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Contraction-Mapping Algorithm for the Equilibrium Problem over the Fixed Point Set of a Nonexpansive Semigroup

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Abstract. In this paper, we consider the proximal mapping of a bifunction. Under the Lipschitz-type and the strong monotonicity conditions, we prove that the proximal mapping is contractive. Based on this result, we construct an iterative process for solving the equilibrium problem over the fixed point sets of a nonexpansive semigroup and prove a weak convergence theorem for this algorithm. Also, some preliminary numerical experiments and comparisons are presented.

Keywords: bilevel optimization, contractive mapping, nonexpansive semigroup, equilibrium problem, strong monotonicity.

AMS Subject Classification: 47H05; 47H09; 47J25; 65K10; 90C25.

1 Introduction

Let C be a nonempty, closed, convex subset of a real Hilbert space H and $f : C \times C \to \mathbb{R}$ be a bifunction such that $f(x, x) = 0$ for all $x \in C$. Such a bifunction is called an equilibrium bifunction. The equilibrium problem for f on C can be formulated as

$$\text{find } x^* \in C \text{ such that } f(x^*, y) \geq 0, \quad \forall y \in C. \qquad (EP(f, C))$$

It is well-known that equilibrium problems include, as special cases, many important problems, such as optimization problems, variational inequalities, saddle point problems, Nash equilibrium problems, fixed point problems, and others; see, for instance, [1, 9, 10, 11, 12, 14, 19, 20, 25] and references therein. Equilibrium problems have been generalized and extensively studied in many directions due to its importance. For example, in [2, 24], the authors consider the problem of finding a common element of the solution set of equilibrium problems and the solution set of fixed point problems.

$$\text{Find } x^* \in C \text{ such that } \begin{cases} f_i(x^*, y) \geq 0, & \forall y \in C, i = 1, \ldots, p, \\ T_j(x^*) = x^*, & \forall j = 1, \ldots, q, \end{cases}$$

where $f_i : C \times C \to \mathbb{R}$, $i = 1, \ldots, p$, are equilibrium bifunctions and $T_j : C \to C$, $j = 1, \ldots, q$, are nonexpansive mappings. This approach is very interesting from the theoretical point of view, it gives us the ability to combine techniques in the area of fixed point theory and the one of the equilibrium problems.

Another approach of equilibrium problems comes from real-world problems. In [11, 12], Iiduka considered the power control problem for the CDMA data networks. This problem can be modeled as an equilibrium problem with an implicit constraint set. In this situation, all existing solving method for equilibrium problems cannot be applied directly. To overcome this difficulty, the author introduced a nonexpansive mapping T, the fixed point set of which is coincident with the constraint set. The initial equilibrium problem now becomes

$$\text{find } x^* \in Fix(T) \text{ such that } f(x^*, y) \geq 0 \text{ for all } y \in Fix(T), \tag{1.1}$$

where $Fix(T) := \{x \in H : T(x) = x\}$ is the fixed point set of T. Although this problem is very interesting, it is hard to solve. To the best of our knowledge, there are very few methods for solving (1.1). In [13], the authors proposed a subgradient method:

$$\begin{cases} \text{Choose } x^1 \in \mathbb{R}^m. \text{ Set } \rho_1 := \|x^1\| \text{ and } n = 1. \\ \text{Find } y^n \in K_n := \{x \in \mathbb{R}^m : \|x\| \leq \rho_n + 1\} \text{ such that} \\ \qquad f(x^n, y^n) \geq 0 \text{ and } \max_{y \in K_n} f(y, x^n) \leq f(y^n, x^n) + \epsilon_n, \\ x^{n+1} = T(x^n - \lambda_n f(y^n, x^n)\xi_n); \ \rho_{n+1} = \max\{\rho_n, \|x^{n+1}\|\}. \end{cases}$$

Under the assumptions that $\{\xi_k\}$ is bounded, i.e., $\|\xi_k\| \leq N$, $\forall n \geq 1$, $\lambda_n \in [a, b] \subset (0, 2/N^2)$, $\forall n \geq 1$ and the set $\Omega := \{u \in Fix(T) : f(y^n, u) \leq 0, \ n \geq 1\}$ is nonempty, the authors proved that the sequence $\{x^n\}$ generated by the above algorithm converges to a solution of (1.1). However, as we can see, the assumptions of this algorithm are very difficult to verify.

Very recently, in [8], Hai introduced a contraction-mapping algorithm for solving (1.1). Under the assumptions that f is strongly monotone and Lipschitz type continuous, the author proved that the mapping U_λ defined by

$$U_\lambda : C \to C, \quad x \mapsto \text{argmin}\Big\{\lambda f(x, y) + \frac{1}{2}\|y - x\|^2 : y \in C\Big\}$$

is contractive when λ is small enough. Moreover, the sequence $\{x^n\}$ generated by

$$x^0 \in C, \quad x^{n+1} = U_{\lambda_n} T x^n \tag{1.2}$$

converges to a solution of (1.1). Note that Algorithm (1.2) is still applicable in the case when C is the set of common fixed points of a finite family of nonexpansive mappings $\{T_j\}_{j=1}^n$. In this case, it is sufficient to set $T = \sum_{j=1}^n w_j T_j$, where $w_j \in (0,1)$, $\sum_{j=1}^n w_j = 1$. However, this is not true if the family $\{T_j\}$ is infinite and how to solve the equilibrium problem over the fixed point set of an infinite of family of nonexpansive mappings is still an open question.

In this paper, we will find the answer to this question in a special case, when the constraint set C is given by the fixed point set of a nonexpansive semigroup. More precisely, let $T(s)_{s\in\mathbb{R}^+}$ be a nonexpansive semigroup on H with nonempty fixed point set $\bigcap_{s\in\mathbb{R}^+} Fix(T(s))$. We consider the problem:

$$\text{find } x^* \in \bigcap_{s\in\mathbb{R}^+} Fix(T(s)) \text{ such that } f(x^*, y) \geq 0 \text{ for all } y \in \bigcap_{s\in\mathbb{R}^+} Fix(T(s)). \quad (1.3)$$

Motivated by the results in [8], we introduce a contraction algorithm for solving problem (1.3). In contrast to [8], the main contribution of this paper is twofold: the considered problem is more general and the convergence conditions are weakened. In this paper, we assume that the bifunction f is Lipschitz-type continuous on bounded sets instead of the whole space H. Moreover, to prove the contraction of the proximal mapping U_λ, we introduce a new type of the Lipschitz continuity, which is a relaxation of the one used in [8].

The rest of the paper is organized as follows. Section 2 briefly explains the necessary mathematical background. In Section 3, we prove the contraction of the proximal mapping. In the next section, we introduce a new algorithm for solving (1.3) and perform a convergence analysis on it. We close the paper with some computational experiments in Section 5.

