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Some Generalizations of Kannan's Theorems via σ_c-function and its Application

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Abstract. In this article, we go on to discuss various proper extensions of Kannan's two different fixed point theorems, and introduce the new concept of σ_c function, which is independent of the three notions of simulation function, manageable functions, and R-functions. These results are analogous to some well-known theorems, and extend several known results in this literature. An application of the new results to the integral equation is also provided.

Keywords: fixed point, coincidence point, Kannan's mapping, simulation function, R-function, manageable function, σ_c-function.

AMS Subject Classification: 47H10; 54H25.

1 Introduction

The fixed point theory is one of the most useful and essential tools of nonlinear analysis. Banach [1] has given the most important and fundamental theorem of this branch by defining the concept of contraction operators. After that, so many theorems and generalizations of it has been made over the course of time. Recently, Khojasteh et al. [9] introduced the notion of simulation function and Du and Khojasteh [3] presented a very close but the independent concept of manageable function, both of which give a new way to extend the Banach's fixed point result. However, López de Hierro and Shahzad [2] has given the concept of R-function (the generalized concept of both simulation and manageable function) to obtain the extension of Banach's theorem for the R-contraction operator. A multi-valued version of such generalization can be found in [6]. On the other hand, Kannan [7, 8] found a particular type of operators which are not necessarily continuous, but satisfies the fixed point property on complete metric spaces. The class of operators found by Kannan and that of Banach are independent of each other [7, 8]. Here, in this article, we prove several proper generalizations of Kannan's theorems, by finding fixed points and coincidence points for two set of operators, via the new concept of σ_c-functions. These new generalizations also extend several known theorems like Koparde-Waghmode theorem [10] and Patel-Deheri's theorem [13] by finding the analogous results of Malceski theorem [12].

2 Preliminaries

Let $T, S\colon (X, d) \to (X, d)$ be two operators on metric space (X, d). Then, T is said to have a fixed point c in X, if $Tc = c$. The point c is said to be a coincidence point of the pair (T, S), if $Tc = Sc$. The space X is said to satisfy the coincidence property with respect to the pair (T, S) if there is at least one point c for which $Tc = Sc$. The iterates of the self-mapping T is the set $\{T^n\colon X \to X\}_{n\in\mathbb{N}\cup\{0\}}$, where $T^0 = Id_X$, and $T^{n+1} = T \circ T^n$ for all $n \in \mathbb{N}$. Given a point $x_0 \in X$, the Picard sequence of T based on x_0 is the sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ given by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Clearly, $x_n = T^n x_0$ for all $n \in \mathbb{N}$. The mapping T is said to be asymptotically regular at point $x \in X$ if $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$. The mapping T is said to be sequentially convergent if, for each sequence $\{x_n\}$ the following holds true: if $\{Tx_n\}$ converges, then $\{x_n\}$ also converges. We say that a sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ is S-bounded if $\{Sx_n\}_{n\in\mathbb{N}\cup\{0\}}$ is bounded and S-Cauchy if $\{Sx_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a Cauchy sequence.

We now state the Kannan's two theorems, for which we find the generalizations.

Theorem 1. *(Kannan, [7]) If T is an operator on a complete metric space (X, d), satisfying the condition that $d(Tx, Ty) \le \alpha[d(Tx, x) + d(Ty, y)]$, for all $x, y \in X$, where $0 < \alpha < 1/2$, then T has unique fixed point in X.*

Theorem 2. *(Kannan, [8]) Let X be a metric space with d as metric. Let T be a map of X into itself such that*

(i) $d(Tx, Ty) \le \alpha[d(Tx, x) + d(Ty, y)]$ *for all $x, y \in X$, where $0 < \alpha < 1/2$;*

(ii) T is continuous at a point $c \in X$; and

(iii) There exists a point $p \in X$ such that the sequence of iterates $\{T^n(p)\}$ has a subsequence $\{T^{n_i}(p)\}$ converging to c.

Then c is the unique fixed point of T.

We state some definitions; starting with the concept of simulation functions, which was initiated by Khojasteh et al. [9], to show a new way to study, fixed point theory.

Definition 1. (Simulation function, [9]) Let $\zeta\colon [0,\infty)\times[0,\infty)\to\mathbb{R}$ be a mapping, then ζ is said to be a simulation function if it satisfies the following conditions:

(ζ1) $\zeta(0,0)=0$;

(ζ2) $\zeta(t,s)<s-t$ for all $t,s>0$;

(ζ3) if $\{t_n\}$ and $\{s_n\}$ are sequences in $(0,\infty)$ such that, if $\lim_{n\to\infty} t_n=\lim_{n\to\infty} s_n>0$, then $\limsup_{n\to\infty}\zeta(t_n,s_n)<0$.

