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The Global Strong Solutions of the 3D Incompressible Hall-MHD System with Variable Density

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Abstract. In this paper, we focus on the well-posedness problem of the three-dimensional incompressible viscous and resistive Hall-magnetohydrodynamics system (Hall-MHD) with variable density. We mainly prove the existence and uniqueness issues of the density-dependent incompressible Hall-magnetohydrodynamic system in critical spaces on $\mathbb{R}^3$.

Keywords: variable density, Hall-MHD, global solution, critical spaces.

AMS Subject Classification: 35Q30; 35Q35; 76D03.

1 Introduction

The concept of a magneto plasma has attracted interests of engineer and mathematician ever since it was first introduced by Alfvén [1]. In [1], the author proposed the basic equations of magnetohydrodynamics (MHD) and showed the existence of waves in magnetized plasmas. Since then, MHD has developed into a broad and mature scientific fields, with applications ranging from solar physics and astrophysical dynamos, to fusion plasmas and dusty laboratory plasmas. Meanwhile, a growing interest in what is known as Hall-MHD has appeared in recent years. The Hall-MHD theory has been used to describe and explain some kinds of interesting physical phenomenons. To our best knowledge, the first systematic study on Hall-MHD was completed by Lighthill in [25] and then by Campos in [4]. The overview of underlying physics associated with Hall plasma, applications of Hall-MHD to space and laboratory plasma have been given in [21]. For more explanation on the physical background of Hall-MHD system, we refer to [19, 22, 26, 27, 31].

In this article, we investigate the well-posedness problem of incompressible viscous and resistive Hall-MHD with variable density in critical functional spaces (the compressible system was studied in [2]):

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \\ \rho \partial_t u + \rho u \cdot \nabla u - \mu \Delta u + \nabla p = (\nabla \times B) \times B, \\ \partial_t B - \nabla \times (u \times B) + h \nabla \times \left(\frac{(\nabla \times B) \times B}{\rho} \right) = \nu \Delta B, \\ \operatorname{div} u = \operatorname{div} B = 0, \\ (\rho, u, B)|_{t=0} = (\rho_0, u_0, B_0), \end{cases} \tag{1.1}$$

where $\rho = \rho(t, x) \in \mathbb{R}^+$ stands for the density, $u = u(t, x) \in \mathbb{R}^3$, $B = B(t, x) \in \mathbb{R}^3$ and scalar function $p = p(t, x)$ with $t \geq 0$ and $x \in \mathbb{R}^3$ represent the velocity field, the magnetic field and the scalar pressure, respectively. The parameters μ and ν are the fluid viscosity and the magnetic resistivity. $h \nabla \times \left(\frac{(\nabla \times B) \times B}{\rho} \right)$ is the Hall term, it reflects that in a moving conductive fluid, the magnetic field can also induce currents, and which can in turn polarize the fluid and change the magnetic field. While the dimensionless number h measures the magnitude of the Hall effect compared to the typical length scale of the fluid.

The mathematical researching on the Hall-MHD system has been started only rather recently, despite its physical relevance. Let us briefly recall some known results on it. The authors in [2] had derived the Hall-MHD equations from a two-fluid Euler-Maxwell system, and it also provided a kinetic formulation for the Hall-MHD. By a Galerkin method, authors in [2] proved the global existence of weak solutions in the periodic domain. While for the whole spaces $\mathbb{R}^3$, Chae, Degond and Liu in [5] proved the global existence of classical solutions under the smallness condition on the initial data. Dumas and Sueur [14] established the weak-strong uniqueness property and proposed a sufficient condition to guarantee the magneto-helicity identity. An analyticity of mild solutions to the 3D incompressible Hall-MHD system was constructed by Duan in [13] by introducing a Lei-Lin ([24]) type functional space. Global well-posedness in critical spaces was studied by Danchin and Tan in [11, 12], in which the proof based on an important observation on the special structure of the Hall term. The temporal decay estimates for weak and strong solutions are given in [6] by Chae and Schonbek, in [32] by Weng and in [34] by Zhao. The global well-posedness under the axially symmetric coordinate are given in [16]. To give a more complete view of current studies on the global regularity problems, we mention some of other interesting results in [7, 8, 15, 17, 20, 23, 28, 29, 30, 33].

In order to derive an appropriate formulation of the system, in what follows, we introduce some algebraic identites. Elementary calculation implies that

$$(\nabla \times B) \times B = B \cdot \nabla B - \nabla \left(\frac{|B|^2}{2} \right).$$

Hence, setting $\Pi := p + |B|^2/2$, equation

$$\rho\partial_t u + \rho u \cdot \nabla u - \mu\Delta u + \nabla p = (\nabla \times B) \times B$$

can be rewritten as

$$\rho\partial_t u + \rho u \cdot \nabla u - \mu\Delta u + \nabla \Pi = B \cdot \nabla B.$$

Hence, we conclude that system (1.1) recasts in

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \\ \rho\partial_t u + \rho u \cdot \nabla u - \mu\Delta u + \nabla \Pi = B \cdot \nabla B. \\ \partial_t B - \nabla \times (u \times B) + h\nabla \times \left(\frac{(\nabla\times B)\times B}{\rho}\right) = \nu\Delta B, \\ \operatorname{div} u = \operatorname{div} B = 0, \\ (\rho, u, B)|_{t=0} = (\rho_0, u_0, B_0). \end{cases} \tag{1.2}$$

In this paper, we mainly investigate the global well-posedness of the density-dependent Hall-MHD system (1.2) in critical spaces with respect to the scaling introduced below. Let us make a brief scaling analysis on this system with variable density. Firstly, we find out that the system (1.2) does not have any scaling invariance as the classical MHD system. However, if we consider the case $B = 0$, then the system (1.2) becomes the density-dependent incompressible Navier-Stokes system:

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \\ \rho\partial_t u + \rho u \cdot \nabla u - \mu\Delta u + \nabla \Pi = 0, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0. \end{cases} \tag{1.3}$$

Obviously, system (1.3) is invariant for all $\lambda > 0$ under the change

$$\begin{aligned} &(\rho_0(x), u_0(x)) \mapsto (\rho_0(\lambda x), \lambda u_0(\lambda x)) \\ &(\rho(t,x), u(t,x), \Pi(t,x)) \mapsto (\rho(\lambda^2 t, \lambda x), \lambda u(\lambda^2 t, \lambda x), \lambda^2 \Pi(\lambda^2 t, \lambda x)). \end{aligned}$$

The global existence of solutions in critical space was obtained in [9] with small data. Secondly, if the velocity vanished in (1.2), then the magnetic fields B governed by the following Hall equation:

$$\begin{cases} \partial_t B - \nu\Delta B + h\nabla \times \left(\frac{(\nabla\times B)\times B}{\rho}\right) = 0, \\ \operatorname{div} B = 0. \end{cases} \tag{1.4}$$

One can check that (1.4) are invariant for all $\lambda > 0$ by the rescaling

$$\rho(t,x) \mapsto \rho(\lambda^2 t, \lambda x), \quad B(t,x) \mapsto B(\lambda^2 t, \lambda x),$$

under the assumption that the data B_0 is rescaled by

$$B_0(x) \mapsto B_0(\lambda x).$$

Finally, we get an intuitively conclusion that, when the hall effect constant $h > 0$, without considering the Lorentz force in the momentum equations, u

and ∇B have the same regularity level. Whilst, when the hall effect constant $h = 0$, (1.2) is nothing but an inhomogeneous resistive MHD system in which u and B share the same regularity. Based on these observations, we can choose our suitable working spaces such that u and ∇B, u and B have the same scaling invariance property, respectively.

In the present paper, we consider the perturbation of the equation near equilibrium state $(1, 0, 0)$. Denote $a = 1/\rho - 1$. For fluids with positive density, we can write the perturbation system as:

$$\begin{cases} \partial_t a + u \cdot \nabla a = 0, \\ \partial_t u + u \cdot \nabla u - \mu(1+a)\Delta u + (1+a)\nabla \Pi = (1+a)B \cdot \nabla B, \\ \partial_t B - \nu\Delta B - \nabla \times (u \times B) + h\nabla \times ((a+1)(\nabla \times B) \times B) = 0, \\ \operatorname{div} u = \operatorname{div} B = 0. \end{cases}$$

For the Hall equation, considering the curl of Hall, we introduce an additional unknown denoted by $J = \nabla \times B$. By an elementary computation, we have $\Delta B = -\nabla \times J$, and then the magnetic B can be rewritten as

$$B = \operatorname{curl}^{-1} J := (-\Delta)^{-1} \nabla \times J,$$

where the -1 order homogeneous Fourier multiplier curl^{-1} is defined in the sense of Fourier transformation

$$\mathcal{F}(\operatorname{curl}^{-1} J)(\xi) := \frac{i\xi \times \hat{J}(\xi)}{|\xi|^2}.$$

Altogether, in what follows we consider the following perturbation Cauchy problem:

$$\begin{cases} \partial_t a + u \cdot \nabla a = 0, \\ \partial_t u + u \cdot \nabla u - \mu(1+a)\Delta u + (1+a)\nabla \Pi = (1+a)B \cdot \nabla B, \\ \partial_t B - \nu\Delta B = \nabla \times (u \times B) - h\nabla \times \Big((a+1)(J \times B)\Big), \\ \partial_t J - \nu\Delta J = \nabla \times \nabla \times \Big((u \times \operatorname{curl}^{-1} J) - h(a+1)(J \times \operatorname{curl}^{-1} J)\Big), \\ \operatorname{div} u = \operatorname{div} B = 0, \\ (a, u, B, J)|_{t=0} = (a_0, u_0, B_0, J_0), \end{cases} \tag{1.5}$$

where $a_0 = 1 + \frac{1}{\rho_0}$ and $J_0 = \nabla \times B_0$. In this paper, we study the case of positive general coefficients μ, ν, h, for data $a_0 \in \dot{B}^{\frac{3}{2}}_{2,1}$, $(u_0, B_0) \in \dot{B}^{\frac{1}{2}}_{2,1}$ and $J_0 \in \dot{B}^{\frac{1}{2}}_{2,1}$ (the definition of the homogeneous and non-homogeneous Besov spaces will be given in the next section). Motivated by the paper [9] that dedicated to the incompressible Navier-Stokes equations, we then define our working spaces. Firstly, we introduce a local version in the form of non-homogeneous Besov spaces:

$$X(T) := \{ f \in \mathcal{C}([0,T]; B^{\frac{1}{2}}_{2,1}), \quad \nabla^2 f \in L^1(0,T; B^{\frac{1}{2}}_{2,1}), \quad \operatorname{div} f = 0 \ \}$$

for $T > 0$, and its norm is defined by

$$\|f\|_{X(T)} = \|f\|_{\widetilde{L}^\infty(0,T;B^{\frac{1}{2}}_{2,1})} + \|\nabla^2 f\|_{L^1(0,T;B^{\frac{1}{2}}_{2,1})}.$$

To prove the global existence result, we define

$$E := \{ f \in \mathcal{C}_b(\mathbb{R}^+; \dot{B}^{\frac{1}{2}}_{2,1}), \quad \nabla^2 f \in L^1(\mathbb{R}^+; \dot{B}^{\frac{1}{2}}_{2,1}) \quad \operatorname{div} f = 0 \ \}$$

if $T = \infty$ and the norm of E is given by

$$\|f\|_E = \|f\|_{\widetilde{L}^\infty(\mathbb{R}^+; \dot{B}^{\frac{1}{2}}_{2,1})} + \|\nabla^2 f\|_{L^1(\mathbb{R}^+; \dot{B}^{\frac{1}{2}}_{2,1})}.$$

Now we can state our main result.

