Mathematical Modelling and Analysis
Volume 24, Issue 1, 95–104, 2019
https://doi.org/10.3846/mma.2019.007

http://mma.vgtu.lt
ISSN: 1392-6292
eISSN: 1648-3510

Global Strong Solutions of the Density Dependent Incompressible MHD System with Zero Resistivity in a Bounded Domain

Jishan Fan[a], Bessem Samet[b] and Yong Zhou[c]

[a] *Department of Applied Mathematics, Nanjing Forestry University*
Nanjing 210037, China
[b] *Department of Mathematics, College of Science, King Saud University*
P.O. Box 2455, Riyadh 11451, Saudi Arabia
[c] *School of Mathematics (Zhuhai), Sun Yat-Sen University*
Zhuhai, Guangdong 519082, China

E-mail(*corresp.*): zhouyong3@mail.sysu.edu.cn

E-mail: fanjishan@njfu.edu.cn

Received March 13, 2018; revised November 19, 2018; accepted November 20, 2018

Abstract. In this paper, we first establish a regularity criterion for the strong solutions to the density-dependent incompressible MHD system with zero resistivity in a bounded domain. Then we use it and the bootstrap argument to prove the global well-posedness provided that the initial data u_0 and b_0 satisfy that $(d-2)\|\nabla u_0\|_{L^2} + \|b_0\|_{W^{1,p}}$ are sufficiently small with $d < p < \frac{2d}{d-2}$ $(d = 2, 3)$. We do not assume the positivity of initial density, it may vanish in an open subset (vacuum) of Ω.

Keywords: MHD, zero resistivity, bounded domain.

AMS Subject Classification: 35Q30; 35Q35; 76D03.

1 Introduction

Magnetohydrodynamics (MHD) studies the interaction of electromagnetic fields and conducting fluids. In this paper, we consider the following density-dependent incompressible MHD system:

$$\partial_t\rho + \mathrm{div}(\rho u) = 0, \tag{1.1}$$

$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla\Big(\pi + \frac{1}{2}|b|^2\Big) - \mu\Delta u = (b\cdot\nabla)b, \tag{1.2}$$

$$\partial_t b + u\cdot\nabla b - b\cdot\nabla u = \eta\Delta b, \tag{1.3}$$

$$\mathrm{div}u = \mathrm{div}b = 0 \quad \text{in} \quad \Omega\times(0,\infty), \tag{1.4}$$

$$u = 0, \eta b\cdot n = 0, \eta \mathrm{rot} b\times n = 0 \quad \text{on} \quad \partial\Omega\times(0,\infty),$$

$$(\rho, u, b)(\cdot, 0) = (\rho_0, u_0, b_0)(\cdot) \quad \text{in} \quad \Omega\subset \mathrm{R}^d \ (d=2,3). \tag{1.5}$$

Here ρ denotes the density, u the velocity field, π the pressure, and b the magnetic field, respectively. μ is the viscosity coefficient and η is the resistivity coefficient. Ω is a bounded domain in R^d with smooth boundary $\partial\Omega$, n is the unit outward normal vector to the boundary $\partial\Omega$. We will assume that the initial data satisfy the following compatibility condition:

$$-\mu\Delta u_0 + \nabla\left(\pi_0 + \frac{1}{2}|b_0|^2\right) - b_0\cdot\nabla b_0 = \sqrt{\rho_0}g \tag{1.6}$$

with $g\in L^2(\Omega)$.

Wu [1] shows the local well-posedness of strong solutions to the problem (1.1)–(1.5) under the condition (1.6). When $\eta > 0$ and $d = 2$, Huang and Wang [5] (also see [6]) prove the global well-posedness of the strong solutions. Fan-Li-Nakamura [2] showed a regularity criterion. Fan-Zhou [3] proved the uniform-in-$\mu(\eta)$ local well-posedness of smooth solutions when $\Omega := \mathrm{R}^d$. The aim of this paper is to prove some similar results when $\eta = 0$. We will prove

