Convergence Analysis of Iterative Scheme and Error Estimation of Positive Solution for a Fractional Differential Equation

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Abstract. In this paper, we focus on the iterative scheme and error estimation of positive solutions for a class of p-Laplacian fractional order differential equation subject to Riemann-Stieltjes integral boundary condition. Under a weaker growth condition of nonlinearity, by using a monotone iterative technique, we first establish a new result on the sufficient condition for the existence of a unique positive solution to the above problem, then construct an iterative scheme which converges to the unique positive solution, and then present an error estimation and the exact convergence rate of the approximate solution.

Keywords: uniqueness, fractional p-Laplacian equation, monotone iterative technique, error estimation.

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1   Introduction

Fractional order differential equations are generalization of integer order differential equations to non-integer order ones. Due to the nonlocality of fractional order derivatives, the generalization has a wide range of interesting applications in science and engineering. For example, many physical phenomena with long time memory behavior in diffusive interaction, electrochemistry, polymer rheology can be described successfully by fractional order mathematical models [1, 18, 26, 27, 31, 36, 41]. Thus fractional differential equations are regarded as an excellent tool for modeling various complex processes in natural sciences and engineering such as bioprocess [15, 37] and viscoelasticity [11, 22, 25, 40, 42].

In this paper, we focus on the iterative scheme and error estimation of positive solutions for the following p-Laplacian fractional order differential equation subject to Riemann-Stieltjes integral boundary condition

$$
\begin{cases}
-\boldsymbol{\mathcal{D}_t}^\beta \left(\varphi_p \left(-\boldsymbol{\mathcal{D}_t}^\alpha z\right)\right)(t) = f(t, z(t), \boldsymbol{\mathcal{D}_t}^\gamma z(t)), & t \in (0, 1), \\
\boldsymbol{\mathcal{D}_t}^\alpha z(0) = \boldsymbol{\mathcal{D}_t}^{\alpha+1} z(0) = \boldsymbol{\mathcal{D}_t}^\gamma z(0) = 0, \\
\boldsymbol{\mathcal{D}_t}^\alpha z(1) = 0, \ \boldsymbol{\mathcal{D}_t}^\gamma z(1) = \int_0^1 \boldsymbol{\mathcal{D}_t}^\gamma z(s) dA(s),
\end{cases}
\tag{1.1}
$$

where $\boldsymbol{\mathcal{D}_t}^\alpha, \boldsymbol{\mathcal{D}_t}^\beta, \boldsymbol{\mathcal{D}_t}^\gamma$ are the Riemann-Liouville fractional derivatives of order α, β, γ with $0 < \gamma \le 1 < \alpha \le 2 < \beta < 3$, $\alpha - \gamma > 1$, $\int_0^1 \boldsymbol{\mathcal{D}_t}^\gamma z(s) dA(s)$ denotes a Riemann-Stieltjes integral, A is a function of bounded variation. The p-Laplacian operator is defined as $\varphi_p(s) = |s|^{p-2}s$, $p > 2$, and obviously, $\varphi_p(s)$ is invertible and its inverse operator is $\varphi_q(s)$, where $q = \frac{p}{p-1}$ is the conjugate index of p.

In recent years, many nonlinear functional analysis methods have been developed to solve local or nonlocal nonlinear boundary value problems of differential equations, such as the iterative methods [2, 8, 10, 12, 17, 19, 20, 23, 24, 28, 29, 30, 32, 34, 38], upper and lower solution methods [17, 22, 23],the variational method [7, 8, 9, 13, 33] and the fixed point theorems [4, 5, 6, 14, 24, 35, 39]. These tools and methods not only serve for various applied mathematics problems but also are used to solve many practical problems from physics and engineering. In [16], Webb and Zima established the existence of multiple positive solutions for the nonlocal boundary value problem

$$
\begin{cases}
z''(t) + k^2 z = f(t, z(t)), & 0 < t < 1, \\
z(0) = 0, \ z(1) = \int_0^1 z(s) dA(s),
\end{cases}
\tag{1.2}
$$

where dA can be a signed measure. By means of the monotone iterative technique, Mao et. al [10] considered the uniqueness of the positive solution for the nonlocal problem (1.2) under the following growth condition:

(**H**$_1$) $f(t, z)$ is nondecreasing in z, and there exists a constant $0 < b < 1$ such that, for any $(t, x) \in (0, 1) \times [0, +\infty)$ and $r \in (0, 1)$,

$$
f(t, rz) \ge r^b f(t, z).
\tag{1.3}
$$

In [21], a more general condition is used to obtain a necessary and sufficient condition of positive solutions for a nonlinear singular differential system, namely

($\mathbf{H_2}$) $f(t, z)$ is nondecreasing in z; for any $r \in (0, 1)$, there exists $\eta(r) = m(r^{-\kappa} - 1)$ such that, for all $(t, z) \in (0, 1) \times [0, +\infty)$, $0 < m \le 1, 0 < \kappa < 1$,

$$f(t, rz) \ge r(1 + \eta(r))f(t, z). \tag{1.4}$$

The aim of this paper is to find weaker conditions than ($\mathbf{H_1}$) and ($\mathbf{H_2}$), and then to establish the uniqueness of the positive solution for the general p-Laplacian fractional order differential equation (1.1). The iterative scheme converging to the unique solution will also be constructed. Based on the convergence sequence, estimates on the error and the convergence rate of approximate solution are also obtained.