2 Preliminaries

In this section, we present some basic concepts, properties, and notations that we will use in the sequel. The reader is referred to, for example, [7, 21] for more details. Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$, respectively. Let C be a nonempty closed convex set in H. For a given bifunction $f : C \times C \to \mathbb{R}$, consider the problem of finding $x^* \in C$ such that

$$f(x^*, y) \geq 0, \quad \forall y \in C, \quad (2.1)$$

which is known as the *equilibrium problem*, considered and investigated by Fan [6], Blum, Muu and Oettli [5, 17]. If we have $f(x,y) := \langle F(x), y - x\rangle$, where $F : C \to C$ is a mapping, then problem (2.1) collapses to the problem

$$\text{find } x^* \in C \text{ such that } \langle F(x^*), y - x^*\rangle \geq 0, \quad \forall y \in C,$$

which is called the *variational inequality*, introduced and studied by Stampacchia [23]. Let us start with some well known definitions arising from algorithms for variational inequalities and equilibrium problems.

Definition 1. [20] (I) The mapping $F : C \to H$ is said to be

1. γ-strongly monotone on C iff there exists a constant $\gamma > 0$ such that for all $x, y \in C$,
$$\langle F(x) - F(y), x - y\rangle \geq \gamma\|x - y\|^2;$$

2. monotone on C iff for all $x, y \in C$,
$$\langle F(x) - F(y), x - y\rangle \geq 0;$$

3. γ-strongly pseudomonotone on C iff there exists a constant $\gamma > 0$ such that for all $x, y \in C$,
$$\langle F(y), x - y\rangle \geq 0 \Rightarrow \langle F(x), x - y\rangle \geq \gamma\|x - y\|^2;$$

4. pseudomonotone on C iff for all $x, y \in C$,
$$\langle F(y), x - y\rangle \geq 0 \Rightarrow \langle F(x), x - y\rangle \geq 0.$$

(II) The bifunction $f : C \times C \to \mathbb{R}$ is said to be

1. γ-strongly monotone on C iff there exists a constant $\gamma > 0$ such that for all $x, y \in C$,
$$f(x, y) + f(y, x) \leq -\gamma\|x - y\|^2;$$

2. monotone on C iff for all $x, y \in C$,
$$f(x, y) + f(y, x) \leq 0;$$

3. γ-strongly pseudomonotone on C iff there exists a constant $\gamma > 0$ such that for all $x, y \in C$,
$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq -\gamma\|x - y\|^2;$$

4. pseudomonotone on C iff for all $x, y \in C$,
$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0.$$

The definitions of monotonicity for bifunctions are generalizations of the ones for mappings. If the mapping F is γ-strongly monotone (monotone, γ-strongly pseudomonotone, pseudomonotone) then so is the bifunction $f(x, y) := \langle F(x), y - x\rangle$.

DEFINITION 2. The mapping $F : C \to H$ is said to be Lipschitz continuous on C iff there exists a constant $L > 0$ such that for all $x, y \in C$,
$$\|F(x) - F(y)\| \leq L\|x - y\|. \tag{2.2}$$

If $L = 1$, then F is said to be nonexpansive and if $L \in (0, 1)$, then we call F contractive.

Definition 3. The bifunction $f : C \times C \to \mathbb{R}$ is said to be Lipschitz-type continuous on C iff it satisfies one of the following conditions:

$$f(x,y) + f(y,z) - f(x,z) \geq \sum_{i=1}^{p} \langle u_i(x,y), v_i(y-z) \rangle, \quad \forall x,y,z \in C,$$

$$f(x,y) + f(y,z) - f(x,z) \geq \sum_{i=1}^{p} \langle u_i(y,z), v_i(x-y) \rangle, \quad \forall x,y,z \in C, \tag{2.3}$$

where $v_i : C \to C$, $(i = 1, \ldots, p)$ are α_i-Lipschitz mappings satisfying $v_i(0) = 0$, $u_i : C \times C \to C$, $(i = 1, \ldots, p)$ are mappings satisfying $u_i(x,y) + u_i(y,x) = 0$ and there exist $\beta_i > 0$ such that $\|u_i(x,y)\| \leq \beta_i \|x-y\|,\ \forall x,y \in C,\ i = 1, \ldots, p$. The constant $L := \sum_{i=1}^{p} \alpha_i \beta_i$ is called the Lipschitz constant of the bifunction f.

We note that there exist several definitions of the Lipschitz-type continuity of bifunctions [1, 4, 8, 15]. The Lipschitz-type condition used in this paper is a relaxation of the one introduced in [8]. Moreover, if $f(x,y) := \langle F(x), y - x\rangle$, where $F : C \to C$ is a mapping, then the Lipschitz-type condition (2.3) collapses into the classical one (2.2). In the next Proposition, we establish the relationship between the constants of the Lipschitz-type continuity and the strong monotonicity of a bifunction.

Proposition 1. *Let $f : C \times C \to \mathbb{R}$ be a L-Lipschitz-type continuous and γ-strongly monotone equilibrium bifunction. Then $\gamma \leq L$.*

Proof. In the condition (2.3), let $x = z \neq y$. It implies that

$$-\gamma\|x-y\|^2 \geq f(x,y) + f(y,x) \geq -\sum_{i=1}^{p} \alpha_i \beta_i \|x-y\| \|y-x\| = -L\|x-y\|^2,$$

which implies the desired result. □

Let $f : C \times C \to \mathbb{R}$ be a bifunction such that for all $x \in C$, the function $f(x,.)$ is proper, convex and lower semicontinuous. For a constant $\lambda > 0$, the *proximal mapping* U_λ of the bifunction f is defined as

$$U_\lambda : C \to C, \quad U_\lambda(x) := \operatorname{argmin}\Big\{\lambda f(x,y) + \frac{1}{2}\|x-y\|^2 : y \in C\Big\}. \tag{2.4}$$

The term "proximal mapping" was first used by Moreau [16] in 1963 to describe the mapping

$$\operatorname{Prox}_g(z) = \operatorname{argmin}\Big\{g(u) + \frac{1}{2}\|u-z\|^2 : u \in C\Big\},$$

where g is a lower semicontinuous, proper and convex function. This mapping is known to be a very useful tool in optimization. In practice, the proximal mapping can be computed easily by the Matlab Optimization Toolbox.

Let $g : C \to \mathbb{R}$ be a function. We call the set

$$\partial g(x) := \{w \in H : g(y) - g(x) \geq \langle w, y - x\rangle, \quad \forall y \in C\}.$$

the subdifferential of g at x . The normal cone $N_C(x)$ of C at $x \in C$ is

$$N_C(x) := \{q \in H : \langle q, y - x\rangle \leq 0, \ \forall y \in C\}.$$

It is well known that $N_C(x)$ is nonempty, closed and convex. For all $x \in H$, the following problem

$$\min\{\|y - x\| : y \in C\}$$

has a unique solution, denoted by $P_C(x)$. The mapping $x \mapsto P_C(x)$ is called the *projection onto* C.

DEFINITION 4. A family $\{T(s) : \ s \in \mathbb{R}_+\}$ of mappings from C into itself is called a *nonexpansive semigroup* on C iff:

(i) The mapping $T(s)$ is nonexpansive on C for all $s \in \mathbb{R}_+$;

(ii) $T(0)x = x$, $\forall x \in C$;

(iii) $T(s_1 + s_2) = T(s_1) \circ T(s_2)$, $\forall s_1, s_2 \in \mathbb{R}_+$;

(iv) for each $x \in C$, the function $t \mapsto T(t)x$ is continuous.

The following lemmas are needed for further investigation.

Lemma 1. [21] *Let $f : C \to \mathbb{R}$ be convex and subdifferentiable on C. Then, x^* is a solution of the problem* $\min\{f(x) : x \in C\}$ *if and only if* $0 \in \partial f(x^*) + N_C(x^*)$.

