)^{\frac{\theta}{2}}\right)^2 \sum_{n=1}^{\infty}\left(\lambda_n^{\frac{\theta}{2}}\right)^2 |(\varphi_0-u(0),e_n)|^2.$$

Putting

$$\sum_{n=1}^{\infty}\left(\lambda_n^{\frac{\theta}{2}}\right)^2 |(\varphi_0-u(0),e_n)|^2 = \|\varphi_0-u(0)\|^2_{H^{\frac{\theta}{4}}} = E^2$$

and using Lemma 9, we obtain

$$\begin{aligned}\|u_k(x,.)-u(x,.)\|^2 &\le (1/(\eta_1\omega))^\theta \left(\tau_{\frac{\theta}{2}}(k+1)^{-\frac{\theta}{2}}\right)^2 \|\varphi_0-u(0)\|^2 \\ &\le (1/(\eta_1\omega))^\theta E^2 \left(\tau_{\frac{\theta}{2}}(k+1)^{-\frac{\theta}{2}}\right)^2,\end{aligned}$$

which implies that

$$\|u_k(x,.)-u(x,.)\| \le (\frac{1}{\eta_1\omega})^{\frac{\theta}{2}} E\tau_{\frac{\theta}{2}}(k+1)^{-\frac{\theta}{2}} \le C_\theta \frac{E}{(k+1)^{\frac{\theta}{2}}} \to 0 \text{ as } k\to\infty,$$

where $C_\theta = \tau_{\frac{\theta}{2}}\,(1/\eta_1\omega)^{\theta/2}$. $\square$

Practically, most of the data taken from the experiment is affected by errors, which generates the difference in value between them and the exact data ψ. This difference is called the noise level that we note δ. Thereafter, one considers the disturbed case, with inexact data ψ^δ verifying the following inequality

$$\left\|\psi-\psi^\delta\right\| < \delta. \tag{5.5}$$

Theorem 8. *Let $0<\omega<\omega^*$ and $\varphi_0(y)$ be an arbitrary initial data element for the iterative procedure proposed above such that $(\varphi_0(y)-u(0,y))\in H^{\frac{\theta}{4}}$, $\theta>0$, let $u_k(x,y)$ be the k-th approximation solution for the exact data $\psi(y)$ and let $u_k^\delta(x,y)$ be the k-th approximation solution corresponding to the perturbed data $\psi^\delta(y)$ such that (5.5) holds. Then one has the following estimate:*

$$\|u(x,.)-u_k^\delta(x,.)\| \le C_\theta E/(k+1)^{\theta/2}+\omega\delta k. \tag{5.6}$$

Proof. Using the triangle inequality, we have

$$\left\|u(x,.)-u_k^\delta(x,.)\right\| \le \|u(x,.)-u_k(x,.)\| + \left\|u_k(x,.)-u_k^\delta(x,.)\right\|.$$

From (5.4), we write

$$\|u(x,.)-u_k(x,.)\| \le \sup_{x\ge 0}\|u(x,.)-u_k(x,.)\| \le C_\theta E/(k+1)^{\theta/2}. \tag{5.7}$$

On the other hand,

$$\begin{aligned}\left\|u_k(x,.)-u_k^\delta(x,.)\right\| &= \left\|K(x)(I-\omega K(L))^k(\varphi_k-\varphi_k^\delta)\right\| \\ &= \left\|\omega K(x)(I-\omega K(L))^k\sum_{i=0}^{k-1}(I-\omega K(L))^i(\psi-\psi^\delta)\right\| \\ &\le \left\|\omega\sum_{i=0}^{k-1}(I-\omega K(L))^i(\psi-\psi^\delta)\right\| \le \omega k\delta. \qquad (5.8)\end{aligned}$$

Combining (5.7) and (5.8), we obtain the estimate (5.6). □

Remark 3. If the number of iterations k depends on level noise δ such that $\omega k\delta \to 0$ as $\delta \to 0$, we obtain

$$\sup_{x\in[0,\infty)}\|u(x,.)-u_k^\delta(x,.)\| \xrightarrow[k\to\infty]{} 0.$$

