



# Numerical Solution of Falkner-Skan Equation by Iterative Transformation Method

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**Abstract.** In this paper, we study the nonlinear boundary-layer equation of Falkner-Skan defined on a semi-infinite domain. An iterative finite difference (IFD) scheme is proposed to numerically solve such nonlinear ordinary differential equation. A computational iterative scheme is developed based on Newton-Kantorovich quasi-linearization. At every iteration, the obtained linearized differential equation is numerically solved using the standard finite difference method. Numerical experiments show the accuracy and efficiency of the method compared to existing solvers. The computation is performed for different parameter values, including the special case of Blasius problem.

**Keywords:** Falkner-Skan equation, Blasius equation, quasi-linearization, iterative finite difference method.

**AMS Subject Classification:** 34B15; 65D15; 65L10; 65L12.

## 1 Introduction

The numerical solution for nonlinear differential equations is a major problem in computational mathematics. Specifically, nonlinear boundary value problems on infinite domain attract a major attention of the scientific community. One known nonlinear differential equation defined on a semi-infinite domain is the so-called Blasius equation which appeared in the literature in 1908 [4]. Blasius equation describes the viscous flow in a laminar boundary layer over a flat plate. Because of its interesting application to fluid flow, physicists as well as engineers have been always keenly interested in solving this nonlinear ordinary

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differential equation rich in physical, mathematical, and numerical challenges. Blasius equation is part of the set of more general nonlinear problems known as the Falkner-Skan equations [7]. The third-order nonlinear equation of Falkner-Skan type is a famous example of nonlinear boundary value problems on semi-infinite domains arising from the boundary layer flow with stream-wise pressure gradient. It is governed by the following equation:

$$\frac{d^3 f}{d\eta^3} + \beta_0 f \frac{d^2 f}{d\eta^2} + \beta_1 \left[ 1 - \left( \frac{df}{d\eta} \right)^2 \right] = 0, \quad 0 < \eta < \infty \quad (1.1a)$$

subject to the boundary conditions

$$f(0) = 0, \quad f'(0) = \gamma, \quad \lim_{\eta \rightarrow +\infty} f'(\eta) = 1, \quad (1.1b)$$

where  $\gamma$  is the movement velocity ratio of the plate to the mainstream.

In fact, when  $0 < \gamma < 1$ , the speed of oncoming fluid is larger than the speed of the plate; however, when  $\gamma > 1$ , the speed of the moving plate is faster than the speed of the oncoming fluid. Furthermore, the case when  $\gamma < 0$  corresponds to a movement of the plate and the mainstream in opposite directions. Particularly, the special case  $\gamma = 0$  (for a fixed plate) and  $\beta_1 = 0$  leads to Blasius problem [4].

Due to its challenging physical properties, many researchers have attempted to solve Falkner-Skan equation with various methods. Among these attempts, we can cite the work of Hartree [11] who showed that under some conditions there exists a family of unique solutions of the problem whose first-order derivative tends to 1 exponentially. Howarth [14] attempted also to solve the boundary layer problem in a retarded region by reducing the problem to the solution of a first order differential equation.

Later, Falkner-Skan problem has attracted the attention of more researchers such as Cebeci and Keller [5], Asaithambi [2], and Sher and Yakhot [27] who applied various numerical approaches to solve this problem. Moreover, Liao [15, 16] applied the homotopy analysis method to solve the Falkner-Skan equation which provided an explicit analytical solution with the same boundary conditions as those of above. Additionally, Salama [26] proposed a higher-order method for solving such free boundary problems. Sachdev et al [25] proposed also an exact solution to Falkner-Skan equation subject to appropriate physical boundary conditions arising from boundary layer theory. Besides, Liu and Chang [17] investigated numerically the solutions of the Falkner-Skan equation with the boundary conditions of resting and impermeable wall. Moreover, Zhu et al. [30] presented two iterative methods for solving the Falkner-Skan equation based on the quasilinearization technique and a cubic spline solver. The original problem has been formulated as a new free boundary value problem. Additionally, an adaptive algorithm for the Thomas-Fermi equation by means of the moving mesh finite element method has been developed [31].

