Mathematical Modelling and Analysis
Volume 29, Issue 1, 46–56, 2024
https://doi.org/10.3846/mma.2024.17193

Discrete Group Method for a Mathematical Model of the Diffusion in Swelling Gelatin

Parviz Darania

Department of Mathematics, Faculty of Science, Urmia University
P.O.Box 165 Urmia, Iran
E-mail(*corresp.*): p.darania@urmia.ac.ir

Received June 22, 2022; accepted January 12, 2024

Abstract. In this article, we will introduce the mathematical model of the diffusion process in a swelling medium that has already been modeled. The main purpose of this paper is introducing a new class of group transformations for solving the nonlinear emulsion equation. Thus, we introduce this method as the discrete group methods based on solvable class orbits. One of the crucial advantages of this method is that a transformation is sought, which reduces the equation being investigated to some standard form for which the methods of integration are known. In developing this approach, one may construct all canonical forms of solvable equations.

Keywords: gelatin swelling, mathematical model, discrete group method, Emden-Fowler equation, Abel equations.

AMS Subject Classification: 35C05; 92F05; 34L30; 47E05; 35D35.

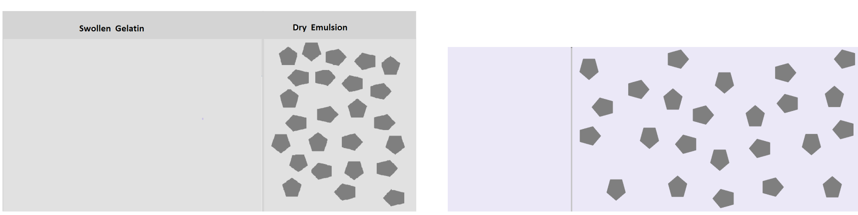
1 Introduction

Most laws related to natural phenomena, especially physical laws and engineering sciences, are modeled based on differential equations. To describe the behavior of these models, simulation is a valuable tool. Simulation is also an imitation of actual system performance over time. So the evolution of these systems occurs over time and the time describing variable is significant. These models include equations called evolution equations. Studying and calculating the analytical and approximate solutions of this type of equation is one of the challenges of the last three decades for researchers. Also, industrial process modeling has certain problems. One of these problems is how to analyze these models. It should be noted that simulation alone is not enough to examine these models. To examine these simulated models and their validity, we must be able to obtain the solutions of these models by numerical or analytical methods. Now, to express the future behavior of the model, we must obtain its solutions by solving the equations of the simulation system and comment based on its approximate or analytical solutions. Nevertheless, there are models such as emulsion equations whose exact solutions have not been obtained due to the nature of the equations.

In what follows, in Section 2, we will introduce the mathematical model of swelling medium and mathematics of diffusion equations which have been so far obtained. In Section 3.1, we will be mainly concerned with the dry emulsion equations, in order to transforming them into the solvable orbits of classical Emden-Fowler equations. By using the discrete group methods, some applicable transformations for obtaining the exact solutions of solvable equations are presented in Section 3.2.

2 Preliminaries

Emulsion layers, discussed previously in [6], consisting of silver halide in grain forms as well as oil droplets. Once a photo is taken, light is captured by the silver halide grains. When the film is developed, the oil droplets containing dye couplers, produce color. In addition, some other ingredients exist in emulsion layers. A considerable proportion of the emulsion is occupied by the silver halide grains; however, the volume of other materials is not significant. When the photographic film is being produced, paper is being manufactured, and the film is being developed, aqueous solutions disperse inside and outside emulsion layers. Understanding the diffusion process is very important. In this regard, when a dry emulsion layer is dispersed in an aqueous solution Figure 1(a), water diffuses into the emulsion, which makes it swell Figure 1(b). Whenever the left end of gelatin is continuously in contact with water, some part of the left end will be occupied by pure water. Because the medium is swelling and changing as a result, the basic laws of diffusion are not valid anymore.



a) before b) after

Figure 1. Dry emulsion diffusion in solvent.