Lemma 2. [22] *Let C be a nonempty bounded closed convex subset of H and let $\{T(s) : \ s \in \mathbb{R}_+\}$ be a nonexpansive semigroup on C. Then, for any $h \geq 0$*

$$\lim_{s\to\infty} \sup_{y\in C} \left\| T(h)\Big(\frac{1}{s}\int_0^s T(t)y \, \mathrm{d}t\Big) - \frac{1}{s}\int_0^s T(t)y \, \mathrm{d}t \right\| = 0$$

Lemma 3. [18] *Let $\{\alpha_k\}$, $\{\beta_k\}$, $\{\lambda_k\}$ be sequences of nonnegative numbers satisfying*

$$\alpha_{k+1} \leq (1 - \lambda_k)\alpha_k + \lambda_k t_k + \beta_k, \ \forall k \geq 1.$$

If $\lambda_k \in (0, 1)$, $\forall k \geq 1$, $\sum_{k=1}^\infty \lambda_k = \infty$, $\lim t_k = 0$ and $\sum_{k=1}^\infty \beta_k < \infty$, then $\lim_{k\to\infty} \alpha_k = 0$.

3 Contraction of the proximal mapping

In this section, we give a sufficient condition for the contraction of the proximal mapping U_λ defined by (2.4). Note that if $f(x, y) := \langle F(x), y - x\rangle$ then $U_\lambda = P_C(x - \lambda F(x))$ for all $x \in C$. We first recall the contraction of the mapping $x \mapsto P_C(x - \lambda F(x))$.

Proposition 2. *Suppose that the mapping $F : C \to C$ is γ-strongly monotone and L-Lipschitz continuous. Then for $\lambda \in (0, \frac{2\gamma}{L^2})$, the mapping $x \mapsto P_C(x - \lambda F(x))$ is contractive.*

This result can be generalized as follows.

Theorem 1. *Let the bifunction $f : C \times C \to \mathbb{R}$ be γ-strongly monotone and L-Lipschitz-type continuous on C. Suppose that $\lambda \in (0, \frac{\gamma}{L^2})$. Then for all $x, y \in C$, we have*

$$\|U_\lambda(x) - U_\lambda(y)\| \leq (1 - \lambda\gamma/8)\|x - y\|.$$

Proof. Since $U_\lambda(x)$ is the unique solution of the problem

$$\min\{\lambda f(x, y) + 0.5\|y - x\|^2 : y \in C\},$$

from Lemma 1, it follows that there exist $w \in \partial f(x, .)(U_\lambda(x))$, $q \in N_C(U_\lambda(x))$ such that

$$0 = \lambda w + U_\lambda(x) - x + q.$$

From the definition of $N_C(U_\lambda(x))$, for all $y \in C$ we have

$$\langle x - U_\lambda(x) - \lambda w, y - U_\lambda(x)\rangle \leq 0.$$

Since $f(x, .)$ is convex, applying the definition of $\partial f(x, .)(U_\lambda(x))$, we obtain

$$\langle x - U_\lambda(x), y - U_\lambda(x)\rangle \leq \langle \lambda w, y - U_\lambda(x)\rangle \leq \lambda \left[f(x, y) - f(x, U_\lambda(x))\right]. \quad (3.1)$$

In (3.1), taking $y = U_\lambda(y) \in C$, we deduce

$$\langle x - U_\lambda(x), U_\lambda(y) - U_\lambda(x)\rangle \leq \lambda \left[f(x, U_\lambda(y)) - f(x, U_\lambda(x))\right].$$

Analogously,

$$\langle y - U_\lambda(y), U_\lambda(x) - U_\lambda(y)\rangle \leq \lambda \left[f(y, U_\lambda(x)) - f(y, U_\lambda(y))\right].$$

Adding the last two inequalities, we get

$$\begin{aligned}\|U_\lambda(x) - U_\lambda(y)\|^2 \leq \lambda &\left[f(y, U_\lambda(x)) - f(y, U_\lambda(y)) + f(x, U_\lambda(y)) - f(x, U_\lambda(x))\right] \\ &+ \langle U_\lambda(x) - U_\lambda(y), x - y\rangle.\end{aligned}$$

We consider two cases:

Case 1: $f(x, y) + f(y, z) - f(x, z) \geq \sum_{i=1}^{p} \langle u_i(x, y), v_i(y - z)\rangle, \quad \forall x, y, z \in C.$

It follows that

$$f(y, U_\lambda(x)) - f(x, U_\lambda(x)) \leq f(y, x) + \sum_{i=1}^{p} \langle u_i(x, y), v_i(x - U_\lambda(x))\rangle,$$

$$f(x, U_\lambda(y)) - f(y, U_\lambda(y)) \leq f(x, y) + \sum_{i=1}^{p} \langle u_i(y, x), v_i(y - U_\lambda(y))\rangle.$$

Adding these inequalities and using the γ-strong monotonicity of the bifunction f, we arrive at

$$\begin{aligned}\|U_\lambda(x)-U_\lambda(y)\|^2 &\le \lambda[f(x,y)+f(y,x)+\sum_{i=1}^{p}\langle u_i(y,x),v_i(y-U_\lambda(y))\\ &\quad -v_i(x-U_\lambda(x))\rangle] \quad +\langle U_\lambda(x)-U_\lambda(y),x-y\rangle\\ &\le \lambda[-\gamma\|x-y\|^2+\sum_{i=1}^{p}\alpha_i\beta_i\|x-y\|\|x-y-U_\lambda(x)+U_\lambda(y)\|]\\ &\quad +\frac{1}{2}[\|x-y\|^2+\|U_\lambda(x)-U_\lambda(y)\|^2-\|x-y-U_\lambda(x)+U_\lambda(y)\|^2].\end{aligned}$$

Hence, using the fact that $0<\lambda<\frac{\gamma}{L^2}$, we have

$$\begin{aligned}\|U_\lambda(x)-U_\lambda(y)\|^2 &\le (1-\lambda\gamma)\|x-y\|^2-\|x-y-U_\lambda(x)+U_\lambda(y)\|^2\\ &\quad +L\lambda\|x-y\|\|x-y-U_\lambda(x)+U_\lambda(y)\|\\ &=(1-\frac{3}{4}\lambda\gamma)\|x-y\|^2-[\|x-y-U_\lambda(x)+U_\lambda(y)\|-\frac{1}{2}L\lambda\|x-y\|]^2\\ &\quad +\frac{1}{4}\lambda(L^2\lambda-\gamma)\|x-y\|^2\le(1-\frac{3}{4}\lambda\gamma)\|x-y\|^2. \end{aligned}\tag{3.2}$$

Case 2: $f(x,y)+f(y,z)-f(x,z)\ge\sum_{i=1}^{p}\langle u_i(y,z),v_i(x-y)\rangle, \quad \forall x,y,z\in C.$
It follows that

$$\begin{aligned}f(y,U_\lambda(x))-f(y,U_\lambda(y)) &\le f(U_\lambda(y),U_\lambda(x))\\ &\quad +\sum_{i=1}^{p}\langle u_i(U_\lambda(x),U_\lambda(y)),v_i(y-U_\lambda(y))\rangle,\\ f(x,U_\lambda(y))-f(x,U_\lambda(x)) &\le f(U_\lambda(x),U_\lambda(y))\\ &\quad +\sum_{i=1}^{p}\langle u_i(U_\lambda(y),U_\lambda(x)),v_i(x-U_\lambda(x))\rangle.\end{aligned}$$