2 Preliminaries and lemmas

Before we give a detailed description of some preliminaries and lemmas, we first introduce a weaker condition than ($\mathbf{H_1}$) and ($\mathbf{H_2}$), which is to be used in the rest of the paper.

($\mathbf{A}$) $f : ((0, 1) \times [0, +\infty) \times [0, +\infty) \to [0, +\infty))$ is continuous and nondecreasing in the second and the third variables; and for any $(t, x, y) \in (0, 1) \times [0, +\infty) \times [0, +\infty)$ and $r \in (0, 1)$, there exists a function $\sigma : [0, 1] \to [0, +\infty)$ with $\sigma(r) > r$ for $r \in (0, 1)$ such that

$$f(t, rx, ry) \ge \sigma(r)f(t, x, y). \tag{2.1}$$

Remark 1. Obviously, the assumption ($\mathbf{A}$) includes assumptions ($\mathbf{H_1}$) and ($\mathbf{H_2}$) as special cases, that is, the condition (2.1) is weaker than the conditions (1.3) and (1.4).

Remark 2. If ($\mathbf{A}$) holds, then for any $c \ge 1$ and $(t, x, y) \in (0, 1) \times [0, +\infty) \times [0, +\infty)$, inequality (2.1) is equivalent to

$$f(t, cx, cy) \le \frac{1}{\sigma(c^{-1})}f(t, x, y). \tag{2.2}$$

Proof. In fact, for any $c \ge 1$ and $(t, x, y) \in (0, 1) \times [0, +\infty) \times [0, +\infty)$, it follows from (2.1) that

$$f(t, x, y) = f(t, c^{-1}cx, c^{-1}cy) \ge \sigma(c^{-1})f(t, cx, cy),$$

i.e, inequality (2.2) holds. $\square$

Remark 3. The function f includes a large number of nonlinear functions and particularly covers the standard type of sublinear functions

$$f(t, x, y) = \sum_{i=1}^{n}[a_i(t)x^{\alpha_i} + b_i(t)y^{\beta_i}],$$

where $0 < \alpha_i, \beta_i < 1$ and $a_i, b_i \in C((0, 1), [0, +\infty))$, $i = 1, 2, ..., n$.

Now we give some basic notations and concepts on the sense of Riemann-Liouville fractional derivatives [11].

The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $x : (0, +\infty) \to \mathbb{R}$ is given by

$$I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds$$

provided that the right-hand side is pointwise defined on $(0, +\infty)$.

The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $x : (0, +\infty) \to \mathbb{R}$ is given by

$$\boldsymbol{D_t}^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^{(n)} \int_0^t (t-s)^{n-\alpha-1} x(s) ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

In order to simplify the p-Laplacian fractional order differential equation (1.1) to a more convenient form, for $x \in C[0, 1]$, we make a change of variable by $z(t) = I^\gamma x(t)$. Since $1 < \alpha \le 2$, $1 < \alpha - \gamma < 2$, according to the definition of the Riemann-Liouville calculus, one has

$$\boldsymbol{D_t}^\alpha z(t) = \frac{d^2}{dt^2} I^{2-\alpha} z(t) = \frac{d^2}{dt^2} I^{2-\alpha} I^\gamma x(t) = \frac{d^2}{dt^2} I^{2-\alpha+\gamma} x(t) = \boldsymbol{D_t}^{\alpha-\gamma} x(t),$$

$$\boldsymbol{D_t}^{\alpha+1} z(t) = \frac{d^3}{dt^3} I^{3-\alpha-1} z(t) = \frac{d^3}{dt^3} I^{3-\alpha-1} I^\gamma x(t) = \frac{d^3}{dt^3} I^{3-\alpha-1+\gamma} x(t)$$

$$= \boldsymbol{D_t}^{\alpha-\gamma+1} x(t), \quad \boldsymbol{D_t}^\gamma z(t) = \boldsymbol{D_t}^\gamma I^\gamma x(t) = x(t). \tag{2.3}$$

Consequently, it follows from (2.3) that the p-Laplacian fractional order differential equation (1.1) reduces to the following equivalent boundary value problem

$$\begin{cases} -\boldsymbol{D_t}^\beta \varphi_p \left(-\boldsymbol{D_t}^{\alpha-\gamma} x(t) \right) = f(t, I^\gamma x(t), x(t)), \\ \boldsymbol{D_t}^{\alpha-\gamma} x(0) = \boldsymbol{D_t}^{\alpha-\gamma+1} x(0) = \boldsymbol{D_t}^{\alpha-\gamma} x(1) = 0, \\ x(0) = 0, \ x(1) = \int_0^1 x(s) dA(s). \end{cases} \tag{2.4}$$