Theorem 1. *Let $d = 2, \mu = 1, \eta = 0, u_0 \in H_0^1\cap H^2, 0 \le \rho_0 \in W^{1,q}, b_0 \in W^{1,p}$ with $2 < q, p < \infty$ and $\mathrm{div}u_0 = \mathrm{div}b_0 = 0$ in Ω. If b satisfies $b\in L^\infty(0,T;W^{1,p})$ for some $2 < p < \infty$, then*

$$\begin{aligned} &u\in L^\infty(0,T;H^2)\cap L^2(0,T;W^{2,p}), u_t\in L^2(0,T;H^1), \sqrt{\rho}u_t\in L^\infty(0,T;L^2),\\ &\rho\in L^\infty(0,T;W^{1,q}), \rho_t\in L^\infty(0,T;L^q),\\ &b\in L^\infty(0,T;W^{1,p}), b_t\in L^\infty(0,T;L^p) \end{aligned} \tag{1.7}$$

for any given $T > 0$.

Theorem 2. *Let $d = 2, \mu = 1, \eta = 0, u_0 \in H_0^1\cap H^2, 0 \le \rho_0 \in W^{1,q}, b_0 \in W^{1,p}$ with $2 < q, p < \infty$ and $\mathrm{div}u_0 = \mathrm{div}b_0 = 0$ in Ω. If $\|b_0\|_{W^{1,p}}$ is sufficiently small, then the problem* (1.1)–(1.5) *has a unique strong solution* (ρ, u, b) *satisfying* (1.7).

Remark 1. Here we do not assume smallness of the initial velocity u_0.

Remark 2. We denote $C_1 := \int_0^T \|u\|_{W^{2,p}}\mathrm{d}t$, then we can take

$$\|b_0\|_{W^{1,p}} = \frac{\delta}{2}\exp(-C_1) =: \delta_1.$$

We need not assume that C_1 is uniformly bounded as $\delta\to 0$, say, we take $C_1 = \frac{1}{\delta}$, then we have $\delta_1 \to 0$ as $\delta\to 0$. Although it is not difficult to prove that C_1 is uniformly bounded as $\delta\to 0$ and we omit the details here.

Theorem 3. *Let $d = 3, \mu = 1, \eta = 0, u_0 \in H_0^1 \cap H^2, 0 \leq \rho_0 \in W^{1,q}, b_0 \in W^{1,p}$ with $3 < q, p < 6$ and $divu_0 = divb_0 = 0$ in Ω. If u and b satisfy*

$$\nabla u \in L^\infty(0, T; L^2), b \in L^\infty(0, T; W^{1,p})$$

with $3 < p < 6$, then (1.7) holds true.

Theorem 4. *Let $d = 3, \mu = 1, \eta = 0, u_0 \in H_0^1 \cap H^2, 0 \leq \rho_0 \in W^{1,q}, b_0 \in W^{1,p}$ with $3 < q, p < 6$ and $divu_0 = divb_0 = 0$ in Ω. If $\|\nabla u_0\|_{L^2} + \|b_0\|_{W^{1,p}}$ is sufficiently small, then the problem (1.1)–(1.5) has a unique strong solution (ρ, u, b) satisfying (1.7).*

Remark 3. Our results also hold true when $\Omega := \mathbb{R}^d \ \ (d = 2, 3)$ without any difference and difficulty. Concerning regularity criteria for the MHD system, we refer to [4, 7, 8] and references therein.

Remark 4. Our results also hold true for compressible MHD flows without resistivity and thus we omit the details here.

Remark 5. In [3], they proved the following regularity criterion $\nabla u \in L^1(0, T; L^\infty(\Omega))$, or $u \in L^2(0, T; L^\infty(\mathbb{R}^d))$ and $\nabla u \in L^1(0, T; L^\infty(\mathbb{R}^d))$, which is different from ours. We are unable to use it to prove a global small result. The novelty of this paper is that we can use our regularity criterion to show a global small result by a bootstrap argument.

To prove Theorems 2 and 4, we will use the following abstract bootstrap argument or continuity argument [9, Page 20] (see also [10, 12]).

Lemma 1. *([9]). Let $T > 0$. Assume that two statements $C(t)$ and $H(t)$ with $t \in [0, T]$ satisfy the following conditions:*

- *(a) If $H(t)$ holds for some $t \in [0, T]$, then $C(t)$ holds for the same t;*
- *(b) If $C(t)$ holds for some $t_0 \in [0, T]$, then $H(t)$ holds for t in a neighborhood of t_0;*
- *(c) If $C(t)$ holds for $t_m \in [0, T]$ and $t_m \to t$, then $C(t)$ holds;*
- *(d) $C(t)$ holds for at least one $t_1 \in [0, T]$.*

Then $C(t)$ holds for all $t \in [0, T]$.