Adding these inequalities and applying analogous arguments to *Case 1*, we obtain

$$\begin{aligned}\|U_\lambda(x)-U_\lambda(y)\|^2 &\le \|x-y\|^2-\lambda\gamma\|U_\lambda(x)-U_\lambda(y)\|^2-\|x-y-U_\lambda(x)+U_\lambda(y)\|^2\\ &\quad +L\lambda\|U_\lambda(x)-U_\lambda(y)\|\|x-y-U_\lambda(x)+U_\lambda(y)\|\\ &\le\|x-y\|^2-\frac{3}{4}\lambda\gamma\|U_\lambda(x)-U_\lambda(y)\|^2.\end{aligned}$$

Hence,

$$\|U_\lambda(x)-U_\lambda(y)\|^2\le\big(1-3\lambda\gamma/(4+3\lambda\gamma)\big)\|x-y\|^2. \tag{3.3}$$

Combining (3.2) and (3.3), we deduce that the function U_λ is contractive with the constant $\tau=\sqrt{1-\frac{3\lambda\gamma}{4+3\lambda\gamma}}$. On the other hand, from Proposition 1, we have

$0 < \lambda < \frac{\gamma}{L^2} < \frac{4}{3\gamma}$. Hence,

$$\sqrt{1 - \frac{3\lambda\gamma}{4 + 3\lambda\gamma}} < 1 - \frac{\lambda\gamma}{4 + 3\lambda\gamma} < 1 - \frac{\lambda\gamma}{8}.$$

We obtain the desired result. □

4 Contraction algorithm for the equilibrium over fixed point sets

Let $f : H \times H \to \mathbb{R}$ be an equilibrium bifunction and $\{T(s)\}_{s\in\mathbb{R}^+}$ be a nonexpansive semigroup of H. In this section, we introduce a new algorithm for solving the problem

$$\text{find } x^* \in \bigcap_{s\in\mathbb{R}^+} Fix(T(s)) \text{ such that } f(x^*, y) \geq 0 \text{ for all } y \in \bigcap_{s\in\mathbb{R}^+} Fix(T(s)). \tag{4.1}$$

Assumption 4.1 *We consider problem* (4.1) *under the following assumptions.*

(A1) *The bifunction f is γ- strongly monotone and L-Lipschitz-type continuous on H.*

(A2) *For all $x \in H$, the function $f(x, .)$ is lower semicontinuous, convex and the function $f(., x)$ is weakly upper semicontinuous on H.*

(A3) *For any bounded sequences $\{x^n\}$ and $\{y^n\}$, we have*

$$\sup\left\{\frac{f(x^n, y^n)}{\|x^n - y^n\|} : n \in \mathbb{N}, \|x^n - y^n\| \neq 0\right\} < \infty.$$

(A4) *The set $\bigcap_{s\in\mathbb{R}^+} Fix(T(s))$ is nonempty.*

Under Assumption 4.1, problem (4.1) has a unique solution x^*. To find this solution, we propose the following algorithm:

Algorithm 1. (Contraction algorithm for the equilibrium problem)

Step 0 (initialization) Set $n = 0$.

Choose $x^0 \in H$ arbitrarily and the sequences $\{\lambda_n\}, \{s_n\} \subset (0, 1)$ satisfying: $\sum_{n=1}^{\infty} \lambda_n = \infty$, $\lim_{n\to\infty} \lambda_n = 0$, $\lim_{n\to\infty} s_n = \infty$, $\lim_{n\to\infty} \frac{\lambda_{n-1}-\lambda_n}{\lambda_n^2} = 0$, $\lim_{n\to\infty} \frac{s_n - s_{n-1}}{s_n \lambda_n^2} = 0$, $\lambda_n < \lambda_{n-1}$, $s_{n-1} < s_n$, $\forall n \geq 1$.

Step 1 (iterative step) Given x^n, compute x^{n+1} as follows:

$$\begin{cases} y^n = \frac{1}{s_n} \int_0^{s_n} T(t)x^n \, \mathrm{d}t; \\ x^{n+1} = U_{\lambda_n} y^n. \end{cases}$$

Step 2 Set $n := n + 1$ and go back to **Step 1**.

Recall that U_λ is the proximal mapping of f, defined by

$$U_\lambda : H \to H, \quad U_\lambda(x) := \text{argmin}\left\{\lambda f(x, y) + 0.5\|y - x\|^2 : y \in H\right\}.$$

Before proving the convergence theorem, we need the following lemma:

Lemma 4. [8] *Suppose that the bifunction f satisfies assumption* **(A2)** *and* $\lim_{n\to\infty}\lambda_n = 0$*, then for all $x \in H$, we have*

$$\lim_{n\to\infty} U_{\lambda_n} x = x.$$

Now we are in a position to prove our convergence theorem.

Theorem 2. *Suppose that Assumption 4.1 is satisfied. Then the sequence $\{x^n\}$ generated by Algorithm 1 weakly converges to the unique solution of problem* (4.1).

Proof. Let

$$T_n : H \to H, \quad T_n x := \frac{1}{s_n}\int_0^{s_n} T(t)x\,\mathrm{dt}, \quad \forall x \in H,\ n \geq 1.$$

Since $\lambda_n \to 0$, without loss of generality, we can assume that $\lambda_n \in (0, \frac{2\gamma}{L^2})$, and hence, from Theorem 1, it follows that the mapping $U_{\lambda_n}T_n$ is contractive with the constant $r_n = 1 - \lambda_n \frac{\gamma}{8}$ for all $n \geq 1$. Let z^n be the unique fixed point of the mapping $U_{\lambda_n}T_n$. First, we investigate the behavior of the sequence $\{z^n\}$.
Claim 1. The sequences $\{z^n\}$ and $\{T_n z^n\}$ are bounded.
Let $p \in \bigcap_{s\in\mathbb{R}^+} Fix(T(s))$. It is easy seen that $p \in Fix(T_n)$ for all $n \geq 1$. We have

$$\begin{aligned}\|z^n - p\| &\leq \|U_{\lambda_n}T_n z^n - U_{\lambda_n}T_n p\| + \|U_{\lambda_n}p - p\| \\ &\leq (1-\lambda_n\gamma/8)\|z^n - p\| + \|U_{\lambda_n}p - p\|. \end{aligned} \tag{4.2}$$

Applying a analogous argument to (3.1), we have

$$\langle p - U_{\lambda_n}p, y - U_{\lambda_n}p\rangle \leq \lambda_n[f(p,y) - f(p, U_{\lambda_n}p)], \quad \forall y \in H, n \geq 1.$$