Thus in order to establish the existence and uniqueness of positive solutions for the equation (1.1), we only need to focus on the equivalent problem (2.4) of the equation (1.1). To do this, we define

$$G_\beta(t, s) = \frac{1}{\Gamma(\beta)} \begin{cases} [t(1-s)]^{\beta-1}, & 0 \le t \le s \le 1, \\ [t(1-s)]^{\beta-1} - (t-s)^{\beta-1}, & 0 \le s \le t \le 1, \end{cases} \tag{2.5}$$

then we have the following lemma:

Lemma 1. *(see [22]). Given $g \in L^1(0, 1)$, the following fractional boundary value problem*

$$\begin{cases} -\boldsymbol{D_t}^{\alpha-\gamma} x(t) = g(t), & 0 < t < 1, \\ x(0) = x(1) = 0, \end{cases} \tag{2.6}$$

has a unique solution

$$x(t) = \int_0^1 G_{\alpha-\gamma}(t,s)g(s)ds.$$

On the other hand, it is easy to know that the fractional differential equation

$$\begin{cases} -\mathcal{D_t}^{\alpha-\gamma}x(t) = 0, & 0 < t < 1, \\ x(0) = 0, x(1) = 1, \end{cases} \tag{2.7}$$

has a unique solution $x(t) = t^{\alpha-\gamma-1}$. Thus according to the strategy of [16], by denoting

$$\mathcal{A} = \int_0^1 t^{\alpha-\gamma-1}dA(t), \quad \mathcal{G}_A(s) = \int_0^1 G_{\alpha-\gamma}(t,s)dA(t),$$

we can obtain the Green function

$$H(t,s) = \frac{\mathcal{G}_A(s)}{1-\mathcal{A}}t^{\alpha-\gamma-1} + G_{\alpha-\gamma}(t,s) \tag{2.8}$$

of the following fractional differential equation subject to the Riemann-Stieltjes integral boundary condition

$$\begin{cases} -\mathcal{D_t}^{\alpha-\gamma}x(t) = g(t), \\ x(0) = 0, \ x(1) = \int_0^1 x(s)dA(s). \end{cases} \tag{2.9}$$

With the help of (2.5)–(2.9), we have the following lemmas which have already been proven in [26].

Lemma 2. *Given $g \in L^1[0,1]$, $0 < \gamma \le 1 < \alpha \le 2 < \beta < 3$ and $\alpha - \gamma > 1$, the fractional order p-Laplacian differential equation*

$$\begin{cases} -\mathcal{D_t}^{\beta}\varphi_p\left(-\mathcal{D_t}^{\alpha-\gamma}x(t)\right) = g(t), \\ \mathcal{D_t}^{\alpha-\gamma}x(0) = \mathcal{D_t}^{\alpha-\gamma+1}x(0) = \mathcal{D_t}^{\alpha-\gamma}x(1) = 0, \\ x(0) = 0, \ x(1) = \int_0^1 x(s)dA(s) \end{cases}$$

has a unique solution

$$x(t) = \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G_{\beta}(s,\tau)g(\tau)d\tau\right)ds.$$

Lemma 3. *Assume $0 \le \mathcal{A} < 1$ and $\mathcal{G}_A(s) \ge 0$ for $s \in [0,1]$, then the function $G_{\beta}(t,s)$ and $H(t,s)$ enjoy the following properties:*
(1) $G_{\beta}(t,s) > 0, H(t,s) > 0, \ for \ t,s \in (0,1)$.
(2)

$$\frac{t^{\beta-1}(1-t)s(1-s)^{\beta-1}}{\Gamma(\beta)} \le G_{\beta}(t,s) \le \frac{\beta-1}{\Gamma(\beta)}t^{\beta-1}(1-t), \quad for \ t,s \in [0,1].$$

(3) There exist two positive constants d, e such that

$$dt^{\alpha-\gamma-1}\mathcal{G}_A(s) \le H(t,s) \le et^{\alpha-\gamma-1}, \ t,s \in [0,1].$$

3   Main results

To ensure the validity of Lemma 3, we need the following assumptions.

(**B**) A is a function of bounded variation satisfying $\mathcal{G}_A(s) \geq 0$ for $s \in [0, 1]$ and

$$0 \leq \int_0^1 t^{\alpha - \gamma - 1} dA(t) < 1.$$

Now we carry out our work in the space $E = C[0, 1]$, which is a Banach space endowed with the form $||x|| = \max_{t \in [0,1]} |x(t)|$ for any $x \in E$. Define

$$P = \{x \in C[0, 1] : x(t) \geq 0, t \in [0, 1]\}.$$

Then P is a normal cone with normality constant 1 in the Banach space E. Next define a sub-set of P by

$$D = \{x(t) \in P : \text{there exist two numbers } K_x > 1 > k_x > 0 \text{ such that}$$
$$k_x t^{\alpha - \gamma - 1} \leq x(t) \leq K_x t^{\alpha - \gamma - 1}, \ t \in [0, 1]\}.$$