Theorem 9. *Let $u_k(x,y)$ be the k-th approximation solution for the exact data $\psi(y)$ and let $u_k^\delta(x,y)$ be the k-th approximation solution corresponding to the perturbed data $\psi^\delta(y)$ such that* (5.5) *holds. For a priori choice of regularization parameter, we take $k=k(\delta)=[(E/\delta)^{\frac{2}{2+\theta}}]$, then,*

$$\sup_{x\ge 0}\left\|u(x,.)-u_k^\delta(x,.)\right\| \le (C_\theta+\omega)\,E^{\frac{2}{2+\theta}}\delta^{\frac{\theta}{2+\theta}}.$$

Proof. The proof results directly from the use of the Theorem 8. □

5.1 The convergence error estimate with an a posteriori parameter choice rule

Let $\tau>1$ be a fixed constant and $\varphi_0(y)=0_H$. We choose the regularization parameter as being the first integer $k=k(\delta,\psi^\delta)\in\mathbb{N}^*$, satisfying

$$\left\|u_k^\delta(L,.)-\psi^\delta\right\| \le \tau\delta \le \left\|u_{k-1}^\delta(L,.)-\psi^\delta\right\|, \qquad (5.9)$$

with $\left\|\psi-\psi^\delta\right\|\le\delta$ and $\left\|\psi^\delta\right\|\ge\tau\delta$.
The following lemma shows that there exist a uniqueness solution for inequality (5.9).

Lemma 2. *Let $\Phi(k)=\left\|u_k^\delta(L,.)-\psi^\delta\right\|$, then have the following statements*

1. *$\Phi(k)$ is a continuous function,* 2. $\lim_{k\to 0^+}\Phi(k)=\left\|\psi^\delta\right\|$,
3. $\lim_{k\to\infty}\Phi(k)=0$, 4. *$\Phi(k)$ is a strictly decreasing function for any $k\in(0,\infty)$.*

Lemma 3. *If the inequality* (5.9) *holds, then the regularization parameter $k=k(\delta,\psi^\delta)$ satisfies*

$$k \le C_\theta^{\frac{2}{2+\theta}}\left(E/\left((\tau-1)\delta\right)\right)^{\frac{2}{2+\theta}}. \qquad (5.10)$$

Proof. We have $\|\psi^\delta\| \geq \tau\delta$ and $\|\psi^\delta - \psi\| \leq \delta$. First, we get

$$\begin{aligned} u_{k-1}(L,y)-\psi(y) &= u_{k-1}(L,y) - \psi(y) + u^\delta_{k-1}(L,y) - u^\delta_{k-1}(L,y) + \psi^\delta(y) - \psi^\delta(y) \\ &= \left(u^\delta_{k-1}(L,y) - \psi^\delta(y)\right) - \left(\left(u^\delta_{k-1}(L,y) - u_{k-1}(L,y)\right) - \left(\psi^\delta(y) - \psi(y)\right)\right). \end{aligned}$$

Then, we get

$$\begin{aligned} \|u_{k-1}(L,.) - \psi\| &\geq \left\|u^\delta_{k-1}(L,.) - \psi^\delta\right\| - \left\|\left(u^\delta_{k-1}(L,.) - u_{k-1}(L,.)\right) - \left(\psi^\delta - \psi\right)\right\| \\ &\geq \tau\delta - \left\|\left(u^\delta_{k-1}(L,.) - u_{k-1}(L,.)\right) - \left(\psi^\delta - \psi\right)\right\|. \end{aligned}$$

On the other hand, one can have

$$\begin{aligned} \left(u^\delta_{k-1}(L,.) - u_{k-1}(L,.)\right) - \left(\psi^\delta(y) - \psi(y)\right) = &- \sum_{n=1}^{\infty} (1 - \omega K_n(L))^{k-1} \\ &\times ((\psi^\delta - \psi)(y), e_n) e_n(y) \end{aligned}$$

with $K_n(L) = \left(E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}}(-L^{\alpha+\beta}\sqrt{\lambda_n})\right)$. Since $\|(I - \omega K(L))\| \leq 1$, then

$$\left\|\left(u^\delta_{k-1}(L,.) - u_{k-1}(L,.)\right) - \left(\psi^\delta - \psi\right)\right\|^2 \leq \left\|\psi^\delta - \psi\right\|^2 \leq \delta^2.$$