Let $y = p$, the last inequality becomes

$$\|p - U_{\lambda_n}p\|^2 \leq \lambda_n |f(p, U_{\lambda_n}p)|. \tag{4.3}$$

Since the sequence $\{U_{\lambda_n}p)\}$ is bounded (see Lemma 4), from Assumption **(A3)**, it follows that there exists a constant $M > 0$ such that $|f(p, U_{\lambda_n}p)| \leq M\|p - U_{\lambda_n}p\|$ for all $n \geq 1$. Combining this and (4.3), we arrive at

$$\|p - U_{\lambda_n}p\| \leq M\lambda_n. \tag{4.4}$$

From (4.2) and (4.4), we get

$$\|z^n - p\| \leq (1-\lambda_n\gamma/8)\|z^n - p\| + \lambda_n M,$$

or equivalently,

$$\|z^n - p\| \leq 8M/\gamma.$$

And so, the sequence $\{z^n\}$ is bounded. Moreover, for all $n \geq 1$, we have

$$\|T_n z^n - p\| = \Big\|\frac{1}{s_n}\int_0^{s_n} T(t)z^n\,\mathrm{dt} - p\Big\| \leq \frac{1}{s_n}\int_0^{s_n}\|T(t)z^n - T(t)p\|\,\mathrm{dt} \leq \|z^n - p\|,$$

which implies the boundedness of $\{T_n z^n\}$. Since the sequence $\{z^n\}$ is bounded, there exists a subsequence $\{z^{n_i}\} \subset \{z^n\}$ such that $z^{n_i} \rightharpoonup z^*$.
Claim 2: $z^* \in \bigcap_{s\in\mathbb{R}^+} Fix(T(s))$.
For all $s > 0$, we have

$$\begin{aligned}\|T(s)z^{n_i} - z^{n_i}\| &\le \|T(s)z^{n_i} - T(s)T_{n_i}z^{n_i}\| + \|T(s)T_{n_i}z^{n_i} - T_{n_i}z^{n_i}\| \\ &+ \|T_{n_i}z^{n_i} - z^{n_i}\| \le 2\|z^{n_i} - T_{n_i}z^{n_i}\| + \|T(s)T_{n_i}z^{n_i} - T_{n_i}z^{n_i}\|. \end{aligned} \tag{4.5}$$

By a similar argument as in (4.3), we get

$$\begin{aligned}\|z^{n_i} - T_{n_i}z^{n_i}\|^2 &= \|U_{\lambda_{n_i}}T_{n_i}z^{n_i} - T_{n_i}z^{n_i}\|^2 \\ &\le \lambda_{n_i}|f(T_{n_i}z^{n_i}, U_{\lambda_{n_i}}T_{n_i}z^{n_i})| = \lambda_{n_i}|f(T_{n_i}z^{n_i}, z^{n_i})|.\end{aligned}$$

From the boundedness of the sequences $\{T_{n_i}z^{n_i}\}$, $\{z^{n_i}\}$ and Assumption **(A3)**, it implies that there exists a constant $R > 0$ such that

$$|f(T_{n_i}z^{n_i}, z^{n_i})| \le R\|T_{n_i}z^{n_i} - z^{n_i}\|. \tag{4.6}$$

Hence, $\|z^{n_i} - T_{n_i}z^{n_i}\| \le \lambda_{n_i}R$, which implies that

$$\lim_{i\to\infty} \|z^{n_i} - T_{n_i}z^{n_i}\| = 0. \tag{4.7}$$

On the other hand, using the boundedness of the sequences $\{z^n\}$ and $\{T_n z^n\}$, from Lemma 2, we deduce

$$\lim_{i\to\infty} \|T(s)T_{n_i}z^{n_i} - T_{n_i}z^{n_i}\| = 0. \tag{4.8}$$

Combining (4.5), (4.7) and (4.8), we have

$$\lim_{i\to\infty} \|T(s)z^{n_i} - z^{n_i}\| = 0.$$

Since $z^{n_i} \rightharpoonup z^*$, using the demi-closeness of $T(s)$ at 0, we get $z^* \in Fix(T(s))$ for all $s > 0$.

Claim 3: $\{z^n\}$ weakly converges to the unique solution of (4.1).

Let x^* be the unique solution of (4.1). We will prove that $x^* = z^*$. Indeed, for all $i \ge 1$, we have

$$\begin{aligned}\|z^{n_i} - x^*\|^2 &= \langle z^{n_i} - T_{n_i}z^{n_i}, z^{n_i} - x^*\rangle + \langle T_{n_i}z^{n_i} - T_{n_i}x^*, z^{n_i} - x^*\rangle \\ &\le \langle z^{n_i} - T_{n_i}z^{n_i}, z^{n_i} - x^*\rangle + \|z^{n_i} - x^*\|^2.\end{aligned}$$

Hence, applying a similar argument as in (3.1), we have

$$\begin{aligned}0 &\le \langle z^{n_i} - T_{n_i}z^{n_i}, z^{n_i} - x^*\rangle = \langle U_{\lambda_{n_i}}T_{n_i}z^{n_i} - T_{n_i}z^{n_i}, U_{\lambda_{n_i}}T_{n_i}z^{n_i} - x^*\rangle \\ &\le \lambda_{n_i}\left[f(T_{n_i}z^{n_i}, x^*) - f(T_{n_i}z^{n_i}, U_{\lambda_{n_i}}T_{n_i}z^{n_i})\right].\end{aligned}$$

Since $\lambda_{n_i} > 0$, we get

$$f(T_{n_i}z^{n_i}, x^*) \ge f(T_{n_i}z^{n_i}, U_{\lambda_{n_i}}T_{n_i}z^{n_i}) = f(T_{n_i}z^{n_i}, z^{n_i}). \tag{4.9}$$

From (4.6) and (4.7), we have

$$f(T_{n_i}z^{n_i}, z^{n_i}) \to 0.$$

Combining (4.7) and the fact that $z^{n_i} \rightharpoonup z^*$, we get $T_{n_i}z^{n_i} \rightharpoonup z^*$. In (4.9), letting $i \to \infty$, using the weak upper semicontinuity of $f(., x^*)$, we obtain

$$\begin{aligned} f(z^*, x^*) &\geq \limsup_{i\to\infty} f(T_{n_i}z^{n_i}, x^*) \geq \limsup_{i\to\infty} f(T_{n_i}z^{n_i}, z^{n_i}) \\ &= \lim_{i\to\infty} f(T_{n_i}z^{n_i}, z^{n_i}) = 0. \end{aligned}$$

Since $x^* \in Sol(f, \bigcap_{s\in\mathbb{R}^+} Fix(T(s)))$, $z^* \in \bigcap_{s\in\mathbb{R}^+} Fix(T(s))$ and f is strongly monotone, we get $f(z^*, x^*) \leq 0$. Thus, $f(z^*, x^*) = 0$, and hence, $x^* = z^*$. Let $\{z^{n_j}\}$ be an arbitrary subsequence of $\{z^n\}$ such that $z^{n_j} \rightharpoonup \hat{z}$. Repeating the above argument, we obtain $\hat{z} = x^*$. Thus $z^n \rightharpoonup x^*$.

Claim 4: $\|x^{n+1} - z^n\| \to 0$.