Theorem 1. *Suppose the conditions* (**A**)–(**B**) *hold. If*

$$0 < \int_0^1 f^{q-1}(\tau, \tau^{\alpha - 1}, \tau^{\alpha - \gamma - 1}) d\tau < +\infty, \tag{3.1}$$

then the p-Laplacian fractional order differential equation (1.1) *has a unique positive solution* $z^*(t)$. *Let* $y^*(t) = \mathcal{D}_t^{\gamma} z^*(t)$, *then* $y^* \in D$. *Moreover for any initial value* $y_0 \in D$, *the sequence of functions defined by*

$$y_n = \int_0^1 H(t, s) \varphi_q \left(\int_0^1 G_\beta(s, \tau) f\left(\tau, I^{\gamma} y_{n-1}(\tau), y_{n-1}(\tau)\right) d\tau \right) ds, \ n \geq 1$$

converges uniformly to $y^*(t) = \mathcal{D}_t^{\gamma} z^*(t)$ *on* $[0, 1]$ *as* $n \to +\infty$. *Furthermore there exists an error estimation*

$$||z_n - z^*|| \leq \frac{2}{\Gamma(\gamma + 1)\sqrt{\epsilon}} (1 - \epsilon^{(q-1)^n}) ||y_0||,$$

which has the rate of convergence

$$||z_n - z^*|| = o(1 - \epsilon^{(q-1)^n}),$$

where $0 < \epsilon < 1$ *is a positive constant which is determined by the initial value* y_0.

Proof.   It follows from Lemma 2 that we can define a nonlinear operator $T : D \to E$ by

$$(Tx)(t) = \int_0^1 H(t, s) \varphi_q \left(\int_0^1 G_\beta(s, \tau) f\left(\tau, I^{\gamma} x(\tau), x(\tau)\right) d\tau \right) ds \tag{3.2}$$

and a fixed point x of the operator T is a solution of the fractional order integro-differential equation (2.4). Thus $z(t) = I^\gamma x(t)$ is a solution of the p-Laplacian fractional order differential equation (1.1).

In what follows, we prove that T is well defined and $T : D \to D$. In fact, according to the definition of D, for any $x \in D$, there are two numbers $K_x > 1 > k_x > 0$ such that

$$k_x t^{\alpha-\gamma-1} \le x(t) \le K_x t^{\alpha-\gamma-1}, \ t \in [0,1]. \tag{3.3}$$

Consequently, one has

$$\frac{k_x \Gamma(\alpha-\gamma)}{\Gamma(\alpha)} t^{\alpha-1} = \frac{K_x}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} s^{\alpha-\gamma-1} ds \le I^\gamma x(t) = \frac{1}{\Gamma(\gamma)}$$

$$\times \int_0^t (t-s)^{\gamma-1} x(s) ds \le \frac{K_x}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} s^{\alpha-\gamma-1} ds = \frac{K_x \Gamma(\alpha-\gamma)}{\Gamma(\alpha)} t^{\alpha-1}. \tag{3.4}$$

Thus applying Lemma 3, (3.1) and (3.2)–(3.4), we have

$$\int_0^1 H(t,s)\varphi_q \left(\int_0^1 G_\beta(s,\tau) f(\tau, I^\gamma x(\tau), x(\tau)) d\tau \right) ds$$

$$\le e \left(\frac{\beta-1}{\Gamma(\beta)} \right)^{q-1} \int_0^1 \left[f \left(\tau, \frac{K_x \Gamma(\alpha-\gamma)}{\Gamma(\alpha)} \tau^{\alpha-1}, K_x \tau^{\alpha-\gamma-1} \right) \right]^{q-1} d\tau t^{\alpha-\gamma-1}$$

$$\le e \left(\frac{\beta-1}{\Gamma(\beta)} \right)^{q-1} \sigma^{q-1} \left(K_x \max \left(\frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}, 1 \right) \right)$$

$$\times \int_0^1 f^{q-1}(\tau, \tau^{\alpha-1}, \tau^{\alpha-\gamma-1}) d\tau t^{\alpha-\gamma-1} < \infty. \tag{3.5}$$

On the other hand, we also have

$$\int_0^1 H(t,s)\varphi_q \left(\int_0^1 G_\beta(s,\tau) f(\tau, I^\gamma x(\tau), x(\tau)) d\tau \right) ds$$

$$\ge \frac{d}{\Gamma(\beta)} \int_0^1 \mathcal{G}_A(s) s^{(\beta-1)(q-1)} (1-s)^{q-1} ds \int_0^1 \tau^{q-1} (1-\tau)^{(\beta-1)(q-1)}$$

$$\times f^{q-1} \left(\tau, \frac{k_x \Gamma(\alpha-\gamma)}{\Gamma(\alpha)} \tau^{\alpha-1}, k_x \tau^{\alpha-\gamma-1} \right) d\tau t^{\alpha-\gamma-1}$$

$$\ge \frac{d}{\Gamma(\beta)} \sigma^{q-1} \left(k_x \min \left(\frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}, 1 \right) \right) \int_0^1 \mathcal{G}_A(s) s^{(\beta-1)(q-1)} (1-s)^{q-1} ds$$

$$\times \int_0^1 \tau^{q-1} (1-\tau)^{(\beta-1)(q-1)} f^{q-1}(\tau, \tau^{\alpha-1}, \tau^{\alpha-\gamma-1}) d\tau t^{\alpha-\gamma-1}. \tag{3.6}$$

(3.5)–(3.6) yield that T is well defined and $T(D) \subset D$.