Theorem 1. *Assume that initial data* $a_0 \in B^{\frac{3}{2}}_{2,1}$ *with* $1 + a_0$ *bound away from zero. Let* $(u_0, B_0) \in B^{\frac{1}{2}}_{2,1}$ *with* $\operatorname{div} u_0 = \operatorname{div} B_0 = 0$ *and* $J_0 := \nabla \times B_0 \in B^{\frac{1}{2}}_{2,1}$. *Then there exists a positive time* $T > 0$ *and a unique local solution* (ρ, u, B) *for Cauchy problem* (1.2) *with* $a := \frac{1}{\rho} - 1$, $J := \nabla \times B$ *and*

$$a \in \mathcal{C}([0, T]; B^{\frac{3}{2}}_{2,1}), \quad (u, B, J) \in X(T) \quad \text{and} \quad \nabla \mathit{\Pi} \in L^1(0, T; B^{\frac{1}{2}}_{2,1}).$$

Besides, the following estimate is valid

$$\|a\|_{\widetilde{L}^\infty_T(B^{\frac{3}{2}}_{2,1})} + \|(u, B, J)\|_{X(T)} \leq C \left(\|a_0\|_{B^{\frac{3}{2}}_{2,1}} + \|(u_0, B_0, J_0)\|_{B^{\frac{1}{2}}_{2,1}} \right),$$

where C *is an absolute constant depending only on* μ, ν *and* h.

If in addition, for some small constants $c > 0$ *which depending only on* μ, ν, *and* h *such that*

$$\|a_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|(u_0, B_0, J_0)\|_{\dot{B}^{\frac{1}{2}}_{2,1}} \leq c,$$

then the Cauchy problem (1.2) admits a unique global solution (ρ, u, B) *with*

$$a \in \mathcal{C}(\mathbb{R}^+; \dot{B}^{\frac{3}{2}}_{2,1}), \quad (u, B, J) \in E \quad \text{and} \quad \nabla \mathit{\Pi} \in L^1(\mathbb{R}^+; \dot{B}^{\frac{1}{2}}_{2,1}).$$

Furthermore, the following estimate is valid

$$\|a\|_{\widetilde{L}^\infty(\mathbb{R}^+; \dot{B}^{\frac{3}{2}}_{2,1})} + \|(u, B, J)\|_E \leq 2c.$$

Remark 1. In contrast with the original Hall-MHD system (1.2), our proof will focus on the extend perturbation system (1.5).

Remark 2. We mention here that on the local well-posedness part of the main theorem, the smallness assumption on a_0 is not a necessary condition. The method was first introduced by R. Danchin in [10] when dealing with the well-posedness of the barotropic viscous fluids in critical spaces. And in [18], Fang and the third author in this paper, together with Zhang constructed an important estimate on the linearized momentum equation which we will used in this article.

The rest parts of this paper are structured as follows. In the second section, we recall the definition of Besov spaces and give some estimates for the linearized equations. In Section 3, we concentrate on the existence part of Theorem 1.

2 Preliminaries

In this section, we briefly introduce the Littlewood-Paley decomposition, the definition of the homogeneous Besov space and some related analysis tools. For more details, we refer readers to [3].

Homogeneous Littlewood-Paley decomposition relies upon a dyadic partition of unity: let $\varphi, \chi \in S(\mathbb{R}^+)$ be two radial functions valued in the interval $[0,1]$ and supported in $\mathcal{C} = \{\xi \in \mathbb{R}^3 : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ and $\mathcal{B} = \{\xi \in \mathbb{R}^3 : |\xi| \leq \frac{4}{3}\}$ respectively such that

$$\sum_{j\in Z}\varphi(2^{-j}\xi) = 1 \quad \text{in} \quad \mathbb{R}^3\backslash\{0\}, \quad \chi(\xi) + \sum_{j\geq 0}\varphi(2^{-j}\xi) = 1 \quad \text{in} \quad \mathbb{R}^3.$$

One can easily prove that $\forall\ u \in \mathcal{S}'(\mathbb{R}^3),\ u = \sum_{q\in\mathbb{Z}} \Delta_q u.$

A number of functional spaces may be characterized in terms of Littlewood-Paley decomposition. First, let us give the definition of non-homogeneous Besov spaces.

Definition 1. For $s \in \mathbb{R}$, $(p,r) \in [1,+\infty]^2$, and $u \in \mathcal{S}'(\mathbb{R}^3)$, we set

$$\|u\|_{B^s_{p,r}} = \left(\sum_{q\geq -1} 2^{qsr}\|\Delta_q u\|^r_{L^p}\right)^{1/r}$$

with the usual modification if $r = +\infty$. We then define the non-homogeneous Besov space

$$B^s_{p,r}(\mathbb{R}^3) = \{u \in \mathcal{S}'(\mathbb{R}^3),\ \|u\|_{B^s_{p,r}} < \infty\}.$$

Similar to Definition 1, we can also define the homogeneous dyadic blocks $\dot{\Delta}_q$ and homogeneous Besov spaces. Let φ be a smooth function satisfying

$$\text{Supp}\varphi \subset \left\{\frac{3}{4} \leq |\xi| \leq \frac{8}{3}\right\}, \quad \sum_{q\in\mathbb{Z}}\varphi(2^{-q}\xi) = 1 \quad \text{for} \quad \xi \neq 0.$$

Then the homogeneous dyadic blocks can be defined as follows:

$$\dot{\Delta}_q u = \varphi(2^{-q}D)u = \int_{\mathbb{R}^3} h_q(y)u(x-y)dy \ \text{ for all } \ q \in \mathbb{Z}.$$

And we have the formal decomposition

$$u = \sum_{q\in\mathbb{Z}}\dot{\Delta}_q u, \quad \forall\ u \in \mathcal{S}'(\mathbb{R}^3)/\mathcal{P}[\mathbb{R}^3],$$

where $\mathcal{P}[\mathbb{R}^3]$ is the set of polynomials. Then homogeneous Besov spaces can be defined through the homogeneous decomposition.

Definition 2. For $s \in \mathbb{R}$, $(p,r) \in [1,+\infty]^2$, and $u \in \mathcal{S}'(\mathbb{R}^3)$, we set

$$\|u\|_{\dot{B}^s_{p,r}} = \left(\sum_{q\in\mathbb{Z}} 2^{qsr}\|\dot{\Delta}_q u\|^r_{L^p}\right)^{1/r}$$

with the usual modification if $r=+\infty$.
– For $s<\frac{N}{p}$ or $s=\frac{N}{p}$ if $r=1$, we define

$$\dot{B}^s_{p,r}(\mathbb{R}^3)=\{u\in\mathcal{S}'(\mathbb{R}^3),\ \|u\|_{\dot{B}^s_{p,r}}<\infty\}.$$

– If $k\in\mathbb{N}$ and $\frac{3}{p}+k\le s<\frac{3}{p}+k+1$ or $s=\frac{3}{p}+k+1$ if $r=1$, then $\dot{B}^s_{p,r}(\mathbb{R}^3)$ is defined as the subset of distributions $u\in\mathcal{S}'(\mathbb{R}^3)$ such that $\partial^\beta u\in\dot{B}^{s-k}_{p,r}(\mathbb{R}^3)$ whenever $|\beta|=k$.

We also need the following mixed time-space norm, which was introduced by J. Chemin [3].

Definition 3. Let $T>0, s\in\mathbb{R}$ and $1\le p,r,q\le\infty$. For any tempered distribution u on $(0,T)\times\mathbb{R}^3$, we set

$$\|u\|_{\tilde{L}^q_T(\dot{B}^s_{p,r})}:=\|(2^{js}\|\dot{\Delta}_j u\|_{L^q_T(L^p)})_{j\in\mathbb{Z}}\|_{\ell^r(\mathbb{Z})}.$$

We finished this section by recalling two key estimates for the linearized transport and momentum equations. The linearized transport equation read as

$$\begin{cases}\partial_t f+\operatorname{div}(vf)=F,\\ f|_{t=0}=f_0.\end{cases}$$

The following result ([3], Theorem 3.14) suffices for our purposes.

Proposition 1. *Let v be a solenoidal vector field such that ∇v belongs to $L^1(0,T;B^{\frac{3}{2}}_{2,1})$. Suppose that $f_0\in B^{\frac{3}{2}}_{2,1}, F\in L^1(0,T;B^s_{2,1})$ for all $s>\frac{3}{2}$. There exists a constant C, and such that the following inequality holds true,*

$$\|f\|_{\widetilde{L}^\infty_t(B^s_{2,1})}\le e^{CV(t)}\left(\|f_0\|_{B^s_{2,1}}+\int_0^t e^{-CV(\tau)}\|F(\tau)\|_{B^s_{2,1}}d\tau\right),$$

with $V(t)=\int_0^t\|\nabla v(\tau)\|_{B^{\frac{3}{2}}_{2,1}}d\tau.$

Moreover, f belongs to $C([0,T];B^s_{2,1})$.