Accordingly, a mathematical model was introduced in [5], in which a swelling medium is considered in the diffusion process. According to Fick's law, the flux of a species passing a point x at time t is calculated $-H(u,x,t)\frac{\partial u(x,t)}{\partial x}$, here $H(u,x,t)$ and $u=u(x,t)$ are diffusion coefficient and spatial concentration of a diffusing species, respectively. On the other hand, in the small interval $(x, \Delta x)$, according to the conservation law of mass, we have

$$\frac{\partial}{\partial t}\Big(u(x,t)\Delta x\Big) = H(u,x,t)\Big(\frac{\partial u(x,t)}{\partial x}\Big|_{x+\Delta x} - \frac{\partial u(x,t)}{\partial x}\Big|_{x}\Big). \qquad (2.1)$$

In Equation (2.1), if we let $\Delta x \to 0$, then

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\Big(H(u,x,t)\frac{\partial u(x,t)}{\partial x}\Big).$$

Now, in the coordinate systems $x-y$ (Eulerian-Lagrangian coordinate [1,5]), we can generalize such diffusion equation to be applied in the swelling media as well. To do this, we introduce the following notations and quantities in Table 1.

Table 1. Notations and quantities.

Notations	Quantities
$N(x,t)$	Volume ratio of water to gelatin
$M(x,t)$	Concentration of species M
$P(x)$	Volume ratio of non gelatin emulsion material to gelatin
$k(x,t) = 1 + N(x,t)$	Swell ratio
$s(x,t) = 1 + N(x,t) + P(x)$	Ratio of total volume to gelatin volume

As we want to focus on what is happening to the emulsion layer in the right parts of Figure 1, what is happening to the left end of gelatin is not taken into consideration. The separated gelatin on the left does not disperse in the emulsion on the right. It is only water that can diffuse.

The gelatin-free emulsion has silver halide grains and oil droplets. Since the solution is very dilute in M that the volume it occupies is not considered. Many of these species are found in photographic films. The model introduced below explains such a situation; M is considered as a vector, not a scalar.

It is apparent from the definition s that variable change at time t is given by

$$y = \int_0^x s(\varsigma,t)d\varsigma. \qquad (2.2)$$

From Equation (2.2), we get

$$\frac{\partial}{\partial y} = \frac{1}{s(x,t)}\frac{\partial}{\partial x}.$$

It should be noted that the volume fraction of the quantity c in a continuous phase (gelatin+water) in gelatin is taken and divided by the swelling ratio k.

Now a small interval $(x, x+\Delta x)$ is considered in the emulsion. This contains to second order in Δx, at time t, an amount $N(x,t)\Delta x$ of water. According to Fick's law, the flux passing a point y corresponds to the spatial gradient $\frac{\partial}{\partial y}$ of continuous phase concentration of water N/k, diffusion coefficient $H_N(k(x,t), s(x,t))$ is the coefficient of proportionality. And by conserving mass, to second order in Δx, the following is given

$$\frac{\partial N(x,t)}{\partial t}\Delta x = -\frac{H_N\Big(k(x,t),s(x,t)\Big)}{s(x,t)}\frac{\partial}{\partial x}\frac{N(x,t)}{k(x,t)}\Big|_x \\ \frac{H_N\Big(k(x+\Delta x,t),s(x+\Delta x,t)\Big)}{s(x+\Delta x,t)}\frac{\partial}{\partial x}\frac{N(x,t)}{k(x,t)}\Big|_{x+\Delta x}.$$

When it is divided by Δx, and considering $\Delta x \to 0$, the equation of nonlinear diffusion is obtained

$$\frac{\partial N(x,t)}{\partial t} = \frac{\partial}{\partial x}\Big(\frac{H_N(k(x,t),s(x,t))}{s(x,t)}\frac{\partial}{\partial x}\frac{N(x,t)}{k(x,t)}\Big). \tag{2.3}$$

Note that, in the same manner for M, we have

$$\frac{\partial M(x,t)}{\partial t} = \frac{\partial}{\partial x}\Big(\frac{H_M(k(x,t),s(x,t))}{s(x,t)}\frac{\partial}{\partial x}\frac{M(x,t)}{k(x,t)}\Big). \tag{2.4}$$