Recently, Fang et al. [8, 9] have studied analytically a special case of the Falkner-Skan equation using  $\beta_1 = -1$  with the boundary conditions of mass transfer and wall stretching, as well as the problem of a shrinking sheet with a constant sheet velocity. On the other hand, different numerical methods have been applied in order to find accurate solutions, such as Runge-Kutta

algorithm [6], recursive evaluation of Taylor coefficients [1], and iterative transformation method [10]. Furthermore, other types of similar flows has been recently treated such as the work of Raju et al. who studied the nonlinear flows of Casson and Carreau fluids on various types of wedges [20, 21, 22, 23, 24]. Also, Mukhopadhyay et al. [18] investigated a boundary-layer forced convection flow of a Casson fluid past a symmetric wedge.

Despite these aforementioned research work since the beginning of the previous century, there is no closed-form solutions available for Falkner-Skan problem nor for the special case of Blasius problem. This fact stimulated researchers to investigate for most reliable, efficient, and low-computation cost solutions. In this manuscript, we introduce a new coordinate transformation and we apply it to Falkner-Skan problem (1.1) in order to overcome the difficulty of the semi-infinite domain. Using the aforementioned transformation and the introduced auxiliary function, a system of nonlinear differential equations is obtained leading to a second order nonlinear integro-differential equation subjected to boundary conditions defined on a finite values 0 and 1. In order to solve the resulting nonlinear integro-differential equation, we develop an iterative scheme based on Newton-Raphson-Kantorovich method in function space [3, 28] combined with the standard finite difference method. Numerical simulations will be performed to demonstrate the efficiency and reliability of the proposed numerical solver in producing accurate solutions with low computation cost.

## 2 Transformed problem and iterative finite difference solution

### 2.1 Order reduction and coordinate transformation

In order to perform order reduction of the problem (1.1), we introduce an auxiliary function  $g$  defined by  $g(\eta) = f'(\eta)$  to obtain the following nonlinear system

$$\begin{cases} \frac{d^2g}{d\eta^2} + \beta_0 f \frac{dg}{d\eta} + \beta_1 (1 - g^2) = 0, \\ f' - g = 0, \end{cases} \tag{2.1a}$$

subject to the boundary conditions

$$f(0) = 0, \quad g(0) = \gamma, \quad \lim_{\eta \rightarrow +\infty} g(\eta) = 1. \tag{2.1b}$$

The solution of Falkner-Skan problem is sought on a semi-infinite domain  $\eta \geq 0$ . Moreover, one of the boundary conditions is asymptotically assigned on the first derivative at infinity. These two aforementioned characteristics of the problem are the main challenges to be faced on handling this type of problems. Therefore, we define a coordinate transformation by introducing a new variable  $t$  in order to map the changing physical domain  $[0, \infty)$  to a fixed computational domain  $[0, 1)$ . This new variable is defined as

$$t = \eta / (\eta + 1). \tag{2.2}$$

It is clear that the mapping  $t : [0, \infty) \rightarrow [0, 1), \eta \rightarrow \eta/(\eta + 1)$  is one-to-one. Then, given  $t, \eta$  is obtained as  $\eta = t/(1 - t)$ . We also introduce two functions  $h(t)$  and  $k(t)$  defined by

$$f(\eta) = f(\eta(t)) = f(t/(1 - t)) = h(t), \tag{2.3a}$$

$$g(\eta) = g(\eta(t)) = g(t/(1 - t)) = k(t). \tag{2.3b}$$

As a result, we have

$$\begin{aligned} f'(\eta) &= \frac{df}{dt} \frac{dt}{d\eta} = h'(t) \frac{dt}{d\eta} \\ &= \frac{1}{(\eta + 1)^2} h'(t) = (1 - t)^2 h'(t). \end{aligned}$$