When the density close to a constant, we are led to study the following linearized momentum equations:

$$\begin{cases}\partial_t u+v\cdot\nabla u-\mu b\Delta u+b\nabla\Pi=f,\\ \operatorname{div}u=0,\\ u|_{t=0}=u_0,\end{cases}\tag{2.1}$$

where $b:=a+1$ is bounded below by a positive constant $\underline{b}$. That is, $\inf\limits_{x\in\mathbb{R}^3}b(x)\ge\underline{b}$. Before stating our result, let us introduce the following notation:

$$A_T=1+\underline{b}2^{N_0\alpha}\|\nabla b\|_{B^{\frac{1}{2}}_{2,1}},\ \text{for }\alpha\in(0,1).$$

Proposition 2. *([18]) Let* $s \in (-\frac{1}{2}, \frac{5}{2})$ *and* $0 < \alpha < 1$. *Also we assume* $\alpha < \frac{s-1}{2}$ *if* $s > 1$ *and* $a_0 \in B^{\frac{3}{2}}_{2,1}$. *Let* u_0 *be a divergence-free vector field with coefficients in* $B^{s-1}_{2,r}$ *for* $r \in [1,\infty]$, *and* f *be a time-dependent vector field with coefficients in* $\widetilde{L}^1_T(B^{s-1}_{2,r})$. u, v *are two divergence-free time-dependent vector fields such that* $\nabla v \in L^1(0,T; B^{\frac{3}{2}}_{2,1})$ *and* $u \in \widetilde{C}([0,T]; B^{s-1}_{2,r}) \cap \widetilde{L}^1_T(B^{s+1}_{2,r})$. *In addition, assume that (2.1) is fulfilled for some distribution* Π. *Let* N_0 *be a positive integer such that* $b_{N_0} = 1 + S_{N_0} a$ *satisfies*

$$\inf_{x\in\mathbb{R}^3} b_{N_0} \geq 0.5\underline{b}.$$

Denoting $\underline{\mu} := \mu \inf_{x\in\mathbb{R}^3}(a+1)$, *then there exists a constant* $C = C(s,\mu,\underline{\mu})$ *such that if additionally,*

$$CA_T^{\kappa+1}\|a - S_{N_0}a\|_{\widetilde{L}^\infty_T(B^{\frac{3}{2}}_{2,1})} \leq \min\{\frac{1}{4}\underline{b},\ \frac{1}{4\mu}\underline{\mu}\},$$

the following estimate holds for $k = |s-1|/\alpha$,

$$\begin{aligned}&\|u\|_{\widetilde{L}^\infty_T(B^{s-1}_{2,r})} + \underline{\mu}\|u\|_{\widetilde{L}^1_T(B^{s+1}_{2,r})} + \|\nabla\Pi\|_{\widetilde{L}^1_T(B^{s-1}_{2,r})}\\ &\quad\leq Ce^{CV(T)}\Big(\|u_0\|_{B^{s-1}_{2,r}} + A_T^k(\|f\|_{L^1_T(B^{s-1}_{2,1})} + \mu A_T\|u\|_{L^1_T(B^{s+1-\alpha}_{2,r})})\Big),\end{aligned}$$

with $V(t) = \int_0^t(\|\nabla v(\tau)\|_{B^{\frac{3}{2}}_{2,1}} + 2^{2N_0}\|a\|^{\frac{2}{\alpha}}_{B^{\frac{3}{2}}_{2,1}})d\tau$.

3 Proof of Theorem 1

In this section, we are going to prove Theorem 1. For presentation clarity, we divide the proof into the following several subsections.

3.1 Local well-posedness result

Step 1. Linearized equations of $(\boldsymbol{B}, \boldsymbol{J})$. As the first step, we give the *a priori* estimate of the following system

$$\begin{cases} \partial_t B + v\cdot\nabla B - \nu\Delta B = F,\\ \partial_t J + v\cdot\nabla J - \nu\Delta J = G.\end{cases} \tag{3.1}$$

The result is stated by the following proposition.

Proposition 3. *Assume that* (B,J) *is a pair solution of (3.1) on* $[0,T]$ *with initial data* (B_0, J_0). *Let* $-\frac{1}{2}<s \leq \frac{5}{2}$, $0 < \alpha < 1$ *and* $W(t)=\int_0^t \|\nabla v(\tau)\|_{B^{\frac{3}{2}}_{2,1}} d\tau$. *Then the following estimate holds*

$$\begin{aligned}&\|B(t)\|_{B^{s-1}_{2,1}} + \|J(t)\|_{B^{s-1}_{2,1}} + \nu\int_0^T\Big(\|B(\tau)\|_{B^{s+1}_{2,1}} + \|J(\tau)\|_{B^{s+1}_{2,1}}\Big)d\tau\\ &\quad\leq Ce^{CW(T)}\Big(\|B_0\|_{B^{s-1}_{2,1}} + \|J_0\|_{B^{s-1}_{2,1}} + \|B\|_{L^1_T(B^{s+1-\alpha}_{2,1})} + \|J\|_{L^1_T(B^{s+1-\alpha}_{2,1})}\\ &\qquad + \int_0^T e^{-CV(\tau)}(\|F(\tau)\|_{B^{s-1}_{2,1}} + \|G(\tau)\|_{B^{s-1}_{2,1}})d\tau\Big).\end{aligned}$$

Proof. Applying the operator Δ_j to the first equation and the second equation in (3.1), taking the inner product with $\Delta_j B$, $\Delta_j J$, respectively, we then obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\Delta_j B\|_{L^2}^2+\nu\|\nabla\Delta_j B\|_{L^2}^2 \\ &\qquad\le C\|[v,\Delta_j]\cdot\nabla B\|_{L^2}\|\Delta_j B\|_{L^2}+\|\Delta_j F\|_{L^2}\|\Delta_j B\|_{L^2} \qquad (3.2)\\ &\frac{1}{2}\frac{d}{dt}\|\Delta_j J\|_{L^2}^2+\nu\|\nabla\Delta_j J\|_{L^2}^2 \\ &\qquad\le C\|[v,\Delta_j]\cdot\nabla J\|_{L^2}\|\Delta_j J\|_{L^2}+\|\Delta_j G\|_{L^2}\|\Delta_j J\|_{L^2}. \qquad (3.3)\end{aligned}$$

According to Bernstein inequality, there exists a $\kappa_0>0$ such that for all $j\ge 0$, we have

$$\kappa_0 2^j\|\Delta_j B\|_{L^2}\le\|\nabla\Delta_j B\|_{L^2},\quad \kappa_0 2^j\|\Delta_j B\|_{L^2}\le\|\nabla\Delta_j B\|_{L^2}.$$

Thus, for $j\ge 0$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\Delta_j B\|_{L^2}^2+\nu\kappa_0 2^j\|\Delta_j B\|_{L^2}^2 \\ &\qquad\le C\left(\|[v,\Delta_j]\cdot\nabla B\|_{L^2}+\|\Delta_j F\|_{L^2}\right)\|\Delta_j B\|_{L^2}, \qquad (3.4)\\ &\frac{1}{2}\frac{d}{dt}\|\Delta_j J\|_{L^2}^2+\nu\kappa_0 2^j\|\Delta_j J\|_{L^2}^2 \\ &\qquad\le C\left(\|[v,\Delta_j]\cdot\nabla J\|_{L^2}+\|\Delta_j G\|_{L^2}\right)\|\Delta_j J\|_{L^2}. \qquad (3.5)\end{aligned}$$

On the other hand, for $j=-1$, by (3.2) and (3.3), we have

$$\begin{aligned} &\frac{d}{dt}\|B_{-1}\|_{L^2}^2\le C\|B_{-1}\|_{L^2}\left(\|F_{-1}\|_{L^2}+\|[v,\Delta_{-1}]\cdot\nabla B\|_{L^2}\right), \qquad (3.6)\\ &\frac{d}{dt}\|J_{-1}\|_{L^2}^2\le C\|J_{-1}\|_{L^2}\left(\|G_{-1}\|_{L^2}+\|[v,\Delta_{-1}]\cdot\nabla J\|_{L^2}\right), \qquad (3.7)\end{aligned}$$

where $B_{-1}=\Delta_j B$ if $j=-1$ and the other terms are the same. Adding $\nu\kappa_0 2^{-2}\|B_{-1}\|_{L^2}^2$ and $\nu\kappa_0 2^{-2}\|J_{-1}\|_{L^2}^2$ on the both sides of (3.6) and (3.7), respectively, then we get

$$\begin{aligned} &\frac{d}{dt}\Big(\|B_{-1}\|_{L^2}^2+\|J_{-1}\|_{L^2}^2\Big)+\nu\kappa_0 2^{-2}\Big(\|B_{-1}\|_{L^2}^2+\|J_{-1}\|_{L^2}^2\Big)\\ &\le\nu\kappa_0 2^{-2}\Big(\|B_{-1}\|_{L^2}^2+\|J_{-1}\|_{L^2}^2\Big)+C\|F_{-1}\|_{L^2}\|B_{-1}\|_{L^2} \qquad (3.8)\\ &\quad+\|[v,\Delta_{-1}]\cdot\nabla B\|_{L^2}\|B_{-1}\|_{L^2}+C\|G_{-1}\|_{L^2}\|J_{-1}\|_{L^2}\\ &\quad+\|[v,\Delta_{-1}]\cdot\nabla J\|_{L^2}\|J_{-1}\|_{L^2}.\end{aligned}$$

Combining (3.4), (3.5) with (3.8), integrating over $[0,T]$, we deduce that for all $j\ge -1$,

$$\begin{aligned} &\|\Delta_j B\|_{L^2}+\|\Delta_j J\|_{L^2}+\nu 2^{2j}\int_0^T\left(\|\Delta_j B\|_{L^2}+\|\Delta_j J\|_{L^2}\right)dt\\ &\le C\left(\|\Delta_j B_0\|_{L^2}+\|\Delta_j J_0\|_{L^2}\right)+C\int_0^T\Big(\|F_j\|_{L^2}+\|G_j\|_{L^2}\\ &\quad+\nu\delta_{-1,j}\left(\|\Delta_{-1}B\|_{L^2}+\|\Delta_{-1}J\|_{L^2}\right)+\|[v,\Delta_j]\cdot\nabla B\|_{L^2}+\|[v,\Delta_j]\cdot\nabla J\|_{L^2}\Big)dt,\end{aligned}$$

where $\delta_{i,j}$ stands for the Kronecker symbol on $\mathbb{Z}^2$. By commutator estimates (Lemma 2.100 in [3]) and elementary computations, for some $\alpha \in (0,1)$, we have

$$\begin{aligned}
&\|B\|_{\widetilde{L}^\infty_T(B^{s-1}_{2,r})} + \|J\|_{\widetilde{L}^\infty_T(B^{s-1}_{2,r})} + \nu\Big(\|B\|_{\widetilde{L}^1_T(B^{s+1}_{2,r})} + \|u\|_{\widetilde{L}^1_T(B^{s+1}_{2,r})}\Big)\\
&\le C\Big(\|B_0\|_{B^{s-1}_{2,r}} + \|J_0\|_{B^{s-1}_{2,r}}\Big) + C\Big(\|B\|_{\widetilde{L}^1_T(B^{s+1-\alpha}_{2,r})} + \|J\|_{\widetilde{L}^1_T(B^{s+1-\alpha}_{2,r})}\Big)\\
&\quad + C\int_0^t \|\nabla v\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\Big(\|B\|_{\dot{B}^{s-1}_{2,1}} + \|J\|_{\dot{B}^{s-1}_{2,1}}\Big)d\tau + C\int_0^t\Big(\|F\|_{\dot{B}^{s-1}_{2,1}} + \|G\|_{\dot{B}^{s-1}_{2,1}}\Big)d\tau.
\end{aligned}$$