Consequently,

$$\left\|u^\delta_{k-1}(L,.) - \psi\right\| \geq \delta\tau - \delta = \delta(\tau - 1).$$

We denote $\sigma_n(L) = \omega K_n(L)$, then, we get

$$\begin{aligned} \|u_{k-1}(L,.) - \psi\|^2 &= \sum_{n=1}^{\infty} (1 - \omega K_n(L))^{2(k-1)} |\psi_n|^2 \\ &= \sum_{n=1}^{\infty} (1 - \omega K_n(L))^{2(k-1)} (\omega K_n(L))^{(\theta+2)-(\theta+2)} |\psi_n|^2 \\ &\leq (\omega)^{-(\theta+2)} \left(\frac{\tau_{\theta+2}}{k^{\theta+2/2}}\right)^2 (1/\eta_1)^\theta E^2. \end{aligned}$$

Thus,

$$(\tau - 1)\delta \leq (1/\omega)^{\frac{\theta+2}{2}} \tau_{\theta+2} (1/\eta_1)^{\frac{\theta}{2}} (1/k)^{\frac{\theta+2}{2}} E,$$

so, $k^{\frac{\theta+2}{2}} \leq (C_\theta/(\tau - 1)) E/\delta$, which gives

$$k \leq ((C_\theta/(\tau - 1)) E/\delta)^{\frac{2}{\theta+2}},$$

where $C_\theta = \left(\frac{1}{\omega}\right)^{\frac{\theta+2}{2}} \tau_{\theta+2} \left(\frac{1}{\eta_1}\right)^{\frac{\theta}{2}}$. □

Theorem 10. *Let $u(x,y)$ given by (4.3) be the solution of problem (4.1) and (4.2). Let $u^\delta_k(x,y)$ the k-th regularized solution, such that $\|\psi - \psi^\delta\| \leq \delta$. If $\varphi(y) \in H^{\frac{\theta}{4}}$ and (5.10) holds, then we have*

$$\sup_{x\in\mathbb{R}^+} \left\|u^\delta_k(x,.) - u(x,.)\right\| \leq \left[\omega \left(\frac{C_\theta}{\tau - 1}\right)^{\frac{2}{2+\theta}} + \left(\frac{1+\tau}{\eta_1}\right)^{\frac{\theta}{2+\theta}}\right] E^{\frac{2}{2+\theta}} \delta^{\frac{\theta}{2+\theta}}.$$

Proof. We have

$$\left\|u_k^\delta(x,.)-u(x,.)\right\| \le \left\|u_k^\delta(x,.)-u_k(x,.)\right\| + \|u_k(x,.)-u(x,.)\|.$$

Estimating the right first term. By using (5.8) and (5.10), we get

$$\left\|u_k^\delta(x,.)-u_k(x,.)\right\| \le \omega(C_\theta)^{\frac{2}{2+\theta}}\,(1/(\tau-1))^{\frac{2}{2+\theta}}\,E^{\frac{2}{2+\theta}}\delta^{\frac{\theta}{2+\theta}}. \tag{5.11}$$

To estimate of the right second term, we write

$$\begin{aligned} u_k(L,y)-u(L,y) &= \sum_{n=0}^{\infty} -\left(1-\omega K_n(L)\right)^k (\psi(y),e_n)e_n(y) \\ &= \sum_{n=0}^{\infty} -\left(1-\omega K_n(L)\right)^k ((\psi-\psi^\delta)(y),e_n)e_n(y) \\ &\quad + \sum_{n=0}^{\infty} -\left(1-\omega K_n(L)\right)^k (\psi^\delta(y),e_n)e_n(y). \end{aligned}$$

Then,

$$\|u_k(L,.)-u(L,.)\| \le \left\|\psi-\psi^\delta\right\| + \left\|u_k^\delta(L,.)-\psi^\delta\right\| \le (\tau+1)\delta.$$