Similarly,

$$\begin{aligned} g'(\eta) &= (1 - t)^2 k'(t), \\ g''(\eta) &= [(1 - t)^2 k''(t) - 2(1 - t)k'(t)](1 - t)^2 \\ &= (1 - t)^4 k''(t) - 2(1 - t)^3 k'(t). \end{aligned}$$

Hence, using (2.2) and (2.3) in the system of nonlinear differential equations (2.1) leads to the following

$$\begin{cases} (1 - t)^4 k''(t) - 2(1 - t)^3 k'(t) + \beta_0 h(t)k'(t) - \beta_1 k^2(t) = -\beta_1, \\ (1 - t)^2 h'(t) - k(t) = 0, \end{cases} \tag{2.4a}$$

subject to the boundary conditions

$$h(0) = 0, \quad k(0) = \gamma, \quad k(1) = 1. \tag{2.4b}$$

Using the boundary condition  $h(0) = 0$ , the second equation in the system (2.4a) becomes

$$h(t) = \int_0^t \frac{1}{(1 - \epsilon)^2} k(\epsilon) d\epsilon. \tag{2.5}$$

Thus, by combining both equations of the system (2.4a), the problem (2.4) can be written in the form of a second order nonlinear integro-differential equation defined for  $t \in (0, 1)$  as

$$\begin{aligned} (1 - t)^4 k''(t) - 2(1 - t)^3 k'(t) - \beta_1 k^2(t) + \beta_1 \\ + \beta_0 (1 - t)^2 k'(t) \int_0^t \frac{k(\epsilon)}{(1 - \epsilon)^2} d\epsilon = 0 \end{aligned} \tag{2.6a}$$

and subject to the boundary conditions

$$k(0) = \gamma, \quad k(1) = 1. \tag{2.6b}$$

In summary, we reduced the Falkner-Skan problem (1.1) of third order, defined on a semi-infinite domain with an asymptotic boundary condition, to a second

order nonlinear integro-differential equation (2.6) defined over a finite domain and subjected to two boundary conditions given at finite values 0 and 1. To determine the solution of problem (1.1), we have to solve problem (2.6) for  $k(t)$ , recuperate  $h(t)$  using (2.5), then compute  $f(\eta)$  using the relation

$$f(\eta) = h(t) = h(\eta/(\eta + 1)), \quad \forall \eta \in (0, +\infty).$$

## 2.2 Quasi-linearization of the transformed Falkner-Skan problem

In this section, we apply the Newton-Raphson-Kantorovich approximation in function space introduced by [3, 28] on the newly introduced second order integro-differential equation (2.6) resulting in the previous section. At every iteration, the obtained quasi-linearized differential equation is numerically solved using the finite difference method yielding the solution of the transformed Falkner-Skan’s boundary value problem (2.1).

First, let us recall the Newton-Raphson-Kantorovich approximation method in function space to solve nonlinear differential equations. Let  $\{v_n\}$  be a sequence of approximate solutions of the nonlinear ordinary differential equation  $F(v, v', v'', x) = 0$ , where  $F(v, v', v'', x)$  is a differentiable functional with respect to  $v, v'$  and  $v''$ . Then, the generalized Newton-Raphson-Kantorovich approximation can be reformulated as

$$F(v_{n+1}, v'_{n+1}, v''_{n+1}, x) \approx F(v_n, v'_n, v''_n, x) + (v_{n+1} - v_n)F_v(v_n, v'_n, v''_n, x) + (v'_{n+1} - v'_n)F_{v'}(v_n, v'_n, v''_n, x) + (v''_{n+1} - v''_n)F_{v''}(v_n, v'_n, v''_n, x).$$

As a result, the sequence  $\{v_n\}$  will converge to the solution of the nonlinear ODE:  $F(v, v', v'', x) = 0$ .