Equations (2.3) and (2.4) are completed at boundary conditions like: Evaporation conditions for water

$$\frac{H_N(k(x,t),s(x,t))}{s(x,t)} = \frac{\partial}{\partial x}\frac{N(x,t)}{k(x,t)} = \pm\mathcal{K}_{ev}\Big(\frac{N(x,t)}{k(x,t)} - \mu\Big), \tag{2.5}$$

here $\mathcal{K}_{ev}$ refers to a constant, μ is the equilibrium value of $\frac{N}{k}$, and $\pm$ is a sign where $-$ and $+$ refer to the right endpoint and left endpoint, respectively. Also, at end points we have

$$\frac{\partial}{\partial x}\frac{M(x,t)}{k(x,t)} = 0.$$

The initial data should be specified

$$N(x,0) = N_0(x), \quad M(x,0) = M_0(x). \tag{2.6}$$

3 Main results

3.1 General equations

It should be noted that when the diffusion coefficient $H_N(k,s)$ is identified, Equation (2.3) turns into an equation of nonlinear diffusion. Now, from [9], we set

$$H_N(k,s) = \frac{2H_0k}{3s-k}\exp\left(-\frac{\beta\alpha}{k-\alpha}\right),$$

where $\alpha,\ 0<\alpha<1, H_0$, and β, refer to fitting parameter, diffusion coefficient of species in pure water and physical constant pertaining to water and gelatin, respectively.

Considering the emulsion in direct contact with water, the corresponding boundary condition is as follows

$$N = N_\infty,$$

where N_∞ is very large.

In special cases where emulsion is extremely thick, if the interval $(0,\infty)$ is occupied and

$$P = 0, \qquad H_N(k,s) = 1/\big(1+N(x,t)\big),$$

Considering the originally dry emulsion, Equations (2.3), (2.5) and (2.6), take the following form

$$\frac{\partial N(x,t)}{\partial t} = \frac{\partial}{\partial x}\Big(\frac{1}{1+N(x,t)}\frac{\partial}{\partial x}\frac{N(x,t)}{1+N(x,t)}\Big), x \in (0,\infty), \quad t>0, \tag{3.1}$$

$$N(0,t) = N_\infty, \quad t>0,$$
$$N(x,0) = 0, \quad x \in (0,\infty). \tag{3.2}$$

Now, in Equations (3.1)–(3.2), we set $N_\infty = \infty$ and

$$\Phi(x,t) = 1/(1+N(x,t))^2.$$

Then, we get

$$\begin{aligned}&\frac{\partial \Phi(x,t)}{\partial t} = \Phi^{\frac{3}{2}}(x,t)\frac{\partial^2\Phi(x,t)}{\partial x^2}, \quad x\in(0,\infty), \quad t>0,\\ &\Phi(0,t) = 0, \quad t>0,\\ &\Phi(x,0) = 1, \quad x \in (0,\infty).\end{aligned} \tag{3.3}$$

Once a similarity solution

$$\Phi(x,t) = \Psi\left(\frac{x}{\sqrt{t}}\right) \tag{3.4}$$

is used, it is seen that $\Psi(z)$ satisfies in the generalized Emden-Fowler equation

$$\Psi^{\frac{3}{2}}(z)\Psi''(z) + \frac{z}{2}\Psi' = 0, \quad z \in (0,\infty), \tag{3.5}$$

$$\Psi(0) = 0, \quad \Psi(\infty) = 1. \tag{3.6}$$

Theorem 1. *[5] There exists a unique solution for $\Psi(z)$ in* (3.5) *and* (3.6)*; moreover, $\Psi'(0)$ exists and we have*

$$\begin{cases} \Psi'(0) > 0, & z \in [0,\infty),\\ \Psi''(0) < 0, & z\in(0,\infty),\\ z^{\frac{1}{2}}\Psi''(0) \to \frac{1}{2\Psi'(0)}, & z \to \infty. \end{cases}$$

Note that, whenever Ψ is near $z=0$, its form is as

$$\Psi(z) = Az\Big(1+\sum_{k=1}^{\infty} a_k z^{\frac{k}{2}}\Big),$$

where both $A = \Psi'(0)$ and the a_k depend on A.

Note that, by using the solution of the system (2.3), (2.5), and (2.6) and substituting k and s in Equation (2.4), we get the same results for M. Thus, the new mathematical feature of the system (2.3)–(2.6) is just in the solution to this system. Therefore, in the next section, we will solve this system.