We also have

$$\begin{aligned} \|\varphi_k-\varphi\|_{H^{\frac{\theta}{4}}} &= \left(\sum_{n=1}^{+\infty}(1-\omega E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\sqrt{\lambda_n}L^{\alpha+\beta}))^{2k}|\varphi_n|^2\lambda_n^{\frac{\theta}{2}}\right)^{\frac{1}{2}} \\ &\le \left(\sum_{n=1}^{+\infty}|\varphi_n|^2\lambda_n^{\frac{\theta}{2}}\right)^{\frac{1}{2}} \le E. \end{aligned}$$

Consequently, we derive the following estimate

$$\begin{aligned} \|\varphi_k-\varphi\| &\le (1/\eta_1)^{\frac{\theta}{2+\theta}}\,E^{\frac{2}{2+\theta}}\|K(L)(\varphi_k-\varphi)\|^{\frac{\theta}{2+\theta}} \\ &\le (1\eta_1)^{\frac{\theta}{2+\theta}}\,E^{\frac{2}{2+\theta}}\left((1+\tau)\delta\right)^{\frac{\theta}{2+\theta}}. \end{aligned} \tag{5.12}$$

Since $\|u_k(x,.)-u(x,.)\| \le \|\varphi_k-\varphi\|$, then from (5.11) and (5.12), we obtain

$$\left\|u_k^\delta(x,.)-u(x,.)\right\| \le \left[\omega\left(C_\theta/(\tau-1)\right)^{\frac{2}{2+\theta}} + ((1+\tau)/\eta_1)^{\frac{\theta}{2+\theta}}\right]E^{\frac{2}{2+\theta}}\delta^{\frac{\theta}{2+\theta}}.$$

□

Remark 4. In order to improve the results of the regularized solution of the proposed iterative method, we use a variant of preconditioning technique [25], which is described as follows

$$\varphi_{k+1}(y) = \varphi_k(y) - \omega A^{-r}\left(u_k(L,y)-\psi(y)\right), \ \ r \ge 0, \tag{5.13}$$

where the parameter ω is chosen under the condition

$$0 < \omega < \omega^* = 1/\left\|A^{-r}E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-L^{\alpha+\beta}\sqrt{A})\right\|.$$

Theorem 11. *Let $\varphi_0(y)$ be an arbitrary element for the iterative scheme* (5.13) *and u_k be the k-th approximate solution. If $(\varphi_0 - u(0)) \in H^{\frac{\theta}{2}(r+1)}, r > 0$, then the rate of convergence of the preconditioned method is given by*

$$\left\| u(x,.) - u_k^{r,\delta}(x,.) \right\| \leq C_\theta \frac{E}{(k+1)^{\frac{\theta}{2}}} + \omega \frac{k}{\lambda_1^r} \delta,$$

where $C_\theta = \gamma_{\frac{\theta}{2}} \left(1/(\eta_1 \omega)\right)^{\frac{\theta}{2}}$.

6 Numerical implementation

In this section, a numerical example is given to illustrate the feasibility and efficiency of the proposed method. We take simple academic example to simplify the calculation task in `MATLAB` software and we use the `MATLAB` function $KS(alpha, beta, x, eps0)$ programmed by Richard Magin (2022). (https://www.mathworks.com/matlabcentral/fileexchange/70999-kilbas-and -saigo-function), MATLAB Central File Exchange. Retrieved January 6, 2022.

Now, we consider the problem of finding the function $u(0,y) = \varphi(y)$ in the following elliptic fractional equation

$$\begin{cases} D^{2\alpha} u(x,y) + x^{2\beta} u_{yy}(x,y) = 0, & (x,y) \in]0,\infty[\times (0,\pi),\ \alpha = \frac{3}{4},\ \beta = \frac{3}{8}, \\ u(x,0) = u(x,\pi) = 0, & \\ \lim\limits_{x \to \infty} u(x,y) = 0. & \end{cases}$$

From the supplementary condition $u(x=1,y) = \psi(y)$. Let A defined by

$$A = -\frac{\partial^2}{\partial y^2},\ D(A) = H_0^1(0,\pi) \cap H^2(0,\pi) \subset H = L^2(0,\pi),$$

with the eigenpairs $\lambda_n = n^2$, $e_n(y) = \sqrt{\frac{2}{\pi}} \sin(ny)$, $n \in \mathbb{N}^*$, which form a basis for $L^2(0,\pi)$.