Hence, in order to solve the transformed Falkner-Skan problem (2.6), we consider the functional

$$F(v, v', v'', t) = (1 - t)^4 v'' - 2(1 - t)^3 v' - \beta_1 v^2 + \beta_1 + \beta_0(1 - t)^2 v' \int_0^t \frac{v(\epsilon)}{(1 - \epsilon)^2} d\epsilon.$$

Then, the partial derivatives of  $F$  are

$$F_v(v, v', v'', t) = \beta_0(1 - t)^2 v' \int_0^t \frac{1}{(1 - \epsilon)^2} d\epsilon - 2\beta_1 v, \tag{2.7a}$$

$$F_{v'}(v, v', v'', t) = -2(1 - t)^3 + \beta_0(1 - t)^2 \int_0^t \frac{1}{(1 - \epsilon)^2} v(\epsilon) d\epsilon, \tag{2.7b}$$

$$F_{v''}(v, v', v'', t) = (1 - t)^4. \tag{2.7c}$$

Thus, the resulting quasi-linearized problem (2.6) is given by

$$\begin{aligned}
 (1-t)^4 k''_n &- 2(1-t)^3 k'_n + \beta_0(1-t)^2 k'_n \int_0^t \frac{1}{(1-\epsilon)^2} k_n(\epsilon) d\epsilon - \beta_1 k_n^2 + \beta_1 \\
 &+ (k_{n+1} - k_n)(\beta_0(1-t)^2 k'_n \int_0^t \frac{1}{(1-\epsilon)^2} d\epsilon - 2\beta_1 k_n) \\
 &+ (k'_{n+1} - k'_n)(\beta_0(1-t)^2 \int_0^t \frac{1}{(1-\epsilon)^2} k_n(\epsilon) d\epsilon - 2(1-t)^3) \\
 &+ (k''_{n+1} - k''_n)(1-t)^4 = 0,
 \end{aligned}$$

which can be simplified as

$$\begin{aligned}
 (1-t)^4 k''_{n+1} &+ \left( \beta_0(1-t)^2 \int_0^t \frac{1}{(1-\epsilon)^2} k_n(\epsilon) d\epsilon - 2(1-t)^3 \right) k'_{n+1} \\
 &+ \left( \beta_0 t(1-t) k'_n - 2\beta_1 k_n \right) k_{n+1} = \beta_0 t(1-t) k'_n k_n - \beta_1 k_n^2 - \beta_1, \tag{2.8a}
 \end{aligned}$$

subject to the boundary conditions

$$k_{n+1}(0) = \gamma, \quad k_{n+1}(1) = 1. \tag{2.8b}$$

The main advantage of the proposed iterative scheme is that, given  $k_n$ , the function  $k_{n+1}$  is the solution of a linear ordinary differential problem with known Dirichlet boundary conditions. Thus, at every iteration, given  $k_n$ , the obtained linearized ODE will be numerically solved for  $k_{n+1}$  using the finite difference method.

Provided that the choice of the initial approximation  $k_0$  is crucial for the convergence of the scheme, we propose the following initial solution satisfying the boundary conditions (2.8b):

$$k_0(t) = (1-\gamma)t^2 + \gamma, \quad \text{for } 0 \leq t \leq 1.$$

### 2.3 Iterative finite difference scheme

In order to solve equation (2.8) for  $k_{n+1}$  using the finite difference method, we consider a uniform mesh of size  $h = \frac{1}{M}$  and grid points  $t_i = i * h$  for  $i = 0 \dots M$ . For a fixed  $n > 0$ , we refer to the finite difference solution as  $K_n$ .

To write the finite difference scheme, We use the standard finite difference approximations of the first and second derivatives, respectively, as

$$k'_n(t_i) \approx \frac{K_n(t_{i+1}) - K_n(t_{i-1}))}{2h}, \quad k''_n(t_i) \approx \frac{K_n(t_{i+1}) - 2K_n(t_i) + K_n(t_{i-1}))}{h^2},$$

for  $1 \leq i \leq M - 1$ .