3.2 Solvable orbits

The group analysis is the most popular theoretical and analytical method for solving differential equations. In this section, we introduce some useful transformations by using the concept of the discrete group theory and its applications in exact solution of differential equations. Note that, all of these transformations are invertible and this invertibility allows us to avoid some lengthy computations for the conversion of the initial and boundary condition. Also, under these transformations, the solution of the transformed equation can be converted into the solution of the reference equation [7, 8].

Let D be a class of ordinary differential equation and

$$D(x, y, a) = 0$$

be an equation in this class, where a is a vector parameters. We shall seek the transformations F_i that are closed in the class $D(x, y, a) = 0$, i.e., they change only the vector a:

$$F_i : D(x, y, a) \to D(t, u, b_i).$$

If each F_i has an inverses, then the collection $\{F_i\}$ defines a discrete transformation group on the class $D(x, y, a) = 0$.

All the existing methods of exact solution of ordinary differential equations can be conditionally divided into two groups:

(I) a search for transformation of the original ordinary differential equations in class D to some other class of ordinary differential equations D_1, which belongs to one of the standard classes of ordinary differential equations having known solutions;

(II) a search for a transformation leaving the original ordinary differential equation in D invariant, that is, transformation into "itself", that gives independent information about the solution.

The discrete group method does not operate with a single equation as in the applications of the Lie method, but operates with a class of equations D, depending on a vector a of parameters containing the investigated equation; but contrary to approach (I), one considers the transformations of the given class D which are closed in themselves on a chosen class of ordinary differential equations.

In the literature, there are two methods for searching discrete group transformations, namely, point transformations and Backlund transformations.

Therefore, discrete group transformations are related to point and Backland transformations. In this article, we have introduced a number of useful transformations based on Backland and point transformations. We show that these transformations have all the properties of the discrete group.

The class of generalized Emden-Fowler equations

$$y''(x) = Ax^n y^m (y'_x)^l \tag{3.7}$$

is determined by a three-dimensional parameter vector $a = (n, m, l) \in \mathbb{R}^3$. Application of $RF - pair(X, X)$ to the generalized Emden-Fowler equations

(3.7), we obtain a transformation

$$g:(n,m,l)\longrightarrow(\tfrac{1}{1-l},-\tfrac{n}{n+1},\tfrac{2m+1}{m}),$$

$$\begin{cases} y'_x = & t^{\frac{1}{1-l}}, \\ y = & (u'_t)^{\frac{-1}{m}}, \\ x = & u^{\frac{1}{n+1}}, \end{cases} \qquad \begin{cases} u'_t = & y^{-m}, \\ u = & x^{n+1}, \\ t = & (y'_x)^{l-1}, \end{cases}$$

where $g^3 = E$ and

$$g^{-1}:(n,m,l)\longrightarrow(-\tfrac{m}{m+1},-\tfrac{1}{l-2},\tfrac{n-1}{n}), \tag{3.8}$$

$$\begin{cases} y'_x = & u^{\frac{1}{2-l}}, \\ y = & t^{\frac{1}{m+1}}, \\ x = & (u'_t)^{\frac{1}{n}}, \end{cases} \qquad \begin{cases} u'_t = & x^{n}, \\ u = & (y'_x)^{2-l}, \\ t = & y^{m+1}, \end{cases}$$

which defines the group $G_3\{g|\ g^3 = E\}$, where E unity (the identity transformation). For further details see [4] and references therein).

The parameter subspace $a = (n, m, 0)$ defines the set of classical Emden-Fowler equations. It is well known that the point transformation

$$\begin{aligned} &S:(n,m,0)\longrightarrow(-m-n-3,m,0), \\ &y = u/t, \qquad x = 1/t, \end{aligned} \tag{3.9}$$

which defines the group $G_2\{S|\ S^2 = E\}$.