Example 1. We take the boundary condition $\varphi(y) = (\pi - y)y$, and we compute a finite approximation of the given function by solving the direct problem

$$\begin{cases} D^{2\alpha} u(x,y) + x^{2\beta} u_{yy}(x,y) = 0, & (x,y) \in]0,\infty[\times (0,\pi),\ \alpha = \frac{3}{4},\ \beta = \frac{3}{8}, \\ u(0,y) = y \times (\pi - y), & y \in (0,\pi), \\ u(x,0) = u(x,\pi) = 0, & \\ \lim\limits_{x \to \infty} u(x,y) = 0. & \end{cases}$$

In this case, the finite approximation of the supplementary condition is given by the formula

$$\psi(y) = u(1,y) = \frac{2}{\pi} \sum_{n=1}^{50} E_{\frac{3}{4},\frac{3}{2},\frac{1}{2}}(-n) \left(\int_0^\pi \varphi(y) \sin(ny) dy \right) \sin(ny),$$

where the coefficients $(\varphi(y), \sin(ny)) = \sqrt{\frac{2}{\pi}} \int_0^\pi \varphi(y) \sin(ny) dy$ have been numerically evaluated by the trapezoidal rule to approximate the integral by using an equidistant grid

$$0 = y_0 < y_1 < ... < y_{M=1999} = \pi, \ h = \pi/M.$$

M is a constant parameter used in the computation of the integral, these data are taken as the "exact data" ψ.

Now, we add noise to the data ψ using a random perturbation (obtained by the MATLAB command randn), we obtain the vector ψ^δ:
$noise = randn(size(\psi))$, $noise = \varepsilon \times noise \times norm(\psi)/norm(noise)$, $\psi^\delta = \psi + noise$, where ε indicates the noise level of the measurement data and the function "$randn(.)$" generates arrays of random numbers whose elements are normally distributed with mean 0, variance $\sigma^2 = 1$, and standard deviation $\sigma = 1$. "$randn(size(\psi))$" returns an array of random entries that is the same size as ψ.

For practical problems, the *a priori* bound is very difficult to obtain. We only give numerical effectiveness under the *a posteriori* regularization parameter choice rule. In order to simplify the calculation, we take $\varphi_0 = 0$ and $\omega = 1$. The k-th regularized solution is given by

$$\varphi_k^{r,\delta} = \frac{2}{\pi} \sum_{n=1}^{50} \left[1 - \left(1 - \omega (n^2)^{-r} E_{\frac{3}{4},\frac{3}{2},\frac{1}{2}}(-n) \right)^k \right] \left(E_{\frac{3}{4},\frac{3}{2},\frac{1}{2}}(-n) \right)^{-1} I_n \sin(ny),$$

with $I_n = \int_0^\pi \psi^\delta \sin(ny) dy$.

We will now prove the efficiency and accuracy of the proposed regularization method without and with preconditioning, i.e., for $(r = 0)$ and $(r = 1, 2)$, with different noise levels $\delta = 5\%, \delta = 10\%$. The parameter $k = k(\delta, \psi^\delta)$ is chosen with the method a posteriori $\|u_k^\delta(1) - \psi^\delta\| \leq \delta\tau$, with $\tau = 1.0000001$. Our results are presented in Table 1.

Table 1. Numerical results of the Kozlov Ma'zya method with and without preconditioning.

δ=5%	$r = 0$	$r = 1$	$r = 2$
Iteration	$k = 9$	$k = 59$	$k = 560$
Relative Error	0.048725851178486	0.018972108585086	0.018560750043301
$\|\varphi_k - \varphi\|$	3.925590098131003	1.528484773502784	1.495343740983968
δ=10%	$r = 0$	$r = 1$	$r = 2$
Iteration	$k = 8$	$k = 12$	$k = 108$
Relative Error	0.085877526223264	0.033685999121789	0.033771003571961
$\|\varphi_k - \varphi\|$	6.918708620586942	2.713906917987948	2.720755287381651

The following figures visualize the comparison between the regularized and exact solutions in different cases, i.e., with different noise levels and different pre-conditioner parameters. Figures 1–2 show that the numerical results obtained are almost completely identical to the above in the theoretical convergence results, which means they are satisfactory.