We have

$$\begin{aligned} \|x^{n+1} - z^n\| &= \|U_{\lambda_n}T_nx^n - U_{\lambda_n}T_nz^n\| \leq (1 - \lambda_n\gamma/8)\|x^n - z^n\| \\ &\leq (1 - \lambda_n\gamma/8)\|x^n - z^{n-1}\| + \lambda_n\|z^{n-1} - z^n\|/\lambda_n. \end{aligned}$$

By Lemma 3, to prove $\|x^{n+1} - z^n\| \to 0$, it is sufficient to show that $\|z^{n-1} - z^n\|/\lambda_n \to 0$. We have

$$\begin{aligned} \|z^{n-1} - z^n\| &= \|U_{\lambda_{n-1}}T_{n-1}z^{n-1} - U_{\lambda_n}T_nz^n\| \\ &\leq \|U_{\lambda_n}T_nz^n - U_{\lambda_n}T_nz^{n-1}\| + \|U_{\lambda_n}T_nz^{n-1} - U_{\lambda_{n-1}}T_{n-1}z^{n-1}\| \\ &\leq (1 - \lambda_n\frac{\gamma}{8})\|z^n - z^{n-1}\| + \|U_{\lambda_n}T_nz^{n-1} - U_{\lambda_{n-1}}T_{n-1}z^{n-1}\|, \end{aligned}$$

or equivalently,

$$\|z^{n-1} - z^n\| \leq \frac{8}{\lambda_n\gamma}\|U_{\lambda_n}T_nz^{n-1} - U_{\lambda_{n-1}}T_{n-1}z^{n-1}\|. \tag{4.10}$$

By a similar argument as in (3.1), we get

$$\begin{aligned} &\langle U_{\lambda_n}T_nz^{n-1} - T_nz^{n-1}, U_{\lambda_n}T_nz^{n-1} - U_{\lambda_{n-1}}T_nz^{n-1}\rangle \\ &\quad \leq \lambda_n\left(f(T_nz^{n-1}, U_{\lambda_{n-1}}T_nz^{n-1}) - f(T_nz^{n-1}, U_{\lambda_n}T_nz^{n-1})\right), \\ &\langle U_{\lambda_{n-1}}T_nz^{n-1} - T_nz^{n-1}, U_{\lambda_{n-1}}T_nz^{n-1} - U_{\lambda_n}T_nz^{n-1}\rangle \\ &\quad \leq \lambda_{n-1}\left(f(T_nz^{n-1}, U_{\lambda_n}T_nz^{n-1}) - f(T_nz^{n-1}, U_{\lambda_{n-1}}T_nz^{n-1})\right). \end{aligned}$$

Adding the last two inequalities, we obtain

$$\begin{aligned} &\|U_{\lambda_n}T_nz^{n-1} - U_{\lambda_{n-1}}T_nz^{n-1}\|^2 \leq (\lambda_{n-1} - \lambda_n) \\ &\quad \times \left\{f(T_nz^{n-1}, U_{\lambda_n}T_nz^{n-1}) - f(T_nz^{n-1}, U_{\lambda_{n-1}}T_nz^{n-1})\right\}. \end{aligned} \tag{4.11}$$

From the L-Lipschitz-type continuity of f, it is easy seen that

$$\begin{aligned} &f(T_nz^{n-1}, U_{\lambda_n}T_nz^{n-1}) - f(T_nz^{n-1}, U_{\lambda_{n-1}}T_nz^{n-1}) \\ &\quad \leq f(U_{\lambda_{n-1}}T_nz^{n-1}, U_{\lambda_n}T_nz^{n-1}) \\ &\qquad + L\|U_{\lambda_{n-1}}T_nz^{n-1} - U_{\lambda_n}T_nz^{n-1}\|\|T_nz^{n-1} - U_{\lambda_{n-1}}T_nz^{n-1}\|. \end{aligned} \tag{4.12}$$

Let $p \in \bigcap_{s\in\mathbb{R}^+} Fix(T(s))$. Then $p \in Fix(T_n)$ for all $n \geq 1$. We have

$$\begin{aligned}\|U_{\lambda_{n-1}}T_nz^{n-1}-p\| &\leq \|U_{\lambda_{n-1}}T_nz^{n-1}-U_{\lambda_{n-1}}T_np\|+\|U_{\lambda_{n-1}}p-p\|\\ &\leq \|z^{n-1}-p\|+\|U_{\lambda_{n-1}}p-p\|.\end{aligned}$$

Since $\{z^n\}$ is bounded and $\|U_{\lambda_{n-1}}p-p\| \to 0$, the sequence $\{U_{\lambda_{n-1}}T_nz^{n-1}\}$ is bounded. The boundedness of $\{U_{\lambda_n}T_nz^{n-1}\}$ is obtained analogously. From Assumption **(A3)**, it follows that there exists a constant $N > 0$ such that

$$f(U_{\lambda_{n-1}}T_nz^{n-1}, U_{\lambda_n}T_nz^{n-1}) \leq N\|U_{\lambda_{n-1}}T_nz^{n-1}-U_{\lambda_n}T_nz^{n-1}\|. \tag{4.13}$$

Combining (4.11), (4.12) and (4.13), we arrive at

$$\|U_{\lambda_n}T_nz^{n-1}-U_{\lambda_{n-1}}T_nz^{n-1}\| \leq \xi(\lambda_{n-1}-\lambda_n), \tag{4.14}$$

where $\xi := \sup_{n\geq 1}\{N+L\|T_nz^{n-1}-U_{\lambda_{n-1}}T_nz^{n-1}\|\} < \infty$.

On the other hand, we have

$$\begin{aligned}&\|U_{\lambda_{n-1}}T_nz^{n-1}-U_{\lambda_{n-1}}T_{n-1}z^{n-1}\| \leq \|T_nz^{n-1}-T_{n-1}z^{n-1}\|\\ &\quad\leq \left(\frac{1}{s_{n-1}}-\frac{1}{s_n}\right)\left\|\int_0^{s_{n-1}} T(t)z^{n-1}\,\mathrm{dt}\right\| + \frac{1}{s_n}\left\|\int_{s_{n-1}}^{s_n} T(t)z^{n-1}\,\mathrm{dt}\right\|.\end{aligned}$$

Using the nonexpansivity of $T(t)$, we get

$$\begin{aligned}\left\|\int_0^{s_{n-1}} T(t)z^{n-1}\,\mathrm{dt}\right\| &\leq \int_0^{s_{n-1}} \left(\|T(t)z^{n-1}-T(t)p\|+\|T(t)p\|\right)\mathrm{dt}\\ &\leq \int_0^{s_{n-1}} \|z^{n-1}-p\|+\|p\|\,\mathrm{dt} = s_{n-1}\left(\|z^{n-1}-p\|+\|p\|\right)\end{aligned}$$

and

$$\begin{aligned}\left\|\int_{s_{n-1}}^{s_n} T(t)z^{n-1}\,\mathrm{dt}\right\| &\leq \int_{s_{n-1}}^{s_n} \left(\left\|T(t)z^{n-1}-T(t)p\right\|+\|p\|\right)\mathrm{dt}\\ &\leq \int_{s_{n-1}}^{s_n} \left(\left\|z^{n-1}-p\right\|+\|p\|\right)\mathrm{dt} = (s_n-s_{n-1})\left(\left\|z^{n-1}-p\right\|+\|p\|\right).\end{aligned}$$