Then the Gronwall's Lemma implies that

$$\begin{aligned}
&\|B(t)\|_{B^{s-1}_{2,1}} + \|J(t)\|_{B^{s-1}_{2,1}} + \nu\int_0^T \Big(\|B(\tau)\|_{B^{s+1}_{2,1}} + \|J(\tau)\|_{B^{s+1}_{2,1}}\Big)d\tau\\
&\le Ce^{CW(T)}\Big(\|B_0\|_{B^{s-1}_{2,1}} + \|J_0\|_{B^{s-1}_{2,1}} + \|B\|_{L^1_T(B^{s+1-\alpha}_{2,1})} + \|J\|_{L^1_T(B^{s+1-\alpha}_{2,1})}\\
&\quad + \int_0^T e^{-CV(\tau)}(\|F(\tau)\|_{B^{s-1}_{2,1}} + \|G(\tau)\|_{B^{s-1}_{2,1}})d\tau\Big).
\end{aligned}$$

This finish the proof of this proposition. □

Step 2: *A priori* estimates.

Let $(u_L, \nabla\Pi_L)$ solves the non-stationary Stokes system

$$\begin{cases}
\partial_t u_L - \mu\Delta u_L + \nabla \Pi_L = 0,\\
\operatorname{div} u_L = 0,\\
u_L|_{t=0} = u_0,
\end{cases}$$

and let $B_L := e^{\nu t\Delta}B_0$, $J_L := e^{\nu t\Delta}J_0$.

One can directly derive that $(u_L, B_L) \in C([0,T]; B^{\frac{1}{2}}_{2,1}) \cap L^1(0,T; B^{\frac{5}{2}}_{2,1})$ and $\nabla\Pi_L \in L^1(0,T; B^{\frac{1}{2}}_{2,1})$. Also we can directly verify that

$$\begin{aligned}
\|u_L\|_{\widetilde{L}^\infty_T(B^{\frac{1}{2}}_{2,1})} + \|B_L\|_{\widetilde{L}^\infty_T(B^{\frac{1}{2}}_{2,1})} + \|J_L\|_{\widetilde{L}^\infty_T(B^{\frac{1}{2}}_{2,1})}&\\
\le C(\|u_0\|_{B^{\frac{1}{2}}_{2,1}} + \|B_0\|_{B^{\frac{1}{2}}_{2,1}} + \|J_0\|_{B^{\frac{1}{2}}_{2,1}}&).
\end{aligned}$$

Moreover, assume that T has been chosen so small as to satisfy

$$\mu\|u_L\|_{L^1_T(B^{\frac{5}{2}}_{2,1})} + \nu\|B_L\|_{L^1_T(B^{\frac{5}{2}}_{2,1})} + \nu\|J_L\|_{L^1_T(B^{\frac{5}{2}}_{2,1})} \le \lambda,$$

where λ will be determined later.

Let $\bar{u} = u - u_L, \bar{B} = B - B_L, \bar{J} = J - \nabla\times B_L, \nabla\bar{\Pi} = \nabla\Pi - \nabla\Pi_L$, where $(a, u, B, \nabla\Pi, J)$ satisfies (1.5) on $[0,T]\times\mathbb{R}^3$. Suppose that $a \in C^1([0,T]; B^{\frac{3}{2}}_{2,1})$, $(u, B, J) \in C^1([0,T]; B^{\frac{1}{2}}_{2,1}) \cap L^1_T(B^{\frac{5}{2}}_{2,1})$ and $\nabla\Pi \in L^1(0,T; B^{\frac{1}{2}}_{2,1})$. We can deduce that $(a, \bar{u}, \bar{B}, \bar{J}, \nabla\bar{\Pi})$ satisfies the following system

$$\begin{cases}
\partial_t a + (\bar{u} + u_L)\cdot\nabla a = 0,\\
\partial_t\bar{u} + (\bar{u} + u_L)\cdot\nabla\bar{u} - \mu(a+1)\Delta\bar{u} + (a+1)\nabla\bar{\Pi} = H + I,\\
\partial_t\bar{B} + (\bar{u} - h\bar{J})\cdot\nabla\bar{B} - \nu\Delta\bar{B} = F,\\
\partial_t\bar{J} + (\bar{u} - h\bar{J})\cdot\nabla\bar{J} - \nu\Delta\bar{J} = G,\\
(a, \bar{u}, \bar{B}, \bar{J})|_{t=0} = (a_0, 0, 0, 0),
\end{cases}$$

where

$$\begin{aligned} &H = -(\bar{u} + u_L)\nabla u_L + \mu a \Delta u_L - a \nabla \Pi_L, \quad G = \nabla \times F + M(u, J), \\ &I = (a+1)(\bar{B} \cdot \nabla \bar{B} + B_L \cdot \nabla \bar{B} + \bar{B} \cdot \nabla B_L + B_L \cdot \nabla B_L), \\ &F = -(u_L - hJ_L) \cdot \nabla B + aJ \cdot \nabla B + B \cdot \nabla(u - hJ) - hB \cdot \nabla(aJ), \\ &M(u, J) = \begin{pmatrix} \partial_2 v \cdot \nabla B^3 - \partial_3 v \cdot B^2 \\ \partial_3 v \cdot \nabla B^1 - \partial_1 v \cdot B^3 \\ \partial_1 v \cdot \nabla B^2 - \partial_2 v \cdot B^1 \end{pmatrix} \end{aligned} \tag{3.9}$$

and $v = \bar{u} - h\bar{J}$. Assume that the following inequalities are fulfilled for some suitable λ, N_0, $\widetilde{U}_0$ and T:

$$\begin{cases} \|a\|_{\widetilde{L}^\infty_T(B^{\frac{3}{2}}_{2,1})} \le 2\|a_0\|_{B^{\frac{3}{2}}_{2,1}}, \\ A_T^{\kappa+1}\|a - S_{N_0}a\|_{\widetilde{L}^\infty_T(B^{\frac{3}{2}}_{2,1})} \le \min\{\frac{1}{4C}\underline{b},\ \frac{1}{4C\mu}\underline{\mu}\}, \\ \|(\bar{u}, \bar{B}, \bar{J})\|_{\widetilde{L}^\infty_T(B^{\frac{1}{2}}_{2,1})} + \underline{\mu}\|(\bar{u}, \bar{B}, \bar{J})\|_{L^1_T(B^{\frac{5}{2}}_{2,1})} + \|\nabla \bar{\Pi}\|_{L^1_T(B^{\frac{1}{2}}_{2,1})} \le \lambda \widetilde{U}_0. \end{cases} \tag{3.10}$$

Then, we are going to prove that they are actually satisfied with strict inequalities. Since the left sides of (3.10) depend continuously on the time variable and are satisfied with strict inequalities when $t = 0$, a basic bootstrap argument insures that (3.10) are indeed satisfied for small T. For convenience, we denote

$$\begin{aligned} &U_0 := \|u_0\|_{B^{\frac{1}{2}}_{2,1}} + \|B_0\|_{B^{\frac{1}{2}}_{2,1}} + \|J_0\|_{B^{\frac{1}{2}}_{2,1}}, \\ &\overline{U}(T) = \|(\bar{u}, \bar{B}, \bar{J})\|_{\widetilde{L}^\infty_T(B^{\frac{1}{2}}_{2,1})} + \underline{\mu}\|(\bar{u}, \bar{B}, \bar{J})\|_{L^1_T(B^{\frac{5}{2}}_{2,1})} + \|\nabla \bar{\Pi}\|_{L^1_T(B^{\frac{1}{2}}_{2,1})}, \\ &\overline{E}(T) = \|(\bar{B}, \bar{J})\|_{\widetilde{L}^\infty_T(B^{\frac{1}{2}}_{2,1})} + \nu\|(\bar{B}, \bar{J})\|_{L^1_T(B^{\frac{5}{2}}_{2,1})}, \\ &U_L := \mu\|u_L\|_{L^1_T(B^{\frac{5}{2}}_{2,1})} + \|\nabla \Pi_L\|_{L^1_T(B^{\frac{3}{2}-1}_{2,1})} + \nu\|B_L\|_{L^1_T(B^{\frac{5}{2}}_{2,1})} + \nu\|J_L\|_{L^1_T(B^{\frac{5}{2}}_{2,1})}. \end{aligned}$$

Firstly, we prove $(3.10)_1$ holds with strict inequality. From Propositions 1, we can get that

$$\|a\|_{\widetilde{L}^\infty_T(B^{\frac{3}{2}}_{2,1})} \le e^{C\widetilde{V}(T)}\|a_0\|_{B^{\frac{3}{2}}_{2,1}} \le e^{C(\frac{\lambda}{\mu} + \frac{\lambda}{\underline{\mu}}\widetilde{U}_0)}\|a_0\|_{B^{\frac{3}{2}}_{2,1}},$$

where

$$\widetilde{V}(T) = \int_0^T (\|\nabla \bar{u}\|_{B^{\frac{3}{2}}_{2,1}} + \|\nabla u_L\|_{B^{\frac{3}{2}}_{2,1}})dt.$$

By choosing that λ sufficiently small and satisfying

$$e^{C(\frac{\lambda}{\mu} + \frac{\lambda}{\underline{\mu}}\widetilde{U}_0)} < 2, \tag{3.11}$$

we then conclude that $(3.10)_1$ holds with strict inequality on $[0, T)$.