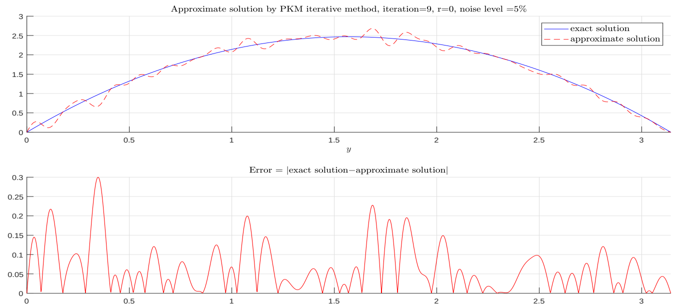

Figure 1. KM iteration method. Number of iterations=9, noise level=5 %, pre-conditioner parameter= 0, relative error= 0.04872.

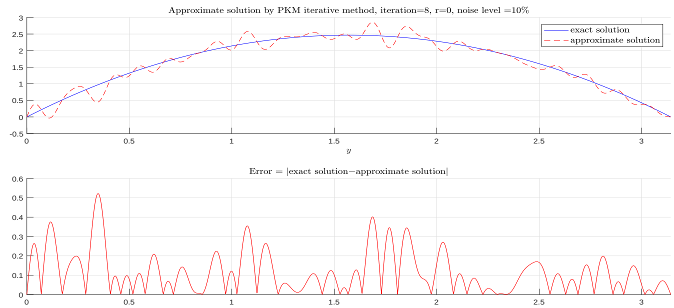

Figure 2. KM iteration method. Number of iterations= 8, noise level=10 %, pre-conditioner parameter= 0, relative error= 0.08587.

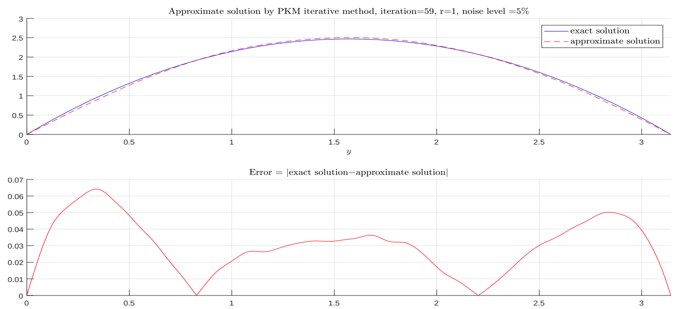

Figure 3. Preconditioning KM iteration method. Number of iterations=59 , noise level=5 %, pre-conditioner parameter=1, relative error= 0.01897.

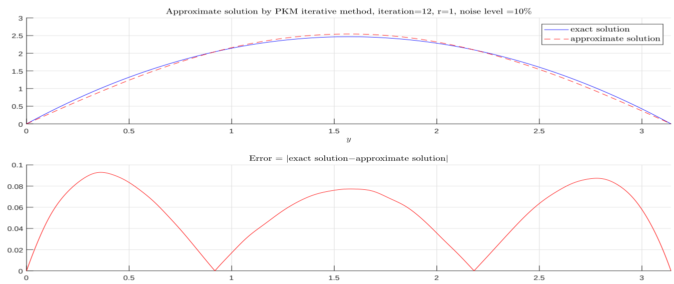

Figure 4. Preconditioning KM iteration method. Number of iterations= 12, noise level=10 %, pre-conditioner parameter=1, relative error=0.03368.

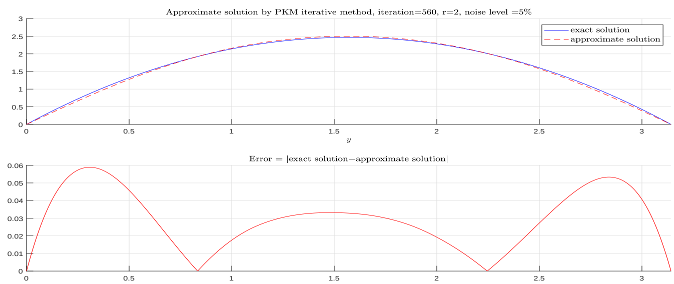

Figure 5. Preconditioning KM iteration method. Number of iterations=560, noise level=5 %, pre-conditioner parameter=2, relative error= 0.01856.