Hence,

$$\|U_{\lambda_{n-1}}T_nz^{n-1}-U_{\lambda_{n-1}}T_{n-1}z^{n-1}\| \leq \frac{s_n-s_{n-1}}{s_n}\eta, \tag{4.15}$$

where $\eta := \sup_{n\geq 1}\{2\left(\|z^n-p\|+\|p\|\right)\} < \infty$. Combining (4.10), (4.14) and (4.15), we have

$$\begin{aligned}&\|U_{\lambda_n}T_nz^{n-1}-U_{\lambda_{n-1}}T_{n-1}z^{n-1}\| \leq \|U_{\lambda_n}T_nz^{n-1}-U_{\lambda_{n-1}}T_nz^{n-1}\|\\ &+\|U_{\lambda_{n-1}}T_nz^{n-1}-U_{\lambda_{n-1}}T_{n-1}z^{n-1}\| \leq \xi(\lambda_{n-1}-\lambda_n)+\frac{s_n-s_{n-1}}{s_n}\eta.\end{aligned} \tag{4.16}$$

Combining (4.10) and (4.16), we obtain

$$\frac{\|z^{n-1}-z^n\|}{\lambda_n} \leq \frac{8}{\gamma}\left(\frac{\lambda_{n-1}-\lambda_n}{\lambda_n^2}\xi+\frac{s_n-s_{n-1}}{s_n\lambda_n^2}\eta\right).$$

Using the conditions $\frac{\lambda_{n-1}-\lambda_n}{\lambda_n^2} \to 0$ and $\frac{s_n-s_{n-1}}{s_n\lambda_n^2} \to 0$, we get $\frac{\|z^{n-1}-z^n\|}{\lambda_n} \to 0$, which completes the proof. $\square$

Remark 1. (a) One example of $\{\lambda_n\}$ and $\{s_n\}$ satisfying the conditions in Algorithm 1 is $\lambda_n = 1/n^\rho$ and $s_n = n^\delta$ with $\rho \in (0, \frac{1}{2})$ and $\delta > 0$.
(b) To implement Algorithm 1, we do not have to calculate the constants of the strong monotonicity and the Lipschitz-type continuity of f.

We can see that **(A1)**, **(A2)** and **(A4)** are usual conditions, which are used in many other works (see for example [8,19]). Let us give an examples to illustrate condition **(A3).**

Example 1. Let $f : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ be a bifunction defined by

$$f(x, y) := \langle \Phi(x, y), y - x\rangle + \varphi(y) - \varphi(x), \quad \forall x, y \ \in \mathbb{R}^m,$$

where the mapping $\Phi : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is bounded on a bounded set and φ is a L-Lipschitz continuous mapping on $\mathbb{R}^m$. The bifunction f in this example is a generalized form of the cost bifunctions in Nash-Cournot models considered in [1,3,8,19,20]. For any bounded sequences $\{x^n\}$ and $\{y^n\}$, we have

$$|f(x^n, y^n)| = |\langle \Phi(x^n, y^n), y^n - x^n\rangle + \varphi(y^n) - \varphi(x^n)| \le \sigma\|x^n - y^n\|,$$

where $\sigma := \sup\{\|\Phi(x^n, y^n)\| + L : n \ge 1\} < \infty$. Hence, f satisfies **(A3)**.

Next, we consider the case when f is Lipschitz-type continuous on bounded sets. More precisely, instead of using condition **(A1)**, we assume that:
(A1') The bifunction f is γ-strongly monotone on H and Lipschitz-type continuous on each bounded set of H.
Let $p \in \bigcap_{s\in\mathbb{R}^+} Fix(T(s))$ and $\{\lambda_n\}$ be a nonnegative sequence satisfying $\lim_{n\to\infty} \lambda_n = 0$. From Assumption **(A3)**, it follows that there exists a constant $\theta > 0$ such that $|U_{\lambda_n} p - p| \le \theta\lambda_n$ for all $n \ge 1$. Denote

$$D := \{z \in H : \|z - p\| \le 8\theta/\gamma\}.$$

Let L be the Lipschitz constant of f on S. Since $\lambda_n \to 0$, there exists a number $n_0 \ge 1$ such that for all $n \ge n_0$, we have $\lambda_n \in (0, \gamma/L^2)$. For all $z \in D$ and $n \ge n_0$, we have

$$\begin{aligned}\|U_{\lambda_n} T_n z - p\| &\le \|U_{\lambda_n} T_n z - U_{\lambda_n} T_n p\| + \|U_{\lambda_n} p - p\| \\ &\le (1 - \lambda_n \frac{\gamma}{8})\|z - p\| + \lambda_n\theta \le (1 - \lambda_n \frac{\gamma}{8})\theta\frac{8}{\gamma} + \lambda_n\theta = \theta\frac{8}{\gamma},\end{aligned}$$

which implies $U_{\lambda_n} T_n(D) \subset D$ for all $n \ge n_0$. Now, consider the sequence $\{x^n\}_{n\ge n_0}$ generated by

$$x^{n_0} \in D, \quad x^{n+1} = U_{\lambda_n} T_n x^n. \tag{4.17}$$

Corollary 1. Suppose that assumptions **(A1'),(A2)–(A4)** hold. Let $\{\lambda_n\}$ and $\{s_n\}$ be the sequences satisfying the conditions in Algorithm 1. Then the sequence $\{x^n\}_{n\ge n_0}$ generated by (4.17) weakly converges to the unique solution of problem (4.1).

Proof. The proof directly follows from Theorem 1 with the notion that z^n and x^n belongs to D for all $n \ge n_0$. $\square$

5 Numerical experiments

In this section, we give some examples to test the effectiveness of the proposed algorithm. Some comparisons of our method with the existing ones are also reported. All codes are written in Matlab 2010 and run on a personal computer with Intel Core2 TM Quad Processor Q9400 2.66Ghz 4GB Ram.

Example 2. Let $H = \mathbb{R}^3$, $f : H \times H \to \mathbb{R}$ is defined by

$$f(x, y) := \langle \mathbf{A}x + \mathbf{B}y, y - x\rangle, \quad \forall x, y \in H,$$

where

$$\mathbf{A} = \begin{pmatrix} 10 & 7 & 5 \\ 6 & 8 & 5 \\ 5 & 7 & 7 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 8 & 6 & 4 \\ 5 & 6 & 4 \\ 4 & 6 & 5 \end{pmatrix}.$$

The nonexpansive semigroup $\{T(t)\}_{t\in\mathbb{R}^+}$ is defined by

$$T(t)x = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \forall x \in \mathcal{R}^3, \quad t > 0.$$

Choose $s_n := n$ for all $n \geq 1$. We have

$$T_n : H \to H, \quad T_n x := \frac{1}{s_n}\int_0^{s_n} T(t)x \,\mathrm{dt} = \frac{1}{n}\begin{pmatrix} x_1 \sin n + x_2(\cos n - 1) \\ x_1(1 - \cos n) + x_2 \sin n \\ nx_3 \end{pmatrix}.$$