Now given $y_0 \in D$, there exist four positive constants $k_{y_0}, K_{y_0}, \widetilde{k}_{y_0}, \widetilde{K}_{y_0}$ such that

$$k_{y_0} t^{\alpha-\gamma-1} \le y_0 \le K_{y_0} t^{\alpha-\gamma-1}, \ \ \widetilde{k}_{y_0} t^{\alpha-\gamma-1} \le T y_0 \le \widetilde{K}_{y_0} t^{\alpha-\gamma-1},$$

which implies that

$$\frac{\widetilde{k}_{y_0}}{K_{y_0}} y_0 \leq T y_0 \leq \frac{\widetilde{K}_{y_0}}{k_{y_0}} y_0.$$

Choose a constant t_0 such that

$$0 < t_0 \leq \min \left\{ \left(\widetilde{k}_{y_0} / K_{y_0} \right)^{\frac{1}{2-q}}, \left(k_{y_0} / \widetilde{K}_{y_0} \right)^{\frac{1}{2-q}} \right\},$$

and then we have $t_0 \in (0, 1)$ and

$$t_0^{2-q} y_0 \leq T y_0 \leq \left(\frac{1}{t_0} \right)^{2-q} y_0. \tag{3.7}$$

Take $u_0 = t_0 y_0$, $v_0 = y_0 / t_0$, $t_0 \in (0, 1)$. Clearly $u_0 \leq v_0$. Now we define the iterative sequence as follows

$$u_n = T u_{n-1}, v_n = T v_{n-1}, \ (n = 1, 2, \ldots).$$

By $(\mathbf{A})$, we know that T is an increasing operator in x and

$$T(rx) = \int_0^1 H(t,s) \varphi_q \left(\int_0^1 G_\beta(s,\tau) f\left(\tau, rI^\gamma x(\tau), rx(\tau)\right) d\tau \right) ds$$

$$\geq \sigma^{q-1}(r) \int_0^1 H(t,s) \varphi_q \left(\int_0^1 G_\beta(s,\tau) f\left(\tau, I^\gamma x(\tau), x(\tau)\right) d\tau \right) ds$$

$$= \sigma^{q-1}(r) T x, \quad 0 < r < 1, \tag{3.8}$$

$$T(rx) = \int_0^1 H(t,s) \varphi_q \left(\int_0^1 G_\beta(s,\tau) f\left(\tau, rI^\gamma x(\tau), rx(\tau)\right) d\tau \right) ds$$

$$\leq \sigma^{1-q} \left(\frac{1}{r} \right) \int_0^1 H(t,s) \varphi_q \left(\int_0^1 G_\beta(s,\tau) f\left(\tau, I^\gamma x(\tau), x(\tau)\right) d\tau \right) ds$$

$$= \sigma^{1-q} \left(\frac{1}{r} \right) T x, \quad r \geq 1. \tag{3.9}$$

It follows from (3.7)–(3.9) that

$$u_1 = T u_0 \geq \sigma^{q-1}(t_0) T y_0 \geq t_0^{q-1} T y_0 \geq t_0 y_0 = u_0,$$

$$v_1 = T v_0 \leq \left(\frac{1}{\sigma(t_0)} \right)^{q-1} T y_0 \leq \left(\frac{1}{t_0} \right)^{q-1} T y_0 \leq \frac{1}{t_0} y_0 = v_0. \tag{3.10}$$

Thus from induction, (3.10) and $u_0 \leq v_0$, one has

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0. \tag{3.11}$$

Since $u_0 = t_0^2 v_0$, we get

$$u_1 = T u_0 = T(t_0^2 v_0) \geq \sigma^{q-1}(t_0^2) T v_0 \geq (t_0^2)^{q-1} v_1.$$

Thus by induction, we have $u_n \geq (t_0^2)^{(q-1)^n} v_n$, $(n = 0, 1, 2, \ldots)$. Since P is a normal cone with normality constant 1, and $u_{n+m} - u_n \leq v_n - u_n$, for any $m \in N$, we have

$$||u_{n+m} - u_n|| \leq ||v_n - u_n|| \leq \left(1 - (t_0^2)^{(q-1)^n}\right) ||v_0|| \to 0, \text{ as } n \to +\infty, \quad (3.12)$$

which implies that $\{u_n\}$ is a Cauchy sequence, and u_n converges to some $y^* \in D$. By (3.12) and

$$||v_n - y^*|| \leq ||v_n - u_n|| + ||u_n - y^*||,$$

we also obtain $v_n \to y^*$. It follows from (3.11) that $y^* \in D$ is a fixed point of T, and $y^* \in [u_0, v_0]$.