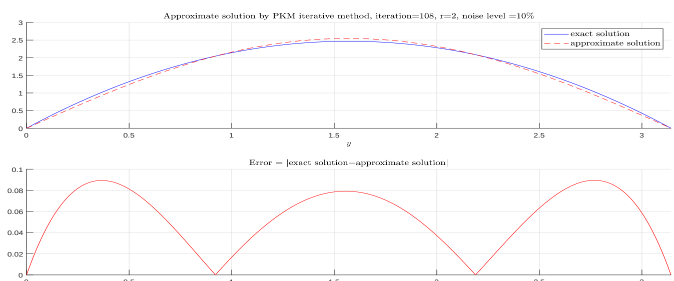

Figure 6. Preconditioning KM iteration method. Number of iterations= 108, noise level=10 %, pre-conditioner parameter=2, relative error= 0.03377.

The representative curves of Figures 3–6, show that the results of the convergence in the case of preconditioning are better than in the case of the adoption of the method without preconditioning (Figures 1–2). In addition, it was shown that in the case of the preconditioning, the convergence results are better whenever the pre-conditioner parameter is greater, of course, provided that the same noise level is maintained.

Example 2. In the second example we take the following piecewise function

$$\varphi(y) = \begin{cases} y^2, & y \in [0, \frac{11\pi}{24}], \\ (\frac{11\pi}{24})^2, & y \in [\frac{11\pi}{24}, \frac{13\pi}{24}], \\ (\pi - y)^2, & y \in [\frac{13\pi}{24}, \pi]. \end{cases}$$

We keep the same values of the parameters of Example 1 and reconstruct the regularized solution according to the a posteriori parameter choice rule.

The results obtained are presented in Table 2 and interpreted by the Figures 7–12.

Table 2. Numerical results of the Kozlov Ma'zya method with and without preconditioning.

δ=5%	$r = 0$	$r = 1$	$r = 2$
Iteration	$k = 14$	$k = 223$	$k = 6053$
Relative Error	0.123434071142749	0.053102037041275	0.054805004550911
$\lVert\varphi_k - \varphi\rVert$	5.908652537574834	2.541935812450566	2.623454984621273
δ=10%	$r = 0$	$r = 1$	$r = 2$
Iteration	$k = 10$	$k = 134$	$k = 1385$
Relative Error	0.182551193348186	0.067015425588387	0.075759759786856
$\lVert\varphi_k - \varphi\rVert$	8.738523827563448	3.207954341889514	3.626535953699377

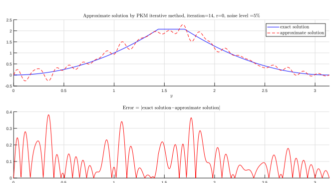

Figure 7. Pre-conditioner parameter=0, noise level=5 %.

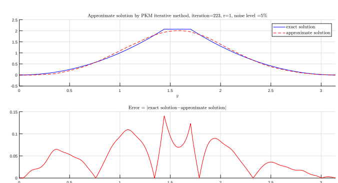

Figure 8. Pre-conditioner parameter=1, noise level=5 %.

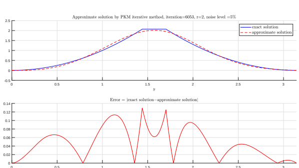

Figure 9. Pre-conditioner parameter=2, noise level=5 %.

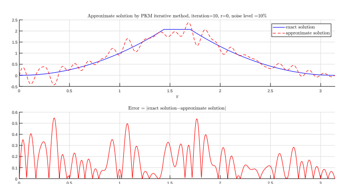

Figure 10. Pre-conditioner parameter=0, noise level=10 %.

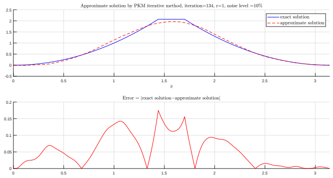

Figure 11. Pre-conditioner parameter=1, noise level=10 %.