According to the estimate (3.14) in [3] (page 134), we get that

$$\|\Delta_j a\|_{L^\infty_T(L^2)} \le \|\Delta_j a_0\|_{L^2} + Cc_j 2^{-j\frac{3}{2}} A_0 \|\nabla(\bar{u} + u_L)\|_{L^1_T(B^{\frac{3}{2}}_{2,1})},$$

where the ℓ^1 norm of c_j equals to 1 and $A_0 = 1 + \|a_0\|_{B^{\frac{3}{2}}_{2,1}}$. By the definition of Besov norm, we see that

$$\begin{aligned}
\|a - S_{N_0}a\|_{\widetilde{L}^\infty_T(B^{\frac{3}{2}}_{2,1})} &= \sum_{q\geq -1} 2^{q\frac{3}{2}} \|\Delta_q(a - S_{N_0}a)\|_{L^\infty_T(L^2)} \\
&\leq \sum_{j=N_0}^{\infty} \sum_{|q-j|\leq 2} 2^{q\frac{3}{2}} \|\Delta_q \Delta_j a\|_{L^\infty_T(L^2)} \\
&\leq C \sum_{j=N_0}^{\infty} \sum_{|q-j|\leq 2} 2^{(q-j)\frac{3}{2}} \Big(2^{j\frac{3}{2}} \|\Delta_j a_0\|_{L^2} + A_0\Big(\frac{\lambda}{\mu} + \frac{\lambda}{\underline{\mu}}\widetilde{U}_0\Big)c_j\Big) \\
&\leq C \sum_{j=N_0}^{\infty} 2^{j\frac{3}{2}} \|\Delta_j a_0\|_{L^2} + CA_0\Big(\frac{\lambda}{\mu} + \frac{\lambda}{\underline{\mu}}\widetilde{U}_0\Big).
\end{aligned}$$

Since $a_0 \in B^{\frac{3}{2}}_{2,1}$ and $A_T \leq 2A_0$, we can select N_0 sufficiently large such that

$$C \sum_{j=N_0}^{\infty} 2^{j\frac{3}{2}} \|\Delta_j a_0\|_{L^2} < (2A_0)^{-(\kappa+1)} \min\Big\{\frac{1}{16}\underline{b},\ \frac{1}{16\mu}\underline{\mu}\Big\}.$$

Thus, $(3.10)_2$ holds with strict inequality provide

$$CA_0\Big(\frac{\lambda}{\mu} + \frac{\lambda}{\underline{\mu}}\widetilde{U}_0\Big) < (2A_0)^{-(\kappa+1)} \min\Big\{\frac{1}{16}\underline{b},\ \frac{1}{16\mu}\underline{\mu}\Big\}. \tag{3.12}$$

Finally, we set T sufficiently small such that $4C2^{2N_0}\|a_0\|^{\frac{2}{\alpha}}_{B^{\frac{3}{2}}_{2,1}} T \leq \log 2$, which combination with (3.11) implies $e^{CV(T)} \leq 4$, where $V(T)$ is defined in Proposition 2. Thus, from Proposition 2, we obtain that

$$\begin{aligned}
\overline{U}(T) &\leq 4CA_T^\kappa\Big(\|-(\bar{u}+u_L)\nabla u_L + \mu a\Delta u_L\|_{L^1_T(B^{\frac{1}{2}}_{2,1})} + \|a\nabla \Pi_L\|_{L^1_T(B^{\frac{1}{2}}_{2,1})} \\
&+ \|(a+1)\bar{B}\cdot\nabla\bar{B}\|_{L^1_T(B^{\frac{1}{2}}_{2,1})} + \|(a+1)\bar{B}\cdot\nabla B_L\|_{L^1_T(B^{\frac{1}{2}}_{2,1})} \\
&+ \|(a+1)B_L\cdot\nabla\bar{B}\|_{L^1_T(B^{\frac{1}{2}}_{2,1})} + \|(a+1)B_L\cdot\nabla B_L\|_{L^1_T(B^{\frac{1}{2}}_{2,1})} \\
&+ \mu A_T\|\bar{u}\|_{L^1_T(B^{\frac{5}{2}-\alpha}_{2,1})}\Big) \leq 4CA_T^\kappa\Big(\overline{U}(T)\|\nabla u_L\|_{L^1_T(B^{\frac{3}{2}}_{2,1})} + U_0\|\nabla u_L\|_{L^1_T(B^{\frac{3}{2}}_{2,1})} \\
&+ 2\mu\|a_0\|_{B^{\frac{3}{2}}_{2,1}}\|u_L\|_{L^1_T(B^{\frac{5}{2}}_{2,1})} + 2(\|a\|_{B^{\frac{3}{2}}_{2,1}} + 1)\bar{E}^2(T) \\
&+ 2(\|a_0\|_{B^{\frac{3}{2}}_{2,1}} + 1)\left(\frac{\lambda}{\nu}\bar{E}(T) + \frac{\lambda^2}{\nu^2}\right) + 2\|a_0\|_{B^{\frac{3}{2}}_{2,1}}\frac{\lambda}{\nu} \\
&+ 2C\mu A_T^{\kappa\alpha+1}T^{\frac{\alpha}{2}}\|\bar{u}\|_{\widetilde{L}^1_T(B^{\frac{5}{2}}_{2,1})} + \frac{1}{8CA_T^\kappa}\|\bar{u}\|_{L^\infty_T(B^{\frac{1}{2}}_{2,1})}\Big).
\end{aligned}$$

If we assume $\lambda \leq \frac{1}{16C}\mu$, then we have

$$\overline{U}(t) \leq 32C\lambda\Big(U_0 + \big(1 + \|a_0\|_{B^{\frac{3}{2}}_{2,1}}\big)\big(\lambda\widetilde{U}_0^2 + 1 + \lambda/\mu\widetilde{U}_0\big)\Big)$$

$$
\begin{aligned}
&+ 64C2^{N_0\alpha}\mu\frac{1}{\underline{\mu}}(1+\|a_0\|_{B^{\frac{3}{2}}_{2,1}})^{\kappa\alpha+1}T^{\frac{\alpha}{2}}\lambda\widetilde{U}_0\\
&\le C_0(U_0+1)\lambda + C_0 2^{N_0\alpha}T\lambda\widetilde{U}_0 + C_0\lambda^2\widetilde{U}_0^2,
\end{aligned}
\tag{3.13}
$$

where C_0 is a general constant depending only on $\|a_0\|_{B^{\frac{3}{2}}_{2,1}}$, μ and $\underline{\mu}$.

Next, we turn to the *a priori* estimate of B and J. From Proposition 3, setting $v = \bar{u} - h\bar{J}$, denoting

$$
W(T) = \int_0^T \Big(\|\nabla\bar{u}\|_{B^{\frac{3}{2}}_{2,1}} + h\|\nabla\bar{J}\|_{B^{\frac{3}{2}}_{2,1}}\Big)dt,
$$

we have

$$
\begin{aligned}
&\bar{E}(T) \le Ce^{CW(T)}\Big(\|(\bar{B},\bar{J})\|_{L^1_T(B^{\frac{5}{2}-\alpha}_{2,1})} + \int_0^T(\|F(\tau)\|_{B^{\frac{1}{2}}_{2,1}} + \|G(\tau)\|_{B^{\frac{1}{2}}_{2,1}})d\tau\Big).\\
&\le \widetilde{C}e^{CW(T)}\Big(T^{\frac{\alpha}{2}}\|\bar{B}\|^{\frac{\alpha}{2}}_{L^\infty_T(B^{\frac{1}{2}}_{2,1})}\|\bar{B}\|^{1-\frac{\alpha}{2}}_{L^1_T(B^{\frac{5}{2}}_{2,1})} + T^{\frac{\alpha}{2}}\|\bar{J}\|^{\frac{\alpha}{2}}_{L^\infty_T(B^{\frac{1}{2}}_{2,1})}\|\bar{J}\|^{1-\frac{\alpha}{2}}_{L^1_T(B^{\frac{5}{2}}_{2,1})}\\
&\quad+\int_0^T\big(\|u_L\|_{B^{\frac{3}{2}}_{2,1}}\|\nabla\bar{B}\|_{B^{\frac{1}{2}}_{2,1}} + \|u_L\|_{B^{\frac{3}{2}}_{2,1}}\|\nabla B_L\|_{B^{\frac{1}{2}}_{2,1}} + \|M(u,B)\|_{B^{\frac{1}{2}}_{2,1}}\big)d\tau\\
&\quad+\int_0^T\big(\|J_L\|_{B^{\frac{3}{2}}_{2,1}}\|\nabla\bar{B}\|_{B^{\frac{1}{2}}_{2,1}} + \|J_L\|_{B^{\frac{3}{2}}_{2,1}}\|\nabla B_L\|_{B^{\frac{1}{2}}_{2,1}} + \|\bar{B}\|_{B^{\frac{3}{2}}_{2,1}}\|\nabla\bar{u}\|_{B^{\frac{1}{2}}_{2,1}}\big)d\tau\\
&\quad+\int_0^T\big(\|\bar{B}\|_{B^{\frac{3}{2}}_{2,1}}\|\nabla u_L\|_{B^{\frac{1}{2}}_{2,1}} + \|\bar{B}\|_{B^{\frac{3}{2}}_{2,1}}\|\nabla\bar{J}\|_{B^{\frac{1}{2}}_{2,1}} + \|\bar{B}\|_{B^{\frac{3}{2}}_{2,1}}\|\nabla J_L\|_{B^{\frac{1}{2}}_{2,1}}\big)d\tau\\
&\quad+\int_0^T\big(\|B_L\|_{B^{\frac{3}{2}}_{2,1}}\|\bar{u}\|_{B^{\frac{1}{2}}_{2,1}} + \|B_L\|_{B^{\frac{3}{2}}_{2,1}}\|\nabla u_L\|_{B^{\frac{1}{2}}_{2,1}} + \|B_L\|_{B^{\frac{3}{2}}_{2,1}}\|\nabla\bar{J}\|_{B^{\frac{1}{2}}_{2,1}}\big)d\tau\\
&\quad+\int_0^T\big(\|B_L\|_{B^{\frac{3}{2}}_{2,1}}\|\nabla J_L\|_{B^{\frac{1}{2}}_{2,1}} + \|u_L\|_{B^{\frac{3}{2}}_{2,1}}\|\nabla\bar{B}\|_{B^{\frac{3}{2}}_{2,1}} + \|u_L\|_{B^{\frac{3}{2}}_{2,1}}\|\nabla B_L\|_{B^{\frac{3}{2}}_{2,1}}\big)d\tau\\
&\quad+\int_0^T\big(\|J_L\|_{B^{\frac{3}{2}}_{2,1}}\|\nabla\bar{B}\|_{B^{\frac{3}{2}}_{2,1}} + \|J_L\|_{B^{\frac{3}{2}}_{2,1}}\|\nabla\bar{B}\|_{B^{\frac{3}{2}}_{2,1}} + \|\bar{B}\|_{B^{\frac{3}{2}}_{2,1}}\|\nabla\bar{u}\|_{B^{\frac{3}{2}}_{2,1}}\big)d\tau\\
&\quad+\int_0^T\big(\|\bar{B}\|_{B^{\frac{3}{2}}_{2,1}}\|\nabla u_L\|_{B^{\frac{3}{2}}_{2,1}} + \|\bar{B}\|_{B^{\frac{3}{2}}_{2,1}}\|\nabla\bar{J}\|_{B^{\frac{3}{2}}_{2,1}} + \|\bar{B}\|_{B^{\frac{3}{2}}_{2,1}}\|\nabla J_L\|_{B^{\frac{3}{2}}_{2,1}}\big)d\tau\\
&\quad+\int_0^T\big(\|B_L\|_{B^{\frac{3}{2}}_{2,1}}\|\bar{u}\|_{B^{\frac{3}{2}}_{2,1}} + \|B_L\|_{B^{\frac{3}{2}}_{2,1}}\|\nabla u_L\|_{B^{\frac{3}{2}}_{2,1}} + \|B_L\|_{B^{\frac{3}{2}}_{2,1}}\|\nabla\bar{J}\|_{B^{\frac{3}{2}}_{2,1}}\big)d\tau\Big).
\end{aligned}
$$