We also use a numerical computation of the integral  $\int_0^t \frac{k_n(\epsilon)}{(1-\epsilon)^2} d\epsilon$  at the

mesh points  $t_i$ , for  $i = 1, \dots, M - 1$ , using the standard Simpson's rule as

$$\int_0^{t_i} \frac{k_n(\epsilon)}{(1-\epsilon)^2} d\epsilon \approx \sum_{j=0}^i \frac{K_n(t_j)}{(1-t_j)^2} W_j = I_n(t_i), \quad \text{if } i \text{ is even} \tag{2.9}$$

$$\int_0^{t_i} \frac{k_n(\epsilon)}{(1-\epsilon)^2} d\epsilon \approx \frac{h}{2} \left( \frac{K_n(t_1)}{(1-t_1)^2} + \gamma \right) + \sum_{j=1}^i \frac{K_n(t_j)}{(1-t_j)^2} W_j = I_n(t_i), \quad \text{if } i \text{ is odd}, \tag{2.10}$$

where  $W_j$ 's are the known Simpson's weights. Hence, we can write the finite difference scheme for the ODE (2.8) as follows

$$\begin{aligned} & \left( \frac{(1-t_i)^4}{h^2} + \frac{\beta_0(1-t_i)^2 I_n(t_i) - 2(1-t_i)^3}{2h} \right) K_{n+1}(t_{i+1}) \\ & + \left( \beta_0 t_i (1-t_i) DK_n(t_i) - 2\beta_1 K_n(t_i) - 2 \frac{(1-t_i)^4}{h^2} \right) K_{n+1}(t_i) \\ & + \left( \frac{(1-t_i)^4}{h^2} - \frac{\beta_0(1-t_i)^2 I_n(t_i) - 2(1-t_i)^3}{2h} \right) K_{n+1}(t_{i-1}) \\ & = \beta_0 t_i (1-t_i) DK_n(t_i) K_n(t_i) - \beta_1 (K_n(t_i))^2 - \beta_1, \quad (1 \leq i \leq M-1), \end{aligned} \tag{2.11}$$

where  $DK_n$  is the centred finite difference approximation of the first derivative using the computed solution from the previous iteration  $n$ , i.e.,

$$DK_n(t_i) \approx \frac{K_n(t_{i+1}) - K_n(t_{i-1}))}{2h}, \quad \text{for } 1 \leq i \leq M-1.$$

Using the simplified notations  $K_n^i = K_n(t_i)$  and  $I_n^i = I_n(t_i)$ , for  $n \geq 0$ , we can summarize the iterative finite difference scheme (2.11) above as follows:

$$\begin{aligned} & \left( \frac{(1-t_i)^4}{h^2} + \frac{\beta_0(1-t_i)^2 I_n^i - 2(1-t_i)^3}{2h} \right) K_{n+1}^{i+1} \\ & + \left( \beta_0 t_i (1-t_i) DK_n^i - 2\beta_1 K_n^i - 2 \frac{(1-t_i)^4}{h^2} \right) K_{n+1}^i \\ & + \left( \frac{(1-t_i)^4}{h^2} - \frac{\beta_0(1-t_i)^2 I_n^i - 2(1-t_i)^3}{2h} \right) K_{n+1}^{i-1} \\ & = \beta_0 t_i (1-t_i) DK_n^i K_n^i - \beta_1 (K_n^i)^2 - \beta_1, \end{aligned} \tag{2.12a}$$

for  $1 \leq i \leq M - 1$ . The boundary conditions are written as

$$K_n(t_0 = 0) = \gamma, \quad K_n(t_M = 1) = 1. \tag{2.12b}$$

In addition, the initial approximation is given by

$$K_0(t_i) = (1-\gamma)t_i^2 + \gamma \quad \text{for } 0 \leq i \leq M. \tag{2.12c}$$

Then, once the numerical solution  $K_n$  of the integro-differential equation (2.6) is computed, the other numerical solution  $H_n$  of the nonlinear system (2.4) is obtained via a numerical evaluation of (2.5) using (2.9) and (2.10).