Thus for any initial value $y_0 \in D$, it follows from $u_0 \leq y_0 \leq v_0$ that $u_n \leq y_n \leq v_n$, $(n = 1, 2, \ldots)$. So we have

$$||y_n - y^*|| \leq ||y_n - u_n|| + ||u_n - y^*|| \leq 2||v_n - u_n||$$
$$\leq 2\left(1 - (t_0^2)^{(q-1)^n}\right) ||v_0||,$$

which implies that the sequence of functions defined by

$$y_n = \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau) f\left(\tau, I^\gamma y_{n-1}(\tau), y_{n-1}(\tau)\right) d\tau\right) ds, \; n = 1, 2, \ldots$$

converges uniformly to the positive solution $y^*(t)$ of equation (2.4) on $[0, 1]$ as $n \to +\infty$. Furthermore we have the error estimation

$$||y_n - y^*|| \leq \frac{2}{\sqrt{\epsilon}}(1 - \epsilon^{(q-1)^n})||y_0||,$$

which has the rate of convergence

$$||y_n - y^*|| = o(1 - \epsilon^{(q-1)^n}),$$

where $0 < \epsilon = t_0^2 < 1$ is a constant which is determined by y_0.

Next we prove that the positive solution of the problem (2.4) is unique. Actually, suppose $\overline{y} \in D$ is any fixed point of T, then we have $T\overline{y} = \overline{y}$. From $\overline{y}, y^* \in D$ and the definition of D, let $t_1 = \sup\{t > 0 : \overline{y} \geq ty^*\}$, then we have $0 < t_1 < +\infty$. We assert that $t_1 \geq 1$. If not, we have $0 < t_1 < 1$, thus

$$\overline{y} = T\overline{y} \geq T(t_1 y^*) \geq \sigma^{q-1}(t_1) T y^* = \sigma^{q-1}(t_1) y^* \geq t_1^{q-1} y^*,$$

since $1 < q < 2$ and we have $t_1^{q-1} > t_1$, a contradiction. Thus we obtain that $t_1 \geq 1$ and $\overline{y} \geq y^*$. In the same way, we also have $\overline{y} \leq y^*$, that is $\overline{y} = y^*$, and then y^* is a unique fixed point of T in D. Of course, it is also a unique positive solution of the fractional order p-Laplacian differential equation (2.4).

In the end, let $z^*(t) = I^\gamma y^*(t)$, then z^* is unique positive solution of the p-Laplacian fractional order differential equation (1.1). Moreover for any initial value $y_0 \in D$, the sequence of functions defined by

$$y_n = \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau) f\left(\tau, I^\gamma y_{n-1}(\tau), y_{n-1}(\tau)\right) d\tau\right) ds, \; n = 1, 2, \ldots$$

converges uniformly to $y^*(t) = \boldsymbol{D_t}^\gamma z^*(t)$ on $[0, 1]$ as $n \to +\infty$. Furthermore there exists an error estimation

$$\|z_n - z^*\| \leq \frac{2}{\Gamma(\gamma + 1)\sqrt{\epsilon}}(1 - \epsilon^{(q-1)^n})\|y_0\|,$$

which has the rate of convergence $\|z_n - z^*\| = o(1 - \epsilon^{(q-1)^n})$, where $0 < \epsilon < 1$ is a positive constant which is determined by the initial value y_0. □

Remark 4. In Theorem 1, we derive a new result on the sufficient condition for the existence and uniqueness of a positive solution for a fractional order p-Laplacian differential equation with nonlocal boundary value condition. More important, we construct an iterative sequence for the solution and give estimation of the error and the rate of convergence. In particular, the initial value of the iterative sequence can be chosen arbitrarily in D, thus we can choose some simple initial value such 0 or $t^{\alpha-\gamma-1}$, which is very beneficial to our computation.

Remark 5. Theorem 1 is still valid if $(\mathbf{A})$ is replaced by one of the following assumptions:

$(\mathbf{H_1^*})$ $f : ((0,1) \times [0, +\infty) \times [0, +\infty) \to [0, +\infty))$ is continuous and non-decreasing in the second and the third variables; and there exists a constant $0 < b < 1$ such that, for any $(t, x, y) \in (0, 1) \times [0, +\infty) \times [0, +\infty)$ and $r \in (0, 1)$,

$$f(t, rx, ry) \geq r^b f(t, x, y).$$

$(\mathbf{H_2^*})$ $f : ((0,1) \times [0, +\infty) \times [0, +\infty) \to [0, +\infty))$ is continuous and nondecreasing in the second and the third variables; and for any $r \in (0, 1)$, there exists $\eta(r) = m(r^{-\kappa} - 1)$ such that, for all $(t, x, y) \in (0, 1) \times [0, +\infty) \times [0, +\infty)$, $0 < m \leq 1, 0 < \kappa < 1$,

$$f(t, rx, ry) \geq r[1 + \eta(r)]f(t, x, y).$$

Remark 6. The short memory behaviors often happen in various biomedicine processes [3] and porous medium flow with short wave characters [1], here we also comment the p-Laplacian fractional order differential equation (1.1) has unique positive solution with similar iterative properties and estimation of error as Theorem 1 when t is in a short memory measure such as $t \in (1 - \epsilon, 1)$ with $\lim_{\epsilon \to 1} \epsilon = 1$ since the Green function possesses the same characters.