2 Proof of Theorem 1

This section is devoted to the proof of Theorem 1. Since the local strong solutions to the problem (1.1)–(1.5) was established in [1], we only need to show a priori estimates (1.7).

First, it follows from (1.1) and (1.4) that

$$0 \leq \rho \leq M < \infty. \tag{2.1}$$

Testing (1.2) by u and using (1.1) and (1.4), we see that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|u|^2\mathrm{d}x+\int|\nabla u|^2\mathrm{d}x=\int(b\cdot\nabla)b\cdot u\mathrm{d}x. \tag{2.2}$$

Testing (1.3) by b and using (1.4), we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |b|^2\mathrm{d}x=\int(b\cdot\nabla)u\cdot b\mathrm{d}x. \tag{2.3}$$

Summing up (2.2) and (2.3), we get

$$\frac{1}{2}\int(\rho|u|^2+|b|^2)\mathrm{d}x+\int_0^T\int|\nabla u|^2\mathrm{d}x\mathrm{d}t\le\frac{1}{2}\int(\rho_0|u_0|^2+|b_0|^2)\mathrm{d}x. \tag{2.4}$$

Testing (1.2) by u_t, using (1.1), (1.4) and (2.1), we derive that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla u|^2\mathrm{d}x+\int\rho|u_t|^2\mathrm{d}x-\frac{\mathrm{d}}{\mathrm{d}t}\int b\otimes b:\nabla u\mathrm{d}x\\
=\;& -\int\rho u\cdot\nabla u\cdot u_t\mathrm{d}x-\int\partial_t(b\otimes b):\nabla u\mathrm{d}x\\
\le\;& \|\sqrt{\rho}u_t\|_{L^2}\|\sqrt{\rho}\|_{L^\infty}\|u\|_{L^\infty}\|\nabla u\|_{L^2}+C\|b\|_{L^\infty}\|b_t\|_{L^2}\|\nabla u\|_{L^2}\\
\le\;& C\|\sqrt{\rho}u_t\|_{L^2}\|u\|_{L^\infty}\|\nabla u\|_{L^2}+C\|u\cdot\nabla b-b\cdot\nabla u\|_{L^2}\|\nabla u\|_{L^2}\\
\le\;& C\|\sqrt{\rho}u_t\|_{L^2}\|u\|_{L^\infty}\|\nabla u\|_{L^2}+C(\|u\|_{L^\infty}+\|\nabla u\|_{L^2})\|\nabla u\|_{L^2}\\
\le\;& \frac{1}{2}\|\sqrt{\rho}u_t\|_{L^2}^2+C\|u\|_{L^\infty}^2\|\nabla u\|_{L^2}^2+C\|\nabla u\|_{L^2}^2+C.
\end{aligned} \tag{2.5}$$

On the other hand, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int|b\otimes b|^2\mathrm{d}x &\le C\|b\|_{L^\infty}^3\|b_t\|_{L^1}\le C\|u\cdot\nabla b-b\cdot\nabla u\|_{L^1}\\
&\le C\|\nabla u\|_{L^2}\le C\|\nabla u\|_{L^2}^2+C.
\end{aligned} \tag{2.6}$$

We will use the following logarithmic Sobolev inequality [11]:

$$\|u\|_{L^\infty}\le C(1+\|\nabla u\|_{L^2}\log^{\frac{1}{2}}(e+\|u\|_{H^2})). \tag{2.7}$$

Doing (2.5)+(2.6)C_1 with C_1 suitably large and using (2.7), we have

$$\int|\nabla u|^2\mathrm{d}x+\int_{t_0}^t\int\rho|u_t|^2\mathrm{d}x\mathrm{d}s\le C(e+y(t))^{C_0\epsilon}, \tag{2.8}$$

provided that

$$\int_{t_0}^T\|\nabla u\|_{L^2}^2\mathrm{d}t\le\epsilon<<1$$

with $y(t):=\sup_{[t_0,t]}\|u(s)\|_{H^2}$ and C_0 is an absolute constant.