Finally, the actual solutions  $f$  and  $g$  of Falkner–Skan equation are numerically computed at  $\eta_i, i = 0, \dots, M$ , by using the backward transformation (2.3) as follows

$$F(\eta_i) = H(t_i), \quad G(\eta_i) = K(t_i), \quad \eta_i = \frac{t_i}{1 - t_i}.$$

We note that, in order to control the convergence of the proposed IFD scheme (2.12a), we set an error tolerance  $\delta > 0$ . Then, the algorithm is iteratively applied until such accuracy is reached in the sense that  $\|K_n - K_{n-1}\|_\infty \leq \delta$ . We also mention that the convergence rate of the proposed IFD scheme (2.12a) is quadratic, since it is based on Newton-Raphson-Kantorovich approximation [3]. Thus, this scheme is highly efficient in terms of computational time and cost. Such fact will be numerically exhibited in the following Section 3.

### 3 Numerical simulation

In this section, we present several numerical simulations in which we compute the solution of Falkner–Skan equation (1.1) for various values of the parameters  $\beta_0$  and  $\beta_1$  corresponding to different cases of flows. Particularly, we focus on the following special cases: Blasius flow [4]:  $\beta_1 = 0$ ; Pohlhausen flow [19]:  $\beta_0 = 0$  and  $\beta_1 = 1$ ; Hiemenz flow [12]:  $\beta_0 = 1$  and  $\beta_1 = 1$ ; Homann flow [13]:  $\beta_0 = 2$  and  $\beta_1 = 1$ .

In Table 1, we present the numerical solution of Blasius equation ( $\beta_1=0$ ) computed with the proposed IFD method when  $\beta_0 = 0.5$ . We also present a comparison of this computed solution to some of the most accurate results available in the literature [6, 14].

**Table 1.** Numerical solution of Blasius equation with  $\beta_0 = 0.5$  and  $\beta_1=0$  compared to Howarth’s [14] and Cortell’s [6] solutions.

$\eta$	$f(\eta)$	$f(\eta)$ [6]	$f(\eta)$ [14]	$f'(\eta)$	$f'(\eta)$ [6]	$f'(\eta)$ [14]	$f''(\eta)$	$f''(\eta)$ [6]	$f''(\eta)$ [14]
0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.33206	0.33206	0.33206
1	0.16557	0.16557	0.16557	0.32978	0.32978	0.32979	0.32301	0.32301	0.32301
2	0.65003	0.65003	0.65003	0.62977	0.62977	0.62977	0.26675	0.26675	0.26675
3	1.39681	1.39682	1.39682	0.84605	0.84605	0.84605	0.16136	0.16136	0.16136
4	2.30575	2.30576	2.30576	0.95552	0.95552	0.95552	0.06423	0.06423	0.06424
5	3.28333	3.28330	3.28329	0.99154	0.99155	0.99155	0.01591	0.01591	0.01591
6	4.27967	4.27965	4.27964	0.99897	0.99898	0.99898	0.00240	0.00240	0.00240
7	5.27925	5.27927	5.27926	0.99992	0.99993	0.99992	0.00022	0.00022	0.00022
8	6.27929	6.27925	6.27923	0.99999	1.00000	1.00000	0.00001	0.00001	0.00001
9	7.27922	7.27925	7.27923	1.00000	1.00000	1.00000	0.00000	0.00000	0.00000

In Table 2 we present the numerical solution of Blasius equation ( $\beta_1=0$ ) computed with the proposed IFD method for the values of  $\beta_0 = 2/3$  and  $\beta_0 = 1$ . We compare it to the results in [6]. It is clear that our computed solution is accurate and in good agreement with the previously computed solutions of Falkner-Skan problem.

In Table 3, we present the computed values of the wall shear stress  $\alpha = f''(0)$  for Falkner-Skan model when  $\beta_0=1$ , for different values of  $\beta_1$ . We also compare