It is not hard to show that for $n = 1$, the transformation T_1 represents a composition Sg^{-1},

$$Sg^{-1}\equiv T_1:(1,m,l)\longrightarrow(-\frac{2ml+3l-3m-5}{(m+1)(l-2)},\frac{1}{l-2},0),$$

$$\begin{cases} y'_x = & (\frac{u}{t})^{\frac{1}{2-l}}, \\ y = & t^{-\frac{1}{m+1}}, \\ x = & tu'_t - u, \end{cases} \qquad \begin{cases} u'_t = & (y'_x)^{2-l} - \frac{2-l}{m+1}Axy^{m+1}, \\ u = & y^{-m-1}(y'_x)^{2-l}, \\ t = & y^{-m-1}, \end{cases} \tag{3.10}$$

where g^{-1} and S are defined in (3.8) and (3.9) respectively. For further details see [3] and references therein.

Theorem 2. *The class of generalized Emden-Fowler equations $a = (1, m, l)$ admits a general group*

$$G_2\{S|\ S^2 = E\}\otimes G_3\{g|\ g^3 = E\}\sim D_3\{(S,g^{-1})|\ (g^{-1})^3 = S^2 = E\},$$

which is maximal in the Backlund transformation class defined by means of the $RF-pair$ method [9]. This group may be given by the graph depicted in Figure 2. The graph represented in Figure 2 is valid for all values m and l, except for the singular points $m = -1$ and $l = 2$.

$$(1,m,l) \xrightarrow{\quad Sg^{-1} \quad} \left(-\frac{2ml+3l-3m-5}{(m+1)(l-2)}, \frac{1}{l-2}, 0\right).$$

Figure 2. Group D_3.

Now, in the Equation (3.7), we assume that $A=-2, n=1,\ m=-\frac{3}{2}$ and $l=1$. Then, we get

$$y''_{xx}(x) = -2xy^{-\frac{3}{2}}y'. \tag{3.11}$$

By this assumption, in the transformation (3.10), we obtain

$$\mathcal{T}_1 = T_1\Big|_{A=-2,\ a=(1,-\frac{3}{2},1)},$$

where

$$\begin{cases} y'_x = & (\frac{u}{t}), \\ y = & t^2, \\ x = & tu'_t - u, \end{cases} \qquad \begin{cases} u'_t = & (y'_x) - 4xy^{-\frac{1}{2}}, \\ u = & y^{-\frac{5}{2}}y'_x, \\ t = & y^{-\frac{5}{2}}, \end{cases} \tag{3.12}$$

and

$$y''_{xx} = \frac{d}{dx}y'_x = \frac{d}{dx}(\frac{u}{t}) = \frac{d}{dt}(\frac{u}{t})\frac{dt}{dx} = \frac{tu'_t - u}{t^3 u''_{tt}}.$$

By substituting of these relations into Equation (3.11) or by using of the discrete group transformations $Sg^{-1}(1,-\frac{3}{2},1)$, we get

$$u''_{tt} = -2tu^{-1}. \tag{3.13}$$

We consider the discrete group analysis and general exact solutions of the Equation (3.5). This equation, can be rewritten as the following generalized Emden-Fowler form

$$\Psi'' = -\tfrac{1}{2}z\Psi^{-\frac{3}{2}}\Psi'. \tag{3.14}$$

Now, by using the suitable class of group transformations and solvable class of orbits, we will analyze the exact solutions of Equation (3.14). Here, by using of the transformation (3.12), let us introduce a transformation

$$\mathcal{T}_1 : \begin{cases} z = & \tau w'_\tau - w, \\ \Psi = & \tau^2, \\ \Psi'_z = & w/\tau. \end{cases} \tag{3.15}$$

The substitution (3.15) reduces Equation (3.14) to the following form

$$w''_{\tau\tau} = -2\tau w^{-1}. \tag{3.16}$$

We shall now show that the Equation (3.16), by using the transformation

$$\mathcal{T}_2 : \begin{cases} \tau = & \exp(\lambda), \\ w = & X\exp(h\lambda), \end{cases} \tag{3.17}$$

with an appropriate choice of h, leads to an autonomous form. To do this, by substituting the transformation (3.17) in the Equation (3.16) and after some computation, we have

$$\exp\Big((2h-2)\lambda\Big)\Big(X''_{\lambda\lambda}+(2h-1)X'_{\lambda}+h(h-1)X\Big)=-2X^{-1}. \qquad (3.18)$$