4　Numerical results

Example 1. Consider the existence of positive solutions for the nonlinear fractional differential equation

$$\begin{cases} -\boldsymbol{D_t}^{\frac{5}{2}}\left(\varphi_3\left(-\boldsymbol{D_t}^{\frac{3}{2}}z\right)\right)(t) = \dfrac{z^{\frac{1}{2}}(t) + \left(\boldsymbol{D_t}^{\frac{1}{4}}z(t)\right)^{\frac{1}{3}}}{\sqrt{t}}, \ t \in (0, 1), \\[3mm] \boldsymbol{D_t}^{\frac{3}{2}}z(0) = \boldsymbol{D_t}^{\frac{5}{2}}z(0) = \boldsymbol{D_t}^{\frac{1}{4}}z(0) = 0, \\[3mm] \boldsymbol{D_t}^{\frac{1}{4}}z(1) = 0, \ \boldsymbol{D_t}^{\frac{1}{4}}z(1) = \displaystyle\int_0^1 \boldsymbol{D_t}^{\frac{1}{4}}z(s)dA(s), \end{cases} \quad (4.1)$$

where A is a function of bounded variation

$$A(t) = \begin{cases} 0, & t \in [0, 1/2), \\ 3/2, & t \in [1/2, 3/4), \\ 1, & t \in [3/4, 1]. \end{cases}$$

Then by simple computation, the equation (4.1) is equivalent to a 4-point boundary value problem with coefficients of both signs

$$\begin{cases} -\boldsymbol{D_t}^{\frac{5}{2}} \left(\varphi_3 \left(-\boldsymbol{D_t}^{\frac{3}{2}} z \right) \right)(t) = \dfrac{z^{\frac{1}{2}}(t) + \left(\boldsymbol{D_t}^{\frac{1}{4}} z(t) \right)^{\frac{1}{3}}}{\sqrt{t}}, & t \in (0, 1), \\ \boldsymbol{D_t}^{\frac{3}{2}} z(0) = \boldsymbol{D_t}^{\frac{5}{2}} z(0) = \boldsymbol{D_t}^{\frac{1}{4}} z(0) = 0, \boldsymbol{D_t}^{\frac{1}{4}} z(1) = 0, \\ \boldsymbol{D_t}^{\frac{1}{4}} z(1) = 1.5 \boldsymbol{D_t}^{\frac{1}{4}} z(0.5) - 0.5 \boldsymbol{D_t}^{\frac{1}{4}} z(0.75). \end{cases}$$

Thus we only need to study the 4-point boundary value problem (4.2).

Let $f(t, x, y) = (x^{\frac{1}{2}} + y^{\frac{1}{3}})/t$, then $f \in C((0, 1) \times [0, +\infty) \times [0, \infty), [0, +\infty))$, and for any fixed $t \in (0, 1)$, $f(t, x, y)$ is nondecreasing in x and y. Now take $\sigma(r) = r^{\frac{3}{4}}, r \in [0, 1]$, then we have $\sigma(r) > r$ for $r \in (0, 1)$, and for any $r \in (0, 1)$ and $(0, 1) \times [0, +\infty) \times [0, \infty)$, we have

$$f(t, rx, ry) = \frac{r^{\frac{1}{2}} x^{\frac{1}{2}} + r^{\frac{1}{3}} y^{\frac{1}{3}}}{\sqrt{t}} \geq r^{\frac{3}{4}} \frac{x^{\frac{1}{2}} + y^{\frac{1}{3}}}{\sqrt{t}} = r^{\frac{3}{4}} f(t, x, y).$$

Thus the condition (**A**) is satisfied.

Next we verify the condition (**B**). Let $\alpha = \frac{3}{2}, \beta = \frac{5}{2}, \gamma = \frac{1}{4}, p = 3$, then $0 < \gamma \leq 1 < \alpha \leq 2 < \beta < 3$, $\alpha - \gamma > 1$ and

$$G_{\alpha-\gamma}(t, s) = \frac{1}{\Gamma(5/4)} \begin{cases} [t(1-s)]^{\frac{1}{4}} - (t-s)^{\frac{1}{4}} =: G_1(t, s), & 0 \leq s \leq t \leq 1, \\ [t(1-s)]^{\frac{1}{4}} =: G_2(t, s), & 0 \leq t \leq s \leq 1. \end{cases} \quad (4.2)$$