On the other hand, (1.2) can be rewritten as

$$-\Delta u+\nabla\Big(\pi+\frac{1}{2}|b|^2\Big)=f:=b\cdot\nabla b-\rho u_t-\rho u\cdot\nabla u. \tag{2.9}$$

By the H^2-theory of Stokes system, we observe that

$$\begin{aligned}\|u\|_{H^2} &\le C\|f\|_{L^2} \le C\|b\cdot\nabla b - \rho u_t - \rho u\cdot\nabla u\|_{L^2}\\ &\le C + C\|\sqrt{\rho}u_t\|_{L^2} + C\|u\|_{L^4}\|\nabla u\|_{L^4}\\ &\le C + C\|\sqrt{\rho}u_t\|_{L^2} + C\|u\|_{L^2}^{\frac12}\cdot\|\nabla u\|_{L^2}\cdot\|u\|_{H^2}^{\frac12},\end{aligned}$$

which gives

$$\|u\|_{H^2} \le C + C\|\sqrt{\rho}u_t\|_{L^2} + C\|u\|_{L^2}\|\nabla u\|_{L^2}^2, \tag{2.10}$$

whence

$$\int_{t_0}^t \|u\|_{H^2}^2 \mathrm{d}s \le C(e+y(t))^{C_0\epsilon}. \tag{2.11}$$

Taking the operator ∂_t to (1.2), testing by u_t, using (1.1) and (1.4), we have

$$\begin{aligned}\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|u_t|^2\mathrm{d}x + \int|\nabla u_t|^2\mathrm{d}t &= -\int\rho_t|u_t|^2\mathrm{d}x - \int\rho_t u\cdot\nabla u\cdot u_t\mathrm{d}x\\ -\int\rho u_t\cdot\nabla u\cdot u_t\mathrm{d}x &- \int\partial_t(b\otimes b):\nabla u_t\mathrm{d}x =: \sum_{i=1}^4 I_i.\end{aligned} \tag{2.12}$$

We use (2.1), Gagliardo-Nirenberg inequality and the Hölder inequality to bound I_i $(i=1,\dots,4)$ as follows:

$$\begin{aligned}I_1 &= -\int\rho u\cdot\nabla|u_t|^2\mathrm{d}x \le C\|u\|_{L^6}\|\sqrt{\rho}u_t\|_{L^3}\|\nabla u_t\|_{L^2}\\ &\le C\|u\|_{L^6}\|\sqrt{\rho}u_t\|_{L^2}^{\frac12}\|\sqrt{\rho}u_t\|_{L^6}^{\frac12}\|\nabla u_t\|_{L^2} \le C\|u\|_{L^6}\|\sqrt{\rho}u_t\|_{L^2}^{\frac12}\|\nabla u_t\|_{L^2}^{\frac32}\\ &\le \frac{1}{16}\|\nabla u_t\|_{L^2}^2 + C\|u\|_{L^6}^4\|\sqrt{\rho}u_t\|_{L^2}^2,\end{aligned} \tag{2.13}$$

$$\begin{aligned}I_2 &= -\int\rho u\cdot\nabla(u\cdot\nabla u\cdot u_t)\mathrm{d}x \le C\|\sqrt{\rho}u_t\|_{L^6}\|u\|_{L^6}\|\nabla u\|_{L^3}^2\\ &\quad + C\|\sqrt{\rho}u_t\|_{L^6}\|\Delta u\|_{L^2}\|u\|_{L^6}^2 + C\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^6}\|u\|_{L^6}^2\\ &\le C\|\nabla u_t\|_{L^2}\|u\|_{H^1}^2\|u\|_{H^2} \le \frac{1}{16}\|\nabla u_t\|_{L^2}^2 + C\|u\|_{H^1}^4\|u\|_{H^2}^2,\end{aligned} \tag{2.14}$$