Now, if we set $h=1$, then Equation (3.18) reduces to the autonomous equation

$$X''_{\lambda\lambda}+X'_{\lambda}=-2X^{-1}. \qquad (3.19)$$

Having applied the transformation

$$\mathcal{T}_3:\begin{cases} X'_{\lambda}= & -Y(X),\\ X''_{\lambda\lambda}= & Y(X)Y'_X(X),\end{cases} \qquad (3.20)$$

Equation (3.19) is easily reduced to

$$Y(X)Y'_X(X)-Y(X)=-2X^{-1}. \qquad (3.21)$$

Furthermore, by the substitution

$$\mathcal{T}_4:\begin{cases} V= & -2X^{-2},\\ Z= & -Y/X+1,\end{cases} \qquad (3.22)$$

we obtain from Equation (3.21), the Abel equation

$$(V-Z^2+Z)V'_Z=(-2Z+2)V. \qquad (3.23)$$

Finally, by substituting the transformation

$$\mathcal{T}_5:\begin{cases} V= & -2\xi^2\theta^{-2},\\ Z= & \frac{\xi}{\theta}\theta'_{\xi},\end{cases}$$

in the Equation (3.23), we get the following classical Emden-Fowler equation

$$\theta''_{\xi\xi}=-2\theta^{-1}. \qquad (3.24)$$

This equation is integrable as a consequence of which, we have been able to construct some solutions for the nonlinear dry emulsion problem.

Note that, Equation (3.24) is a solvable Emden-Fowller equation with an exact solution

$$\begin{cases} \theta(\eta)= & \beta C_1\exp(\mp\eta^2),\\ \xi(\eta)= & \alpha C_1\displaystyle\int\exp(\mp\eta^2)d\eta+C_2,\end{cases} \qquad (3.25)$$

where $\alpha^2\mp\beta^2=0$, $\alpha,\beta\neq 0$ and C_1,C_2,α and β are constants, which are dependent of initial and boundary conditions [8] and [9].

Now, by using the effect of the inverse of the operators $\mathcal{T}_5-\mathcal{T}_1$ on the Equations (3.25), we can obtain the exact solution of the Equation (3.5). For further details see [2, 8, 10] and references therein. Hence, to integrate the Equation (3.3), we start from Equations (3.24) and (3.25). By using the inverse operator $\mathcal{T}_5$ on the Equations (3.25), the analytical solution of Equation (3.23) will be as follows

$$\mathcal{T}_5^{-1}\begin{pmatrix}\theta(\eta)\\ \xi(\eta)\end{pmatrix}=\begin{pmatrix}Z(\eta)\\ V(\eta)\end{pmatrix},$$

where

$$\begin{cases} Z(\eta)= & \mp\frac{2\beta}{\alpha C_1}\eta\exp(\pm\eta^2)\Big(\alpha C_1\displaystyle\int\exp(\mp\eta^2)d\eta+C_2\Big),\\ V(\eta)= & -\frac{1}{\beta^2C_1^2}\exp(\mp2\eta^2)\Big(\alpha C_1\displaystyle\int\exp(\mp\eta^2)d\eta+C_2\Big)^2. \end{cases} \tag{3.26}$$

Form Equations (3.22) and (3.26), we get the analytical solution of Equation (3.21) in the following form

$$\mathcal{T}_4^{-1}\begin{pmatrix}Z(\eta)\\ V(\eta)\end{pmatrix}=\begin{pmatrix}X(\eta)\\ Y(\eta)\end{pmatrix},$$

where

$$\begin{cases} X^2(\eta)= & \dfrac{2\beta^2C_1^2\exp(\pm2\eta^2)}{\Big(\alpha C_1\displaystyle\int\exp(\mp\eta^2)d\eta+C_2\Big)^2},\\ Y^2(\eta)= & \dfrac{2\beta^2C_1^2\exp(\pm2\eta^2)}{\Big(\alpha C_1\displaystyle\int\exp(\mp\eta^2)d\eta+C_2\Big)^2}\\ & \times\Big(\mp\frac{2\beta}{\alpha C_1}\eta\exp(\pm\eta^2)\Big(\alpha C_1\displaystyle\int\exp(\mp\eta^2)d\eta+C_2\Big)\Big)^2. \end{cases} \tag{3.27}$$

Also, from Equations (3.19), (3.20) and (3.27), we have

$$\mathcal{T}_3^{-1}\begin{pmatrix}X(\eta)\\ Y(\eta)\end{pmatrix}=\begin{pmatrix}X(\lambda)=\displaystyle\int G(\lambda)d\lambda\\ G(\lambda)=Y(X(\lambda))\end{pmatrix}, \tag{3.28}$$

where $G(\lambda)=-Y(X(\lambda))$.