Here, we roughly define that $\widetilde{C} = CA_0$. If we choose λ sufficient small such that

$$
e^{C\lambda\widetilde{U}_0} \le 2,
\tag{3.14}
$$

then there exist an absolute constant C_1 such that

$$
\bar{E}(T) \le C_1\lambda(T^{\frac{\alpha}{2}}\lambda\widetilde{U}_0^2 + \lambda\widetilde{U}_0 + \lambda + \lambda\widetilde{U}_0^2).
\tag{3.15}
$$

Here, we omit the details of the estimate for $M(u,B)$ in (3.9) since it is similar to the terms in F or G. Combining (3.15) with (3.13), we have

$$
\bar{U}(T)+\bar{E}(T) \le C_0U_0\lambda + C_02^{N_0\alpha}T\lambda\widetilde{U}_0 + C_0\lambda^2\widetilde{U}_0^2 + C_1\lambda(T^{\frac{\alpha}{2}}\lambda\widetilde{U}_0^2 + \lambda\widetilde{U}_0 + \lambda + \lambda\widetilde{U}_0^2).
$$

Hence, selecting $\widetilde{U}_0 = 16(C_0 + C_1)(U_0 + 1)$, for fixed λ which determined by (3.11)–(3.14), and in addition we can choose T, λ small enough such that

$$T \leq 1, \quad C_1\lambda\widetilde{U}_0 \leq 1/32, \quad C_0 2^{N_0\alpha} T < 1/4, \quad 4C2^{2N_0}\|a_0\|^{\frac{2}{\alpha}}_{B^{\frac{3}{2}}_{2,1}} T \leq \log 2. \quad (3.16)$$

Then we can find that $(3.10)_3$ holds with strict inequality.

Step 3: Constructing approximate solutions and uniform estimates.

It is based on Friedrichs' method: consider the spectral cut-off operator $\mathbb{E}_n$ defined by

$$\mathcal{F}(\mathbb{E}_n f)(\xi) = 1_{\{n^{-1} \leq |\xi| \leq n\}}(\xi)\mathcal{F}(f)(\xi).$$

We want to solve the following truncated system:

$$\begin{cases} \partial_t a = \mathbb{E}_n(-\mathbb{E}_n u \cdot \mathbb{E}_n \nabla a), \\ \partial_t u - \Delta u = \mathbb{E}_n(\mathbb{E}_n(1+a)\mathbb{E}_n B \cdot \mathbb{E}_n \nabla B - \mathbb{E}_n u \cdot \nabla \mathbb{E}_n u \\ \qquad\qquad + \mathbb{E}_n a(\mathbb{E}_n \Delta u - \mathbb{E}_n \nabla \Pi)), \\ \partial_t B - \Delta B = \nabla \times \mathbb{E}_n(\mathbb{E}_n(u - (1+a)\nabla \times B) \times \mathbb{E}_n B), \\ \partial_t J - \Delta J = \nabla \times \nabla \times \mathbb{E}_n(\mathbb{E}_n(u - (1+a)\nabla \times \mathrm{curl}^{-1} J) \times \mathbb{E}_n \mathrm{curl}^{-1} J), \end{cases} \quad (3.17)$$

supplemented with initial data $(\mathbb{E}_n a_0, \mathbb{E}_n u_0, \mathbb{E}_n B_0, \mathbb{E}_n J_0)$.

Then we can conclude that the ordinary differential equations have a unique solution (a^n, u^n, B^n) in the space $C^1([0, T_n^*); L_n^2)$, where L_n^2 is L^2 functional space with Fourier transforms supported in the ball with center 0 and radius n. T_n^* is the maximum existence time of (a^n, u^n, B^n). Then using the elliptic equation we can get the existence of $\nabla \Pi^n \in C^1([0, T_n^*); L_n^2)$. Being spectrally supported in the annulus $\{n^{-1} \leq |\xi| \leq n\}$, one can also deduce that the solution belongs to $C^1([0, T_n^*); B^s_{2,1})$ for all $s \in \mathbb{R}$. Hence, setting $J^n = \nabla \times B^n$,we see that a^n, u^n, B^n, J^n belong to the space $B^{\frac{1}{2}}_{2,1}$ for all $T < T_n^*$ and fulfill:

$$\begin{cases} \partial_t a^n = \mathbb{E}_n(-u^n \cdot \nabla a^n), \\ \partial_t u^n - \Delta u^n - E_n(a^n \Delta u^n) = \mathbb{E}_n((1+a^n)B^n \cdot \nabla B^n \\ \qquad\qquad - u^n \cdot \nabla u^n - a^n \nabla p)), \\ \partial_t B^n - \Delta B^n = \nabla \times \mathbb{E}_n((u^n - J^n) \times B^n), \\ \partial_t J^n - \Delta J^n = \nabla \times \mathbb{E}_n(\nabla \times (u^n - J^n) \times \mathrm{curl}^{-1} J^n). \end{cases}$$

Now, we want to prove that T_n^* may be bounded from below by the supremum T of all the times satisfying (3.17), a^n and (u^n, B^n, J^n) is uniformly bounded in $\mathcal{C}([0,T]; B^{\frac{3}{2}}_{2,1})$ and $X(T)$, respectively. Since $\mathbb{E}_n$ is an L^2 orthogonal projector, it has no effect on the *a priori* estimates which were obtained in the above steps. Hence, the *a priori* estimates applies to our approximate solution (a^n, u^n, B^n, J^n) which independent of n. And the estimate in (3.10) to (a^n, u^n, B^n, J^n) ensue that it is bounded in $L^\infty(0, T; L_n^2)$. So, the standard continuation criterion for ordinary differential equations implies that T_n^* is greater than any time T satisfying (3.16) and for all $n \geq 1$,

$$\begin{cases} \|a^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}}_{2,1})} \leq 2\|a_0\|_{B^{\frac{3}{2}}_{2,1}}, \\ A_T^{\kappa+1}\|a^n - S_{N_0}a^n\|_{\widetilde{L}^\infty_T(B^{\frac{3}{2}}_{2,1})} \leq \min\{\frac{1}{4C}\underline{b}, \ \frac{1}{4C\mu}\underline{\mu}\}, \\ \|(\bar{u}^n, \bar{B}^n, \bar{J}^n\|_{\widetilde{L}^\infty_T(B^{\frac{1}{2}}_{2,1})} + \|(\bar{u}^n, \bar{B}^n, \bar{J}^n\|_{\widetilde{L}^1_T(B^{\frac{5}{2}}_{2,1})} \leq \lambda\widetilde{U}_0. \end{cases}$$

Step 4: Compactness argument.

Till now we have proved that a^n and (u^n, B^n, J^n) is uniformly bounded in $\mathcal{C}([0,T]; B^{\frac{3}{2}}_{2,1})$ and $X(T)$, respectively. The lower bound of T is determined in last step. A standard compactness argument based on the Ascoli's theorem implies that (a^n, u^n, B^n, J^n) tends to (a, u, B, J) which satisfies (1.5).

3.2 Uniqueness results

In this subsection, we study the problem of uniqueness for a solution with critical regularity. We shall prove the following proposition.

Proposition 4. *Let $(a_1, u_1, B_1, \nabla\Pi_1, J_1)$ and $(a_2, u_2, B_2, \nabla\Pi_2, J_2)$ be two solutions of the density-dependent incompressible Hall-MHD system (1.5) with the same initial data and such that $a_i \in \mathcal{C}([0,T]; B^{\frac{3}{2}}_{2,1})$ and $(u_i, B_i, J_i) \in X(T)$ for $i = 1, 2$. Then, $(a_1, u_1, B_1, \nabla\Pi_1) = (a_2, u_2, B_2, \nabla\Pi_2)$ on $[0,T]$ for some $T > 0$.*