$$\begin{aligned}I_3 &\le C\|\sqrt{\rho}u_t\|_{L^4}^2\|\nabla u\|_{L^2}\\ &\le C\|\sqrt{\rho}u_t\|_{L^2}^{\frac23}\|\sqrt{\rho}u_t\|_{L^8}^{\frac43}\|\nabla u\|_{L^2} \le C\|\sqrt{\rho}u_t\|_{L^2}^{\frac23}\|u_t\|_{L^8}^{\frac43}\|\nabla u\|_{L^2}\\ &\le C\|\sqrt{\rho}u_t\|_{L^2}^{\frac23}\|\nabla u_t\|_{L^2}^{\frac43}\|\nabla u\|_{L^2} \le \frac{1}{16}\|\nabla u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^3\|\sqrt{\rho}u_t\|_{L^2}^2,\end{aligned}$$

$$\begin{aligned}I_4 &\le 2\|b\|_{L^\infty}\|b_t\|_{L^2}\|\nabla u_t\|_{L^2} \le C\|u\cdot\nabla b - b\cdot\nabla u\|_{L^2}\|\nabla u_t\|_{L^2}\\ &\le C(\|u\|_{L^{\frac{2p}{p-2}}}\|\nabla b\|_{L^p} + \|b\|_{L^\infty}\|\nabla u\|_{L^2})\|\nabla u_t\| \le C\|\nabla u\|_{L^2}\|\nabla u_t\|_{L^2}\\ &\le \frac{1}{16}\|\nabla u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2.\end{aligned} \tag{2.15}$$

Inserting the above estimates into (2.12) and integrating it over (t_0, t) and using (2.8), (2.10), and (2.11), we arrive at

$$\int\rho|u_t|^2\mathrm{d}x + \int_{t_0}^t\int|\nabla u_t|^2\mathrm{d}x\mathrm{d}s \le C(e+y(t))^{C_0\epsilon},$$

whence

$$\|u\|_{H^2} \le C(e + y(t))^{C_0\epsilon},$$

and thus

$$\|u\|_{L^\infty(0,T;H^2)} \le C, \quad \|\sqrt{\rho} u_t\|_{L^\infty(0,T;L^2)} + \|u_t\|_{L^2(0,T;H^1)} \le C \tag{2.16}$$

by taking $C_0\epsilon \le \frac{1}{2}$. On the other hand, it follows from (2.9) that

$$\begin{aligned}\|u\|_{W^{2,p}} &\le C\|f\|_{L^p} \le C\|b\cdot\nabla b - \rho u_t - \rho u\cdot\nabla u\|_{L^p} \\ &\le C + C\|\rho u_t\|_{L^p} + C\|u\|_{L^\infty}\|\nabla u\|_{L^p} \le C + C\|\nabla u_t\|_{L^2},\end{aligned}$$

and thus

$$\|u\|_{L^2(0,T;W^{2,p})} \le C. \tag{2.17}$$

Now it is easy to show that

$$\begin{aligned}&\rho \in L^\infty(0,T;W^{1,q}), \rho_t \in L^\infty(0,T;L^q), \\ &b \in L^\infty(0,T;W^{1,p}), b_t \in L^\infty(0,T;L^p).\end{aligned} \tag{2.18}$$

This completes the proof. □

3 Proof of Theorem 2

This section is devoted to the proof of Theorem 2. Since it is easy to prove the existence and uniqueness of local smooth solutions to the problem (1.1)–(1.5), we only need to prove a priori estimates (1.7). To this end, we shall use the bootstrap argument.

Let $\delta > 0$ be a fixed number, say $\|b_0\|_{W^{1,p}} \le \delta$. Denote by $H(t)$ the statement that, for $t \in [0,T]$,

$$\|b\|_{L^\infty(0,t;W^{1,p})} \le \delta \tag{3.1}$$

and $C(t)$ the statement that

$$\|b\|_{L^\infty(0,t;W^{1,p})} \le \delta/2. \tag{3.2}$$

The conditions (b)–(d) in Lemma 1 are clearly true and it remains to verify (a) under the condition that $\|b_0\|_{W^{1,p}}$ is small. Once this is verified then the bootstrap argument would imply that $C(t)$, or (3.2) actually holds for any $t \in [0,T]$ and then we can prove (1.7) hold true.