Now, by using the Equations (3.17) and (3.28), the exact solution of Equation (3.16) as follows

$$\mathcal{T}_2^{-1}\begin{pmatrix}X(\lambda)\\ G(\lambda)\end{pmatrix}=\begin{pmatrix}\tau(\eta)\\ w(\eta)\end{pmatrix},$$

where

$$\begin{cases} \tau= & \exp(\lambda),\\ w= & X\exp(\lambda), \end{cases} \tag{3.29}$$

and from Equations (3.14), (3.15) and (3.29), we have

$$\mathcal{T}_1^{-1}\begin{pmatrix}\tau(\eta)\\ w(\eta)\end{pmatrix}=\begin{pmatrix}z(\eta)\\ \Psi(\eta)\end{pmatrix},$$

where

$$\begin{cases} z(\eta)= & \tau w'_\tau-w,\\ \Psi(\eta)= & \tau^2. \end{cases} \tag{3.30}$$

Finally, from Equations (3.4) and (3.30), we can find the exact solution of the Equation (3.3) as follows

$$\begin{cases} \Phi(x,t) = & \Psi(x/\sqrt{t}), \\ z = & x/\sqrt{t}. \end{cases}$$

4 Conclusions

In this paper, we have successfully developed the combination of method of discrete group method and invertible transformations to obtain the exact solution of gelatin swelling equation. The analytical results obtained show that the results of these methods are in agreement.

References

[1] J. Crank. *The Mathematics of Diffusion.* Clarendon Press, Oxford, 1995.

[2] P. Darania. On the discrete group analysis for the exact solutions of some classes of the nonlinear Abel and Burgers equations. *Khayyam Journal of Mathematics*, **7**(2):257–265, 2021.

[3] P. Darania and A. Ebadian. Discrete group method for nonlinear heat equation. *Kyungpook Mathematical Journal*, **46**(3):329–336, 2006.

[4] P. Darania and M. Hadizadeh. On the RF-pair operations for the exact solution of some classes of nonlinear Volterra integral equations. *Mathematical Problems in Engineering*, **2006**:1–11, 2006. https://doi.org/10.1155/MPE/2006/97020.

[5] A. Friedman. Mathematics in industrial problems. In *The IMA Volumes in Mathematics and its Applications* (Part 3). Springer-Verlag, New York, 1990. https://doi.org/10.1007/978-1-4613-9098-5.

[6] A. Friedman and D.S. Ross. *Mathematical Models in Photographic Science.* Springer-Verlag Berlin Heidelberg, 2003. https://doi.org/10.1007/978-3-642-55755-2.

[7] M. Hadizadeh, A.R. Zokayi and P. Darania. On the discrete group analysis for solving some classes of Emden-Fowler equations. *Applied Mathematics Research eXpress*, **2004**(5):169–178, 2004. https://doi.org/10.1155/S1687120004020155.

[8] K.C. Ng and D.S. Ross. Diffusion in swelling gelatin. *J. of Imaging Science*, **35**(2):356–361, 1991.

[9] V.F. Zaitsev and A.D. Polyanin. *Discrete-Group Methods for Integrating Equations of Nonlinear Mechanics: Theory, Solutions and Applications.* CRC Press, Florida, 1994.

[10] A.R. Zokayi, M. Hadizadeh, P. Darania and A. Rajabi. The relation between the Emden–Fowler equation and the nonlinear heat conduction problem with variable transfer coefficient. *Communications in Nonlinear Science and Numerical Simulation*, **11**(7):845–853, 2006. https://doi.org/10.1016/j.cnsns.2004.12.010.