We will show that all the conditions of Algorithm 1 are satisfied. It is obvious that $\{T(t)\}_{t\in\mathbb{R}^+}$ is a nonexpansive semigroup with nonempty fixed point set Ω. Since $\mathbf{A} - \mathbf{B}$ is positive definite, there exists a constant $\gamma > 0$ such that for all $x, y \in H$, we have

$$\begin{aligned} f(x, y) + f(y, x) &= \langle \mathbf{A}x + \mathbf{B}y, y - x\rangle + \langle \mathbf{A}y + \mathbf{B}x, x - y\rangle \\ &= -\langle(\mathbf{A} - \mathbf{B})(x - y), x - y\rangle \leq -\gamma\|x - y\|^2, \end{aligned}$$

i.e., the bifunction f is strongly monotone. Next, for all $x, y, z \in H$, it holds that

$$\begin{aligned} f(x, y) + f(y, z) - f(x, z) &= \langle \mathbf{A}x + \mathbf{B}y, y - x\rangle + \langle \mathbf{A}y + \mathbf{B}z, z - y\rangle \\ - \langle \mathbf{A}x + \mathbf{B}z, z - x\rangle &= \langle \mathbf{B}(y - z), y - x\rangle + \langle \mathbf{A}(y - x), z - y\rangle. \end{aligned}$$

Hence, f is L-Lipschitz type continuous on H where $L = \|\mathbf{A}\| + \|\mathbf{B}\|$. We apply Algorithm 1 to solve the problem $EP(f, \Omega)$. Note that in this example, the subproblems of Algorithm 1 can be rewritten as

$$y^n = T_n x^n, \quad x^{n+1} = (\mathbf{I} + 2\lambda_n\mathbf{B})^{-1}(\mathbf{I} + \lambda_n\mathbf{B} - \lambda_n\mathbf{A})y^n,$$

where $\mathbf{I}$ is the identity. Choose $\lambda_n = \frac{1}{(n+10)^{1/4}}$, $x^0 = (30, 30, 30)^T$ and the stopping criteria $\|x^n - x^*\| < 10^{-4}$, where $x^* = (0, 0, 0)^T$ is the unique solution of $EP(f, \Omega)$. The result is presented in Table 1. To investigate the effect of starting points, we test Algorithm 1, using the different x^0. The results are presented in Table 2. As we can see, the number of iterations is not affected much by the starting points.

Table 1. Iterations of Algorithm 1 with starting point $x_0 = (30, 30, 30)^T$.

Iter(k)	x_k^1	x_k^2	x_k^3	$\|x^k - x^*\|$
0	30.0000	30.0000	30.0000	51.9615
1	-2.8049	0.3908	-3.2234	4.2907
2	0.1085	0.3396	-0.0957	0.3691
3	-0.0221	0.0538	-0.0275	0.0644
4	0.0008	0.0028	-0.0001	0.0030
5	0.0000	0.0000	0.0001	$7.8171.10^{-5}$

Table 2. Iterations of Algorithm 1 with the different starting points x_0.

x^0	Iterations
$(100, 100, 100)^T$	6
$(-100, -100, -100)^T$	6
$(50, -50, 50)^T$	7
$(10, 50, -100)^T$	6

Example 3. In this example, we compare the performance of our algorithm with the gradient projection method (shortly `GPM`):

$$x^0 \in C, \quad x^{n+1} = \operatorname{argmin}\{\lambda f(x,y) + \frac{1}{2}\|y-x\|^2 : y \in C\}. \tag{5.1}$$

It is well known that under the assumptions that f is L-Lipschitz type continuous, γ-strongly monotone and $\lambda \in (0; \frac{2\gamma}{L^2})$, Algorithm (5.1) linearly converges to the unique solution of $EP(f, C)$ (see [1]).

Consider the problem $EP(f, C)$, where

$$C := \{x \in \mathbb{R}^p, x_1 = x_2 = 0; x_i \in \mathbb{R},\ \forall i = 3, \dots, p\}, \quad f : C \times C \to \mathbb{R},$$
$$f(x,y) := \langle \mathbf{A}x + \mathbf{B}y, y - x\rangle, \quad \mathbf{B} := \mathbf{M} + p\mathbf{I}, \quad \mathbf{A} = \mathbf{B} + p\mathbf{I},$$

with $\mathbf{M} = (m_{ij})_{p\times p}$, $m_{ij} = 1$ for all $i, j = 1, \dots, p$, $\mathbf{I}$ is the identity. It is easy seen that f is $(\|\mathbf{A}\| + \|\mathbf{B}\|)$-Lipschitz-type continuous and p-strongly monotone. Using $\lambda = 1.9p/(\|\mathbf{A}\| + \|\mathbf{B}\|)^2$, we can see that all the conditions of Algorithm (5.1) are satisfied.

Next, using the idea in [26], we choose

$$T(s) : \mathbb{R}^p \to \mathbb{R}^p, \quad T(s)x := \begin{pmatrix} e^{-s} & 0 & 0 & \cdots & 0 \\ 0 & e^{-s} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \cdots \\ x_p \end{pmatrix}$$

for all $s \geq 0$ and $x = (x_1, \dots, x_p)^T \in \mathbb{R}^p$.

The problem $EP(f, \bigcap_{s\in\mathbb{R}^+} Fix(T(s)))$ becomes $EP(f, C)$ and the proposed algorithm can be rewritten as

$$\begin{cases} x^0 \in H; \quad y^n = \left(\frac{1-e^{-n}}{n}x_1; \frac{1-e^{-n}}{n}x_2; x_3; \ldots; x_p\right)^T; \\ x^{n+1} = (\mathbf{I} + 2\lambda_n \mathbf{B})^{-1}(\mathbf{I} + \lambda_n \mathbf{B} - \lambda_n \mathbf{A})y^n. \end{cases}$$

We choose the step size of Algorithm 1: $\lambda_n = \frac{1}{(n+10)^{0.25}}$. In the both algorithms, we use the same starting points x^0, which are randomly generated and use the same stopping criteria $\|x^n - x^*\| \le \epsilon$, where $x^* = (0, \ldots, 0)^T$ is the unique solution of $EP(f, C)$, $\epsilon = 10^{-4}$. The results are presented in Table 3.

Table 3. Comparison of algorithms in Example 3.

	GPM		Algorithm 1	
	CPU times (s)	Iterations	CPU times (s)	Iterations
$p = 5$	0.1681	40	0.0024	7
$p = 10$	0.1796	40	0.0028	9
$p = 50$	0.2021	46	0.0073	14
$p = 100$	0.2777	47	0.0239	15
$p = 200$	0.5428	48	0.1542	15
$p = 500$	3.1449	51	1.1849	16
$p = 1000$	9.6368	52	5.9918	17

From Table 3, we can see that: the computational time of the gradient projection method is greater than that of Algorithm 1. This happens because the constrained optimization problem at each iteration of Algorithm (5.1) is replaced by the unconstrained one in Algorithm 1.

6 Conclusions

In this paper, we give a sufficient condition for the contraction of the proximal mapping of a bifunction. Using this mapping, we propose a contraction algorithm for the equilibrium problem over the fixed point set of a nonexpansive semigroup. To the best of our knowledge, this problem has not been considered before. The main points of the proposed algorithms are the followings:

1. At each step of the algorithm, we avoid using the metric projection, which, in general, is computationally expensive. Instead, we only have to compute the value of a nonexpansive mapping at each step.

2. In our algorithm, the constrained subproblems are replaced by the unconstrained ones, which are easier to solve.

3. The bifunction involved need not to be Lipschitz-type continuous on the whole constraint set. We only require the local Lipschitz-type continuity of this bifunction.

On the other hand, the Lipschitz-type condition used in this paper seems rather complicated. Proving the contraction of the proximal mapping under a simpler condition is more delicate and further investigations are necessary.

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