Now we assume that (3.1) holds true for some $t \in [0,T]$. By Theorem 1, we have

$$u \in L^2(0,T;W^{2,p}). \tag{3.3}$$

Testing (1.3) by $|b|^{p-2}b$ and using (1.4), we infer that

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|b\|_{L^p}^p \le C\|\nabla u\|_{L^\infty}\|b\|_{L^p}^p,$$

whence

$$\frac{\mathrm{d}}{\mathrm{d}t}\|b\|_{L^p} \le C\|\nabla u\|_{L^\infty}\|b\|_{L^p}. \tag{3.4}$$

Taking ∇ to (1.3), testing by $|\nabla b|^{p-2}\nabla b$ and using (1.4), we observe that

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla b\|_{L^p}^p &\leq C\|\nabla u\|_{L^\infty}\|\nabla b\|_{L^p}^p + C\|b\|_{L^\infty}\|\Delta u\|_{L^p}\|\nabla b\|_{L^p}^{p-1}\\ &\leq C\|\nabla u\|_{L^\infty}\|\nabla b\|_{L^p}^p + C(\|b\|_{L^p}+\|\nabla b\|_{L^p})\|\Delta u\|_{L^p}\|\nabla b\|_{L^p}^{p-1},\end{aligned}$$

which implies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla b\|_{L^p} &\leq C\|\nabla u\|_{L^\infty}\|\nabla b\|_{L^p} + C(\|b\|_{L^p}+\|\nabla b\|_{L^p})\|\Delta u\|_{L^p}\\ &\leq C\|u\|_{W^{2,p}}\|b\|_{W^{1,p}}. \end{aligned}\tag{3.5}$$

Summing up (3.4) and (3.5), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|b\|_{W^{1,p}} \leq C\|u\|_{W^{2,p}}\|b\|_{W^{1,p}},$$

which yields

$$\|b\|_{W^{1,p}} \leq \|b_0\|_{W^{1,p}}\exp\left(\int_0^t \|u\|_{W^{2,p}}\mathrm{d}s\right) \leq C\|b_0\|_{W^{1,p}} \leq \frac{\delta}{2}. \tag{3.6}$$

This proves that (3.2) holds true for any $t\in[0,T]$. Thus, we arrive at

$$\|b\|_{L^\infty(0,T;W^{1,p})} \leq \frac{\delta}{2}. \tag{3.7}$$

This completes the proof.

□

4 Proof of Theorem 3

We only need to show a priori estimates (1.7). First, we still have (2.1) and (2.4). Similarly to (2.5), we observe that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla u|^2\mathrm{d}x + \int\rho|u_t|^2\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int b\otimes b:\nabla u\mathrm{d}x\\ =\ & -\int\rho u\cdot\nabla u\cdot u_t\mathrm{d}x - \int\partial_t(b\otimes b):\nabla u\mathrm{d}x\\ \leq\ & \|\sqrt{\rho}u_t\|_{L^2}\|\sqrt{\rho}\|_{L^\infty}\|u\|_{L^6}\|\nabla u\|_{L^3} + 2\|b\|_{L^\infty}\|b_t\|_{L^2}\|\nabla u\|_{L^2}\\ \leq\ & C\|\sqrt{\rho}u_t\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{1}{2}} + C\|u\cdot\nabla b - b\cdot\nabla u\|_{L^2}\\ \leq\ & C\|\sqrt{\rho}u_t\|_{L^2}\|u\|_{H^2}^{\frac{1}{2}} + C(\|u\|_{L^{\frac{2p}{p-2}}}\|\nabla b\|_{L^p} + \|b\|_{L^\infty}\|\nabla u\|_{L^2})\\ \leq\ & C\|\sqrt{\rho}u_t\|_{L^2}\|u\|_{H^2}^{\frac{1}{2}} + C. \end{aligned}\tag{4.1}$$

Similarly to (2.10), it follows from (2.9) that

$$\|u\|_{H^2} \leq C + C\|\sqrt{\rho}u_t\|_{L^2}. \tag{4.2}$$

Inserting (4.2) into (4.1) and integrating it over $(0, T)$, we obtain

$$\|u\|_{L^2(0,T;H^2)} + \|\sqrt{\rho}u_t\|_{L^2(0,T;L^2)} \leq C.$$

We still have (2.12). We bound I_1, I_2, and I_4 by the same method as that in (2.13), (2.14) and (2.15). We bound I_3 as follows.