**Table 2.** Numerical solution of Blasius equation with  $\beta_1=0$  compared to Cortell's [6] solution.

$\beta_0$	$\eta$	$f(\eta)$	$f(\eta)$ [6]	$f'(\eta)$	$f'(\eta)$ [6]	$f''(\eta)$	$f''(\eta)$ [6]
1	0	0.00000	0.00000	0.00000	0.00000	0.46960	0.46960
	1	0.23299	0.23299	0.46063	0.46063	0.43438	0.43438
	2	0.88680	0.88681	0.81669	0.81670	0.25567	0.25567
	3	1.79557	1.79558	0.96906	0.96906	0.06771	0.06771
	4	2.78389	2.78390	0.99777	0.99777	0.00687	0.00687
	5	3.78328	3.78325	0.99994	0.99994	0.00026	0.00026
	6	4.78326	4.78324	1.00000	1.00000	0.00000	0.00000
$\frac{2}{3}$	0	0.00000	0.00000	0.00000	0.00000	0.38342	0.38342
	1	0.19090	0.19090	0.37939	0.37939	0.36747	0.36747
	2	0.74235	0.74235	0.70789	0.70789	0.27420	0.27420
	3	1.56317	1.56317	0.90911	0.90910	0.12858	0.12858
	4	2.51730	2.51730	0.98323	0.98323	0.03313	0.03313
	5	3.51042	3.51038	0.99827	0.99827	0.00444	0.00444
	6	4.50983	4.50979	0.99990	0.99990	0.00030	0.00030
	7	5.50976	5.50976	0.99999	0.99999	0.00001	0.00001
	8	6.50983	6.50977	1.00000	1.00000	0.00000	0.00000

these results obtained by our IFD method to some existing accurate results available in the literature [1, 26].

**Table 3.** The wall shear stress  $\alpha = f''(0)$  of Falkner-skan model with  $\beta_0 = 1$  compared to Asaithambi's [1] and Salama's [26] solutions.

$\beta_1$	$\alpha$	$\alpha$ [1]	$\alpha$ [26]
40	7.314802	7.314785	7.314785
30	6.338220	6.338209	6.338208
20	5.180725	5.180718	5.180718
15	4.491492	4.491487	4.491487
10	3.675237	3.675234	3.675234
2	1.687218	1.687218	1.687218
1	1.232587	1.232589	1.232588
0.5	0.927679	0.927680	0.927680
0	0.469599	0.469600	0.469600

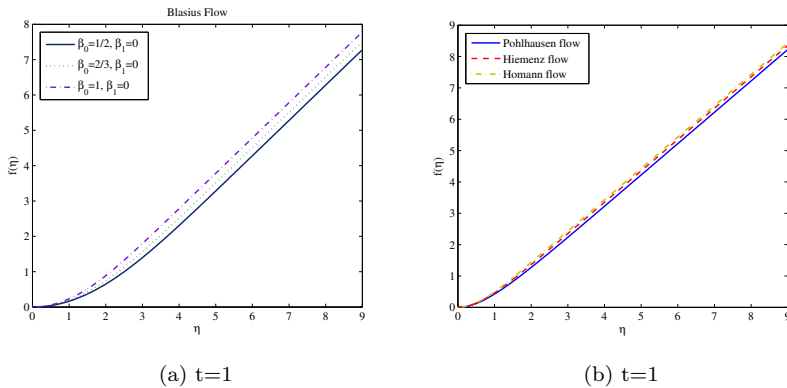
In Table 4 we consider different benchmark solutions for some of the famous practical cases of flows.

Furthermore, in Figure 1, we graphically exhibit the solution of Falkner-Skan problem for different flows (Blasius, Pohlhausen, Hiemenz, and Homann). It is observed that for all these types of flows, the solution of Falkner-Skan problem is increasing with respect to  $\eta$ .