Proof. We remark that the L^∞ norm of a_1 and a_2 is conserved. The equations for

$$(\delta a, \delta u, \delta B, \delta\nabla p, \delta J) := (a_1 - a_2, u_1 - u_2, B_1 - B_2, \nabla\Pi_1 - \nabla\Pi_2, J_1 - J_2)$$

can be written as

$$\begin{cases} \partial_t\delta a + u_2\cdot\nabla\delta a = -\delta u\cdot\nabla a_1, \\ \partial_t\delta u + u_2\cdot\nabla\delta u - (1+a_1)(\mu\Delta\delta u - \nabla\delta p) = \delta H_1 + \delta H_2, \\ \partial_t\delta B - \nu\Delta\delta B = \delta G, \\ \partial_t\delta J - \nu\Delta\delta J = \delta L, \end{cases} \tag{3.18}$$

where denote

$$\begin{aligned} &\delta H_1 := -\delta u\cdot\nabla u_1 + \mu\delta a\Delta u_2 - \delta a\nabla\Pi_2, \\ &\delta H_2 := \delta a B_1\cdot\nabla B_1 + (1+a_2)\delta B\cdot\nabla B_1 + (1+a_2)B_2\cdot\nabla\delta B, \\ &\delta G := \nabla\times\big((\delta u - h\delta a J_1)\times B_1 + (u_2 - h(1+a_2)J_2)\times\delta B - h(1+a_2)\delta J\times B_2\big) \end{aligned}$$

and $\delta L = \nabla\times\delta G$. As usual, when proving uniqueness, the presence of a transport equation is responsible for the loss of one derivative in the estimates involving δa. This induces us to bound $\delta a\times(\delta u, \delta B, \delta J)\times\delta\nabla p$ in the space $C([0,T]; B^{\frac{1}{2}}_{2,1}) \times F^{-\frac{1}{2}}_T \times L^1_T(B^{-\frac{1}{2}}_{2,1})$, where

$$F^{-\frac{1}{2}}_T = C([0,T]; B^{-\frac{1}{2}}_{2,1}) \cap L^1_T(B^{\frac{3}{2}}_{2,1})).$$

Hence, arguing as in Step 2 of the proof gives for all $t\in[0,T]$, we apply Propositions 1 and 2 to system (3.18). Indeed, from Proposition 1, for all $t\in[0,T]$, we have

$$\begin{aligned} \|\delta a(t)\|_{\dot{B}^{\frac{1}{2}}_{2,1}} \le{}& e^{CV(t)}\|a_1 - S_{N_0}a_1\|_{L^\infty_t(B^{\frac{3}{2}}_{2,1})}\|\delta u\|_{L^1_t(B^{\frac{3}{2}}_{2,1})} \\ &+ e^{CV(t}\|S_{N_0}a_1\|_{L^\infty_t(B^{\frac{3}{2}+\eta}_{2,1})} t^{\frac{\eta}{2}}\|\delta u\|^{\frac{\eta}{2}}_{\widetilde{L}^\infty_t(B^{-\frac{1}{2}}_{2,1})}\|\delta u\|^{1-\frac{\eta}{2}}_{L^1_t(B^{\frac{3}{2}}_{2,1})}, \end{aligned}$$

where $V(t) = \int_0^t \|\nabla u_2\|_{B^{\frac{3}{2}}_{2,1}} d\tau$ and $\eta \in (0,1)$. By choosing N_0 sufficiently large and using Young's inequality, we have

$$\|\delta a(t)\|_{\dot{B}^{\frac{1}{2}}_{2,1}} \leq \frac{1}{8}\underline{\mu}\|\delta u\|_{L^1_t(B^{\frac{3}{2}}_{2,1})} + C2^{N_0}T\|\delta u\|_{\widetilde{L}^\infty_t(B^{-\frac{1}{2}}_{2,1})}. \tag{3.19}$$

Next, denoting

$$\delta U(t) = \|\delta u(t)\|_{\dot{B}^{-\frac{1}{2}}_{2,1})} + \underline{\mu}\|\delta u\|_{L^1_t(\dot{B}^{\frac{3}{2}}_{2,1})} + \|\nabla \delta p\|_{L^1_t(\dot{B}^{-\frac{1}{2}}_{2,1})},$$

the estimate for linear momentum equations, Proposition 2, guides us to get

$$\delta U(t) \leq Ce^{CV(T)}A^k_{1,t}\Big(\|\delta H_1\|_{L^1_t(B^{-\frac{1}{2}}_{2,1})} + \|\delta H_2\|_{L^1_t(B^{-\frac{1}{2}}_{2,1})} + \mu A_{1,t}\|\delta u\|_{L^1_t(B^{\frac{3}{2}-\alpha}_{2,1})}\Big),$$

with $A_{1,t} = 1 + \underline{b}2^{N_0\alpha}\|a_1\|_{B^{\frac{3}{2}}_{2,1}}$ and $\alpha \in (0,1)$. By interpolation, the last term can be bounded by

$$\|\delta u\|_{L^1_t(B^{\frac{3}{2}-\alpha}_{2,1})} \leq C\|\delta u\|^{\frac{\alpha}{2}}_{L^1_t(B^{-\frac{1}{2}}_{2,1})}\|\delta u\|^{1-\frac{\alpha}{2}}_{L^1_t(B^{\frac{3}{2}}_{2,1})}.$$

Young's inequality and the uniform bound of the solution imply that

$$\delta U(t) \leq C\int_0^t \Big(\|\delta H_1\|_{B^{-\frac{1}{2}}_{2,1}} + \|\delta H_2\|_{B^{-\frac{1}{2}}_{2,1}} + \|\delta u\|_{B^{-\frac{1}{2}}_{2,1}}\Big)d\tau. \tag{3.20}$$

According to the expressions of δH_1 and δH_2, by using the para-product and remainder estimations in Besov spaces ([3], Chapter 2), one has

$$\begin{aligned}
\|\delta H_1\|_{B^{-\frac{1}{2}}_{2,1}} &\leq C\Big(\|\delta u\|_{B^{-\frac{1}{2}}_{2,1}}\|\nabla u_1\|_{B^{\frac{3}{2}}_{2,1}} + \mu\|\delta a\|_{B^{\frac{1}{2}}_{2,1}}\|\Delta u_2\|_{B^{\frac{1}{2}}_{2,1}}\|\delta a\|_{B^{\frac{1}{2}}_{2,1}}\|\nabla \Pi_2\|_{B^{\frac{1}{2}}_{2,1}}\Big),\\
\|\delta H_2\|_{B^{-\frac{1}{2}}_{2,1}} &\leq C\Big(\|\delta a\|_{B^{\frac{1}{2}}_{2,1}}\|B_1\|^2_{B^{\frac{3}{2}}_{2,1}} + (1 + \|a_1\|_{B^{\frac{3}{2}}_{2,1}})\|\delta B\|_{B^{-\frac{1}{2}}_{2,1}}\|\nabla B_1\|_{B^{\frac{3}{2}}_{2,1}}\\
&\quad + (1 + \|a_1\|_{B^{\frac{3}{2}}_{2,1}})\|\nabla B_2\|_{B^{\frac{3}{2}}_{2,1}}\|\delta B\|_{B^{-\frac{1}{2}}_{2,1}}\Big).
\end{aligned}$$

Plugging the above inequalities into (3.20), and by using the Young's inequality on (3.19), we get that

$$\begin{aligned}
\|\delta a\|_{\widetilde{L}^\infty_T(B^{\frac{1}{2}}_{2,1})} + \delta U &\leq C\int_0^t \big(\|u_1\|_{B^{\frac{5}{2}}_{2,1}} + \|u_2\|_{B^{\frac{5}{2}}_{2,1}} + \|\nabla \Pi_2\|_{B^{\frac{1}{2}}_{2,1}}\big)\delta U(\tau)d\tau\\
&+ C\int_0^t (\|B_1\|^2_{B^{\frac{3}{2}}_{2,1}} + \|B_1\|_{B^{\frac{5}{2}}_{2,1}} + \|B_2\|_{B^{\frac{5}{2}}_{2,1}})\delta U(\tau)d\tau + C2^{N_0}T\delta U(t).
\end{aligned} \tag{3.21}$$

The standard estimates of the heat equation in [3] gives for all $t \in [0,T]$,

$$\|(\delta B, \delta J)(t)\|_{B^{-\frac{1}{2}}_{2,1}} + \|(\delta B, \delta J)(\tau)\|_{\widetilde{L}^2_t(B^{\frac{1}{2}}_{2,1})} \leq \|(\delta G, \delta L)\|_{\widetilde{L}^2_t(B^{-\frac{3}{2}}_{2,1})}. \tag{3.22}$$

For the right hand side, by using the para-product and remainder estimations in Besov spaces, we have

$$\begin{aligned}\|\delta G\|_{\widetilde{L}^2_t(B^{-\frac{3}{2}}_{2,1})} &\le C\Big(\|\delta u\|_{\widetilde{L}^\infty_t(B^{-\frac{1}{2}}_{2,1})}\|B\|_{\widetilde{L}^2_t(B^{\frac{3}{2}}_{2,1})} + \|\delta a\|_{\widetilde{L}^\infty_t(B^{\frac{1}{2}}_{2,1})}\|J_1\|_{\widetilde{L}^\infty_t(B^{\frac{1}{2}}_{2,1})}\\ &\times \|B_1\|_{\widetilde{L}^2_t(B^{\frac{3}{2}}_{2,1})} + \big(\|u_2\|_{\widetilde{L}^2_t(B^{\frac{3}{2}}_{2,1})} + \|J_2\|_{\widetilde{L}^2_t(B^{\frac{3}{2}}_{2,1})}\big)\|\delta B\|_{\widetilde{L}^\infty_t(B^{-\frac{1}{2}}_{2,1})}\\ &+ (1+\|a_2\|_{\widetilde{L}^\infty_t(B^{\frac{3}{2}}_{2,1})})\|B_2\|_{\widetilde{L}^2_t(B^{\frac{3}{2}}_{2,1})}\|\delta J\|_{\widetilde{L}^\infty_t(B^{-\frac{1}{2}}_{2,1})}\Big). \qquad (3.23)\end{aligned}$$

Similarly, by using the identity $B = \mathrm{curl}^{-1}J$, we still have

$$\begin{aligned}\|\delta L\|_{\widetilde{L}^2_t(B^{-\frac{3}{2}}_{2,1})} &\le C\Big(\|\delta u\|_{\widetilde{L}^2_t(B^{\frac{1}{2}}_{2,1})}\|B\|_{\widetilde{L}^\infty_t(B^{\frac{3}{2}}_{2,1})} + \|\delta a\|_{\widetilde{L}^\infty_t(B^{\frac{1}{2}}_{2,1})}\|J_1\|_{\widetilde{L}^\infty_t(B^{\frac{1}{2}}_{2,1})}\\ &\times \|B_1\|_{\widetilde{L}^2_t(B^{\frac{3}{2}}_{2,1})} + \big(\|u_2\|_{\widetilde{L}^2_t(B^{\frac{3}{2}}_{2,1})} + \|J_2\|_{\widetilde{L}^2_t(B^{\frac{3}{2}}_{2,1})}\big)\|\delta B\|_{\widetilde{L}^\infty_t(B^{\frac{1}{2}}_{2,1})}\\ &+ (1+\|a_2\|_{\widetilde{L}^\infty_t(B^{\frac{3}{2}}_{2,1})})\|B_2\|_{\widetilde{L}^2_t(B^{\frac{3}{2}}_{2,1})}\|\delta J\|_{\widetilde{L}^\infty_t(B^{\frac{1}{2}}_{2,1})}\Big). \qquad (3.24)\end{aligned}$$