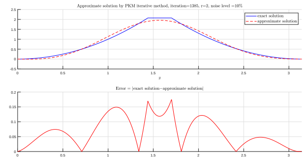

Figure 12. Pre-conditioner parameter=2, noise level=10 %.

Example 3. In the third example we deal with the following piecewise function

$$\varphi(y) = \begin{cases} 0, & y \in [0, \frac{\pi}{4}], \\ \frac{4}{\pi}y - 1, & y \in [\frac{\pi}{4}, \frac{\pi}{2}], \\ 3 - \frac{4}{\pi}y, & y \in [\frac{\pi}{2}, \frac{3\pi}{4}], \\ 0, & y \in [\frac{3\pi}{4}, \pi]. \end{cases}$$

We keep the same values of the parameters of Example 1 and reconstruct the regularized solution according to the a posteriori parameter choice rule.

The results obtained are presented in Table 3 and interpreted by the Figures 13–18. The results obtained in Examples 2 and 3 show that in the case of preconditioning, the convergence results are better than in the case without preconditioning.

Table 3. Numerical results of the Kozlov Ma'zya method with and without preconditioning.

δ=5%	$r = 0$	$r = 1$	$r = 2$
Iteration	$k = 12$	$k = 188$	$k = 4584$
Relative Error	0.287101671366959	0.127704136908131	0.131917716661285
$\|\varphi_k - \varphi\|$	2.330963755440529	2.541935812450566	2.407873571544033
δ=10%	$r = 0$	$r = 1$	$r = 2$
Iteration	$k = 8$	$k = 109$	$k = 1124$
Relative Error	0.398649736026768	0.163394382516553	0.182331945467937
$\|\varphi_k - \varphi\|$	7.276491649309832	2.982412259382508	3.328076651506347

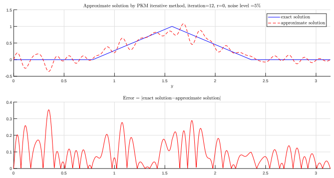

Figure 13. Pre-conditioner parameter=0, noise level=5 %.

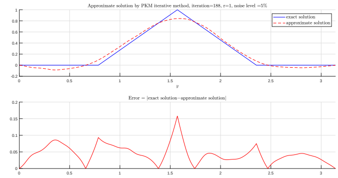

Figure 14. Pre-conditioner parameter=1, noise level=5 %.

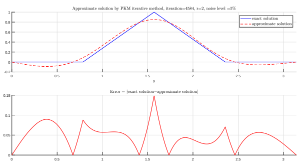

Figure 15. Pre-conditioner parameter=2, noise level=5 %.

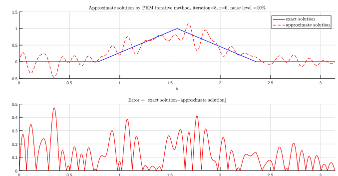

Figure 16. Pre-conditioner parameter=0, noise level=10 %.

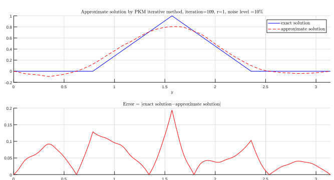

Figure 17. Pre-conditioner parameter=1, noise level=10 %.

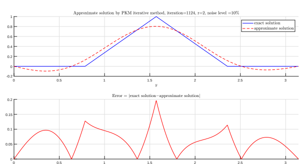

Figure 18. Pre-conditioner parameter=2, noise level=10 %.

7 Conclusions

In this work, we perform an analytical study of an inverse problem for a fractional elliptic equation defined in a banded domain, where, we seek to recover the missing boundary data. This problem is ill-posed, i.e., the solution (if it exists) does not depend continuously on the data. An iterative method was selected and applied to obtain the regularized solution. Moreover, under two regularization parameter choice rules, we establish Hölder type error estimates. Especially, the a posteriori regularization parameter choice is used. Finally, a numerical results show that the iterative method is very effective for this kind of ill-posed problems, in particular, when using the pre-conditioned version.

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