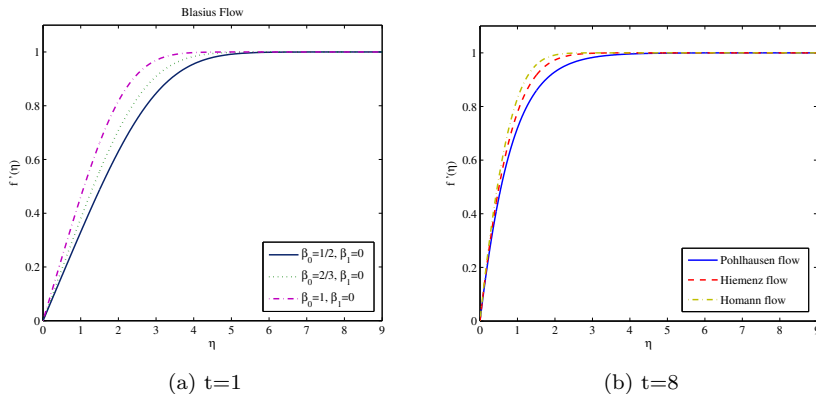
We also display the velocity profile in Figure 2 as well as the shear stress in Figure 3 for different flows. We notice that for all the considered types of flows, the shear stress decays to zero, however the decay in the case of Blasius flow is slower than in the other flow cases.

**Table 4.** The wall shear stress  $\alpha = f''(0)$  of Falkner-skun models compared to different benchmark solutions.

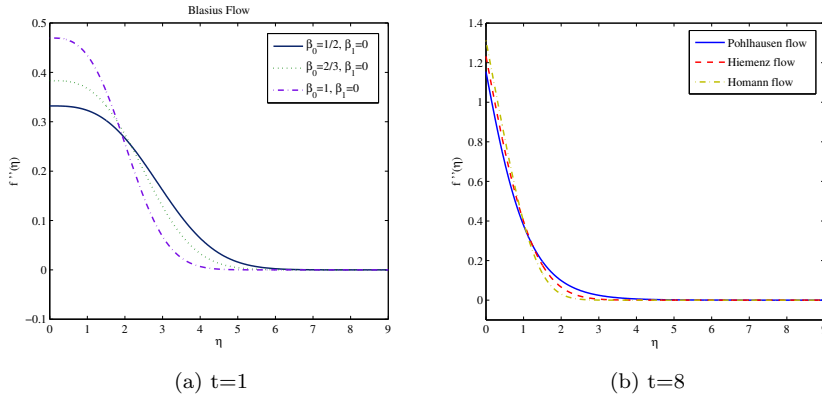
Type of flow	$\beta_0$	$\beta_1$	$\alpha$	
Blasius flow	$\frac{1}{2}$	0	0.33206	0.33206 [6]
Pohlhausen flow	0	1	1.15470	1.15470 [19]
Hiemenz flow	1	1	1.23258	1.23258 [29]
Homann flow	2	1	1.31193	1.31193 [13]



**Figure 1.** The Solution of Falkner-Skan problem for the cases of: Blasius problem  $\beta_1 = 0$  (left) and Pohlhausen, Hiemenz, and Homann flows  $\beta_0 = 0, 1, 2$ , respectively, with  $\beta_1 = 1$  (right).



**Figure 2.** Velocity profile of Falkner-Skan solution for the cases of: Blasius problem  $\beta_1 = 0$  (left) and Pohlhausen, Hiemenz, and Homann flows  $\beta_0 = 0, 1, 2$ , respectively, with  $\beta_1 = 1$  (right).



**Figure 3.** Shear stress of Falkner-Skan solution for the cases of: Blasius problem  $\beta_1 = 0$  (left) and Pohlhausen, Hiemenz, and Homann flows  $\beta_0 = 0, 1, 2$ , respectively, with  $\beta_1 = 1$  (right).

### 4 Conclusions

In this paper, we have presented a new iterative finite difference method to numerically solve a nonlinear problem defined on a semi-infinite domain, arising from the boundary layer flow with stream-wise pressure gradient, known as Falkner-Skan problem.

For that purpose, we have introduced a new coordinate transformation to overcome the difficulty of the semi-infinite domain. Such transformation led to a second order nonlinear integro-differential equation defined on a finite domain and subjected to boundary conditions on finite values.

We have then used Newton-Raphson-Kantorovich quasilinearisation method in function space [3] in combination with the standard finite difference method to solve transformed Falkner-Skan problem. The proposed iterative finite difference method leads to an iterative scheme in which an accurate solution at every iteration is computed using the approximate solution from the previous iteration.

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