Hence, combining (3.21)–(3.23) with (3.24), we finally obtain that

$$\begin{aligned}&\|\delta a\|_{\widetilde{L}^\infty_t(B^{\frac{1}{2}}_{2,1})} + \delta U + \|(\delta B,\delta J)\|_{\widetilde{L}^\infty_t(B^{-\frac{1}{2}}_{2,1})} + \|(\delta B,\delta J)\|_{\widetilde{L}^2_t(B^{\frac{1}{2}}_{2,1})}\\ &\le \frac{1}{2}\Big(\|\delta a\|_{\widetilde{L}^\infty_t(B^{\frac{1}{2}}_{2,1})} + \delta U + \|(\delta B,\delta J)\|_{\widetilde{L}^\infty_t(B^{-\frac{1}{2}}_{2,1})} + \|(\delta B,\delta J)\|_{\widetilde{L}^2_t(B^{\frac{1}{2}}_{2,1})}\Big)\end{aligned}$$

by choosing T sufficiently small. This ensures that on $[0,T]$,

$$(\delta a, \delta u, \delta B, \delta J) \equiv 0.$$

This finish the proof of the uniqueness of the solution. □

3.3 The global estimates for small initial data

In the above section, we have proved that there exists a unique local solution (a,u,B,J) of (1.5) in $C([0,T];B^{\frac{3}{2}}_{2,1}) \times X(T)$. We have used the L^2 estimate for $\nabla\Pi$. That is the reason why we work on the non-homogeneous Besov space. We rewrite it as follows:

$$\underline{b}\|\nabla\Pi\|_{B^\sigma_{2,1}} \le C\|\mathcal{Q}L\|_{B^\sigma_{2,1}} + \|a\|_{B^{\frac{3}{2}}_{2,1}}\|\nabla\Pi\|_{B^\sigma_{2,1}}.$$

While fortunately, the assumption on a_0 in Theorem 1 can avoid the L^2 estimate of $\nabla\Pi$. More precisely, the second term can be absorbed by the left hand side due to the smallness condition on a. Thus using the same method as in Theorem 1, we obtain that there exists a unique local solution (a,u,B,J) of (1.5) in

$$C([0,T];B^{\frac{3}{2}}_{2,1}) \times X(T)$$

for all $T < T^*$, where T^* is the maximum existence time of (a,u,B,J). Denote $\alpha = \|a_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|(u_0,B_0,J_0)\|_{\dot{B}^{\frac{1}{2}}_{2,1}}$. We are going to prove the existence of a positive M such that, if α is small enough, the following bound holds

$$\|a\|_{\widetilde{L}^\infty_T(\dot B^{\frac{3}{2}}_{2,1})} + \|(u,B,J)\|_{\widetilde{L}^\infty_T(\dot B^{\frac{1}{2}}_{2,1})\cap \widetilde{L}^1_T(\dot B^{\frac{5}{2}}_{2,1})} \le M\alpha. \tag{3.25}$$

This estimate is the direct product of the following proposition.

Proposition 5. *If*

$$\|a\|_{\widetilde{L}^\infty_T(\dot B^{\frac{3}{2}}_{2,1})} + \|(u,B,J)\|_{\widetilde{L}^\infty_T(\dot B^{\frac{1}{2}}_{2,1})\cap L^1_T(\dot B^{\frac{5}{2}}_{2,1})} + \|\nabla \Pi\|_{L^1_T(\dot B^{\frac{1}{2}}_{2,1})} \le 2M\alpha,\ T\in(0,T^*),$$

then, we have

$$\|a\|_{\widetilde{L}^\infty_T(\dot B^{\frac{3}{2}}_{2,1})} + \|(u,B,J)\|_{\widetilde{L}^\infty_T(\dot B^{\frac{1}{2}}_{2,1})\cap L^1_T(\dot B^{\frac{5}{2}}_{2,1})} + \|\nabla \Pi\|_{L^1_T(\dot B^{\frac{1}{2}}_{2,1})} \le M\alpha,$$

when α is small enough.

Proof. First, from Proposition 3.1, we obtain

$$\|a\|_{\widetilde{L}^\infty_T(\dot B^{\frac{3}{2}}_{2,1})} \le C_1 e^{\widetilde{V}(T)}\|a_0\|_{\dot B^{\frac{3}{2}}_{2,1}},$$

where $\widetilde{V}(T) = \int_0^T \|\nabla u\|_{\dot B^{\frac{3}{2}}_{2,1}}\,dt$. If we assume α small enough such that $e^{2M\alpha}\le 2$, then we have

$$\|a\|_{\widetilde{L}^\infty_T(\dot B^{\frac{3}{2}}_{2,1})} \le M\alpha$$

for $M = 4C_1$. For the estimate of (u,B,J), we first rewrite the equations of them as

$$\begin{cases} \partial_t u + u\cdot\nabla u - \mu\Delta u + \nabla \Pi = R,\\ \partial_t B - \nu\Delta B = L,\\ \partial_t J - \nu\Delta J = G,\end{cases}$$

where

$$\begin{aligned} R =&\mu a\Delta u - u\cdot\nabla u - a\nabla \Pi + (1+a)B\cdot\nabla B,\\ L =&\nabla\times(u\times B) - \nabla\times(h(1+a)J\times B),\\ G =&\nabla\times(\nabla\times((u-h(1+a)J)\times \mathrm{curl}^{-1}J)).\end{aligned}$$

According to the standard estimates of the linear Stokes equations and heat equations, we obtain that

$$\begin{aligned}&\|(u,B,J)\|_{\widetilde{L}^\infty_T(\dot B^{\frac{1}{2}}_{2,1})\cap L^1_T(\dot B^{\frac{5}{2}}_{2,1})} + \|\nabla \Pi\|_{L^1_T(\dot B^{\frac{1}{2}}_{2,1})}\\ &\qquad \le C\Big(\|(u_0,B_0,J_0)\|_{\dot B^{\frac{1}{2}}_{2,1}} + \|(R,L,G)\|_{L^1_T(\dot B^{\frac{1}{2}}_{2,1})}\Big).\end{aligned} \tag{3.26}$$

In the rest part of this section, we are going to bound $\|(R,L,M)\|_{L^1_T(\dot B^{\frac{1}{2}}_{2,1})}$ term by term. In fact, by the continuity law of the product in Besov space, firstly, we have

$$\begin{aligned}\|R\|_{L^1_T(\dot B^{\frac{1}{2}}_{2,1})} &\le CM\alpha\Big(\|\Delta u\|_{\widetilde{L}^1_T(\dot B^{\frac{1}{2}}_{2,1})}\|\nabla \Pi\|_{\widetilde{L}^1_T(\dot B^{\frac{1}{2}}_{2,1})} + \|B\|^2_{\widetilde{L}^2_T(\dot B^{\frac{3}{2}}_{2,1})}\Big)\\ &\le CM^2\alpha^2 + CM^3\alpha^3,\end{aligned}$$

$$\begin{aligned}\|L\|_{L^1_T(\dot B^{\frac12}_{2,1})} &\le \|\nabla\times(u\times B)\|_{L^1_T(\dot B^{\frac12}_{2,1})} + \|\nabla\times(hJ\times B)\|_{L^1_T(\dot B^{\frac12}_{2,1})}\\ &\le CM\alpha\Big(\|u\|_{\widetilde L^2_T(\dot B^{\frac32}_{2,1})}\|B\|_{L^1_T(\dot B^{\frac32}_{2,1})} + \|J\|_{\widetilde L^2_T(\dot B^{\frac32}_{2,1})}\|B\|_{\widetilde L^2_T(\dot B^{\frac32}_{2,1})}\Big) \le CM^2\alpha^2\end{aligned}$$

and

$$\begin{aligned}\|G\|_{L^1_T(\dot B^{\frac12}_{2,1})} &\le \|\nabla\times((u-h(1+a)J)\times \mathrm{curl}^{-1}J)\|_{L^1_T(\dot B^{\frac32}_{2,1})}\\ &\le C\Big(\|\nabla u\|_{\widetilde L^2_T(\dot B^{\frac32}_{2,1})} + \|\nabla J\|_{\widetilde L^1_T(\dot B^{\frac32}_{2,1})}\Big)\|\mathrm{curl}^{-1}J\|_{\widetilde L^\infty_T(\dot B^{\frac32}_{2,1})}\\ &\quad + \Big(\|u\|_{\widetilde L^2_T(\dot B^{\frac32}_{2,1})} + \|J\|_{\widetilde L^2_T(\dot B^{\frac32}_{2,1})}\Big)\|\nabla\mathrm{curl}^{-1}J\|_{\widetilde L^2_T(\dot B^{\frac32}_{2,1})} \le CM^2\alpha^2.\end{aligned}$$

Plugging the estimates on R, L and M into (3.26), noting that $B = \mathrm{curl}^{-1}J$, we have

$$\|(u,B,J)\|_{\widetilde L^\infty_T(\dot B^{\frac12}_{2,1})\cap L^1_T(\dot B^{\frac52}_{2,1})} + \|\nabla\Pi\|_{L^1_T(\dot B^{\frac12}_{2,1})} \le C_2\Big(\alpha + C_2M^2\alpha^2 + C_2M^3\alpha^3\Big)$$

holds for some positive constant C_2. When $M = 4C_2$ and α satisfies

$$C_2^2M\alpha \le 1/4,\ \ C_2^2M^2\alpha^2 \le 1/4.$$

Then, we finish the proof of Proposition 5 for $M = \max\{4C_1, 4C_2\}$. □

3.4 Proof of global existence result

Now we can give the proof of the global existence. From the standard continuation method and Proposition 5, we easily obtain that (3.25) holds. Combining the local existence, if T^* is finite, then the lifespan of the solution is greater than T^*. Hence $T^* = \infty$ and we finish the proof of Theorem 1.

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This paper consider the local and global well-posedness for the inhomogeneous Hall-MHD system, which is more close to the real situation. This generalized the recent work on the homogeneous system by R. Danchin and J. Tan [11]. Our future work will focus on the well-posedness for the compressible Hall-MHD system. The third author was partially supported by the Zhejiang Province Science Fund (LY21A010009). Finally, we would like to thank Professor Daoyuan Fang for many valuable suggestions.

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