$$\begin{aligned} I_3 &\leq C\|\sqrt{\rho}u_t\|_{L^3}^2\|\nabla u\|_{L^3} \leq C\|\sqrt{\rho}u_t\|_{L^2}\|\sqrt{\rho}u_t\|_{L^6}\|u\|_{H^2} \\ &\leq C\|\sqrt{\rho}u_t\|_{L^2}\|\nabla u_t\|_{L^2}\|u\|_{H^2} \leq \frac{1}{16}\|\nabla u_t\|_{L^2}^2 + C\|u\|_{H^2}^2\|\sqrt{\rho}u_t\|_{L^2}^2. \end{aligned}$$

Inserting the above estimates into (2.12) and using the Gronwall inequality, we arrive at (2.16). We still have (2.17) and (2.18). This completes the proof. □

5 Proof of Theorem 4

This section is devoted to the proof of Theorem 4, which is very similar to that in Section 3. Let $\delta > 0$ be a fixed number, say

$$2\|\nabla u_0\|_{L^2} \leq \delta, \quad 2\|b_0\|_{W^{1,p}} \leq \delta.$$

Denote by $H(t)$ the statement that, for $t \in [0, T]$,

$$\|\nabla u\|_{L^\infty(0,t;L^2)} \leq \delta, \quad \|b\|_{L^\infty(0,t;W^{1,p})} \leq \delta \tag{5.1}$$

and $C(t)$ the statement that

$$\|\nabla u\|_{L^\infty(0,t;L^2)} \leq \delta/2, \quad \|b\|_{L^\infty(0,t;W^{1,p})} \leq \delta/2. \tag{5.2}$$

The conditions (b)–(d) in Lemma 1 are clearly true and it remains to verify (a) under the condition that $\|\nabla u_0\|_{L^2} + \|b_0\|_{W^{1,p}}$ is small enough. Once this is verified then the bootstrap argument would imply that $C(t)$, or (5.2) actually holds for any $t \in [0, T]$ and then we can prove (1.7) hold true.

Now we assume that (5.1) holds true for some $t \in [0, T]$. We still have (3.3), (3.4), (3.5), (3.6) and (3.7). We still have (2.1) and (2.4). Similarly to (4.1), we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla u|^2 \mathrm{d}x + \int \rho |u_t|^2 \mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int b \otimes b : \nabla u \mathrm{d}x \\ \leq\ & \|\sqrt{\rho}u_t\|_{L^2}\|\sqrt{\rho}\|_{L^\infty}\|u\|_{L^6}\|\nabla u\|_{L^3} + 2\|b\|_{L^\infty}\|b_t\|_{L^2}\|\nabla u\|_{L^2} \\ \leq\ & C\|\sqrt{\rho}u_t\|_{L^2}\delta^{\frac{3}{2}}\|u\|_{H^2}^{\frac{1}{2}} + C\delta^2\|u \cdot \nabla b - b \cdot \nabla u\|_{L^2} \\ \leq\ & C\|\sqrt{\rho}u_t\|_{L^2}\delta^{\frac{3}{2}}\|u\|_{H^2}^{\frac{1}{2}} + C\delta^2(\|u\|_{L^{\frac{2p}{p-2}}}\|\nabla b\|_{L^p} + \|b\|_{L^\infty}\|\nabla u\|_{L^2}) \\ \leq\ & C\|\sqrt{\rho}u_t\|_{L^2}\delta^{\frac{3}{2}}\|u\|_{H^2}^{\frac{1}{2}} + C\delta^4. \end{aligned} \tag{5.3}$$

On the other hand, similarly to (4.2), we have

$$\begin{aligned} \|u\|_{H^2} &\leq C\|f\|_{L^2} \leq C\|b \cdot \nabla b - \rho u_t - \rho u \cdot \nabla u\|_{L^2} \\ &\leq C\delta^2 + C\|\sqrt{\rho}u_t\|_{L^2} + C\|u\|_{L^6}\|\nabla u\|_{L^3} \\ &\leq C\delta^2 + C\|\sqrt{\rho}u_t\|_{L^2} + C\delta^{\frac{3}{2}}\|u\|_{H^2}^{\frac{1}{2}}, \end{aligned}$$

which gives

$$\|u\|_{H^2} \le C\delta^2 + C\|\sqrt{\rho}u_t\|_{L^2}.$$

Inserting the above estimates into (5.3) and integrating over $(0,t)$, we conclude that