Thus, we have

$$\begin{aligned} \mathcal{G}_A(s) &= \int_0^1 G_{\alpha-\gamma}(t, s) dA(t) \\ &= \frac{1}{\Gamma(\frac{5}{4})} \begin{cases} 1.5 G_1(0.5, s) - 0.5 G_1(0.75, s), & 0 \leq s < 0.5, \\ 1.5 G_2(0.5, s) - 0.5 G_1(0.75, s), & 0.5 \leq s < 0.75, \\ 1.5 G_2(0.5, s) - 0.5 G_2(0.75, s), & 0.75 \leq s \leq 1, \end{cases} \\ &= \frac{1}{\Gamma(\frac{5}{4})} \begin{cases} \dfrac{3 - \sqrt[4]{3}}{2^5}(1-s)^{\frac{1}{4}} + \dfrac{1}{2}(\dfrac{3}{4} - s)^{\frac{1}{4}} - \dfrac{3}{2}(\dfrac{1}{2} - s)^{\frac{1}{4}}, & 0 \leq s < \dfrac{1}{2}, \\ \dfrac{3 - \sqrt[4]{3}}{2^5}(1-s)^{\frac{1}{4}} + \dfrac{1}{2}(\dfrac{3}{4} - s)^{\frac{1}{4}}, & \dfrac{1}{2} \leq s < \dfrac{3}{4}, \\ \dfrac{3 - \sqrt[4]{3}}{2^5}(1-s)^{\frac{1}{4}}, & \dfrac{3}{4} \leq s \leq 1, \end{cases} \end{aligned}$$

which implies that $\mathcal{G}_A(s) \geq 0$ for $s \in [0, 1]$.

On the other hand, we have

$$\mathcal{A} = \int_0^1 t^{\alpha-\gamma-1} dA(t) = \int_0^1 t^{\frac{1}{4}} dA(t) = 1 - \frac{1}{4}\int_0^1 A(t) t^{-\frac{3}{4}} dt = 0.7961 < 1.$$

Therefore, the condition (**B**) is also satisfied.

Now we check the condition (3.1). In fact,

$$0 < \int_0^1 f^{q-1}(\tau, \tau^{\alpha-1}, \tau^{\alpha-\gamma-1}) d\tau = \int_0^1 \left(\frac{\tau^{\frac{1}{4}} + \tau^{\frac{1}{12}}}{\sqrt{\tau}} \right)^{\frac{3}{2}} d\tau$$

$$< 2^{\frac{3}{2}} \int_0^1 \tau^{-\frac{15}{24}} d\tau < +\infty.$$

Then the p-Laplacian fractional order differential equation (1.1) has a unique positive solution $z^*(t) = I^{\frac{1}{4}} y^*(t)$, $y^* \in D$. Moreover for any initial value $y_0 \in D$, the sequence of functions defined by

$$y_n = \int_0^1 \left(\frac{\mathcal{G}_A(s)}{0.2039} t^{\frac{1}{4}} + G_{\frac{5}{4}}(t,s) \right)$$

$$\times \left(\int_0^1 G_{\frac{5}{2}}(s,\tau) \frac{\left(I^{\frac{1}{4}} y_{n-1}(\tau) \right)^{\frac{1}{2}} + (y_{n-1}(\tau))^{\frac{1}{3}}}{\sqrt{\tau}} d\tau \right)^{\frac{3}{2}} ds, \quad n = 1, 2, \dots \quad (4.3)$$

converges uniformly to $y^*(t)$ on $[0,1]$ as $n \to +\infty$. Furthermore there exists an error estimation

$$||z_n - z^*|| = \max_{t \in [0,1]} |I^{\frac{1}{4}}(y_n(t) - y^*(t))| \le \frac{4}{\Gamma(\frac{1}{4})}||y_n - y^*|| \le \frac{8}{\Gamma(\frac{1}{4})\sqrt{\epsilon}}(1 - \epsilon^{\left(\frac{1}{2}\right)^n})||y_0||,$$

which has the rate of convergence

$$||z_n - z^*|| = o(1 - \epsilon^{\left(\frac{1}{2}\right)^n}),$$

where $0 < \epsilon < 1$ is a positive constant which is determined by the initial value y_0. In particular, if $y_0 = t^{\frac{1}{4}}$, then by computation, we have

$$0.2513 t^{\frac{1}{4}} \le Ty_0 \le 5.6577 t^{\frac{1}{4}}.$$

Take

$$t_0 = \min \left\{ (0.2513)^2, \left(\frac{1}{5.6577} \right)^2 \right\} = \min\{0.0632, 0.0313\} = 0.0313,$$

then we have the error estimation

$$||z_n - z^*|| \le 70.4846 \left(1 - 0.00097969^{\left(\frac{1}{2}\right)^n} \right),$$

and the estimate of convergence rate by

$$||z_n - z^*|| = o \left(1 - 0.00097969^{\left(\frac{1}{3}\right)^n} \right).$$

Table 1 shows the maximum errors of the iterative scheme compared to the exact solution. The numerical results confirm that the scheme (4.3) has the global convergence rate $o(1 - 0.00097969^{\left(\frac{1}{2}\right)^n})$.

Table 1. Numerical results of maximum errors while n increases.

n	Maximum errors for $\alpha = \frac{3}{2}, \beta = \frac{5}{2}, \gamma = \frac{1}{4}, p = 3$
15	0.0149
20	4.6571e-004
25	1.4554e-005
30	4.5480e-007

5   Conclusions

By introducing a more general growth condition, a new result on the uniqueness of positive solutions for the general p-Laplacian fractional order differential equation is established. Through a detailed theoretical analysis, we construct an iterative converging scheme and also obtain maximum error estimates and the convergence rate. The developed algorithms, theoretical and numerical results can also be extended to the problems discussed in [10, 19, 21].

Acknowledgements

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