$$\begin{aligned}\int |\nabla u|^2 \mathrm{d}x &\le \int |\nabla u_0|^2 \mathrm{d}x + 2\int b\otimes b : \nabla u \mathrm{d}x - 2\int b_0 \otimes b_0 : \nabla u_0 \mathrm{d}x + C\delta^4 T \\ &\le \int |\nabla u_0|^2 \mathrm{d}x + \frac{1}{2}\|\nabla u\|_{L^2}^2 + C\delta^4 - 2\int b_0\otimes b_0 : \nabla u_0 \mathrm{d}x,\end{aligned}$$

which gives

$$\int |\nabla u|^2 \mathrm{d}x \quad \le \quad 2\int |\nabla u_0|^2 \mathrm{d}x - 4\int b_0 \otimes b_0 : \nabla u_0 \mathrm{d}x + C\delta^4 \le \frac{\delta^2}{4} \quad (\delta \le 1).$$

This completes the proof. □

Acknowledgements

The authors are indebted to the referees for adding Remarks 2–5. This paper is partially supported by NSFC (No. 11171154). The second author extends his appreciation to Distinguished Scientist Fellowship Program (DSFP) at King Saud University (Saudi Arabia).

References

[1] J. Fan and F. Li. Global strong solutions to the 3D full compressible Navier-Stokes system with vacuum in a bounded domain. *Appl. Math. Letters*, **78**:31–35, 2018. https://doi.org/10.1016/j.aml.2017.11.001.

[2] J. Fan, F. Li and G. Nakamura. Global strong solution to the 2D density-dependent liquid crystal flows with vacuum. *Nonlinear Anal.*, **97**:185–190, 2014. https://doi.org/10.1016/j.na.2013.11.024.

[3] J. Fan, F. Li and G. Nakamura. Regularity criteria for the incompressible magnetohydrodynamic equations with partial viscosity. *Anal. Appl. (Singap.)*, **14**(2):321–339, 2016. https://doi.org/10.1142/S0219530515500074.

[4] J. Fan, H. Malaikah, S. Monaquel, G. Nakamura and Y. Zhou. Global Cauchy problem of 2D generalized MHD equations. *Monatsh. Math.*, **175**(1):127–131, 2014. https://doi.org/10.1007/s00605-014-0652-0.

[5] J. Fan and Y. Zhou. Uniform local well-posedness for the density-dependent magnetohydrodynamic equations. *Appl. Math. Letters*, **24**(11):1945–1949, 2011. https://doi.org/10.1016/j.aml.2011.05.027.

[6] X. Huang and Y. Wang. Global strong solution to the 2D nonhomogeneous incompressible MHD system. *J. Differential Equations*, **254**(2):331–341, 2013. https://doi.org/10.1016/j.jde.2012.08.029.

[7] X. Jia and Y. Zhou. Regularity criteria for the 3D MHD equations involving partial components. *Nonlinear Anal. Real World Appl.*, **13**(1):410–418, 2012. https://doi.org/10.1016/j.nonrwa.2011.07.055.

[8] Z. Jiang, Y. Wang and Y. Zhou. On regularity criteria for the 2D generalized MHD system. *J. Math. Fluid Mech.*, **18**(2):331–341, 2016. https://doi.org/10.1007/s00021-015-0235-4.

[9] T. Ozawa. On critical cases of Sobolev's inequalities. *J. Funct. Anal.*, **127**(2):259–269, 1995. https://doi.org/10.1006/jfan.1995.1012.

[10] T. Tao. *Nonlinear Dispersive Equations: Local and Global Analysis.* American Mathematical Society, Provindence, RI, 2006. https://doi.org/10.1090/cbms/106.

[11] H. Wu. Strong solution to the incompressible magnetohydrodynamic equations with vacuum. *Comput. Math. Appl.*, **61**(9):2742–2753, 2011. https://doi.org/10.1016/j.camwa.2011.03.033.

[12] J. Wu, X. Xu, Q. Jiu, D. Niu and H. Yu. The 2D magnetohydrodynamic equations with magnetic diffusion. *Nonlinearity*, **78**(11):3935–3955, 2015.