Example 1. Let $\zeta:[0,\infty)\times[0,\infty)\to\mathbb{R}$ be such that $\zeta(t,s)=\psi(s)-\phi(t)$ for all $s,t\in[0,\infty)$, where $\psi,\phi\colon[0,\infty)\to[0,\infty)$ are two continuous functions such that $\psi(t)=\phi(t)=0$ if and only if $t=0$ and $\psi(t)<t\le\phi(t)$ for all $t>0$. Then ζ is a simulation function.

More examples of simulation function can be found in [9].

Definition 2. (Manageable function, [3]) A function $\eta:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is said to be manageable if the following conditions hold:

(η1) $\eta(t,s)<s-t$ for all $s,t>0$;

(η2) for any bounded sequence $\{t_n\}\subset(0,\infty)$ and any non-increasing sequence $\{s_n\}\subset(0,\infty)$, we have that $\limsup_{n\to\infty}(t_n+\eta(t_n,s_n))/s_n<1$.

Several examples of manageable functions can be found in [3].

Definition 3. (R-function, [2]) For a nonempty set $A\subseteq\mathbb{R}$, a function $\rho: A\times A\to\mathbb{R}$ is said to be an R-function if it satisfies the following two conditions.

(ρ1) If $\{a_n\}\subset(0,\infty)\cap A$ is such that $\rho(a_{n+1},a_n)>0$, for all $n\in\mathbb{N}$, then $\{a_n\}\to 0$.

(ρ2) If $\{a_n\},\{b_n\}\subset(0,\infty)\cap A$ are two sequences converging to the same limit $L\ge 0$ satisfying that $L<a_n$, and $\rho(a_n,b_n)>0$, for all $n\in\mathbb{N}$, then $L=0$.

In some cases, the following additional property is also considered:

$$(S)\qquad \begin{cases} \text{If } \{a_n\},\{b_n\}\subset(0,\infty)\cap A \text{ are two sequences such that} \\ \{b_n\}\to 0 \text{ and } \sigma(a_n,b_n)>0, \text{ for all } n\in\mathbb{N}, \text{ then } \{a_n\}\to 0. \end{cases}$$

Various examples and properties of R-function can be found in [2].

Remark 1. (see [2]) Every simulation function and manageable function is an R-function that also satisfies the property (S).

Remark 2. (see [5]) A Geraghty function is a function $\varphi\colon[0,\infty)\to[0,1)$ such that if $\{t_n\}\subset[0,\infty)$ and $\{\varphi(t_n)\}\to 1$, then $\{t_n\}\to 0$.

Definition 4. (L-function, [11]) A function $\varphi\colon[0,\infty)\to[0,\infty)$ will be called an L-function if:

(a) $\varphi(0)=0$;

(b) $\varphi(t)>0$ for all $t>0$; and

(c) for all $\varepsilon>0$, there exists $\delta>0$ such that $\varphi(t)\le\varepsilon$ for all $t\in[\varepsilon,\varepsilon+\delta]$.

3 Main Results

Before going into our main theorems, we shall introduce some definitions, as follows:

Definition 5. (σ_c-function) Let $A \subseteq \mathbb{R}$ be a nonempty set and $c \geq 1$ be a fixed constant. A function $\sigma_c\colon A \times A \to \mathbb{R}$ is said to be a σ_c-function if it satisfies the following two conditions:

(σ1) If $\{a_n\} \subset (0,\infty) \cap A$ is a sequence such that $\sigma_c(a_n, a_{n-1} + a_n) > 0$, for all $n \in \mathbb{N}$, then $a_n \to 0$ as $n \to \infty$;

(σ2) If $\{a_n\}, \{b_n\} \subset (0,\infty) \cap A$ are two convergent sequences such that, $cL = \lim_{n\to\infty} b_n \geq \lim_{n\to\infty} a_n = L \geq 0$ and $\sigma_c(a_n, b_n) > 0$ for all $n \in \mathbb{N}$, then $L = 0$.

We denote by Σ_c^A the set of all σ_c functions on A, and we write simply Σ_c for $\Sigma_c^{[0,\infty)}$.

Remark 3. Unlike Remark 1, every simulation function (or manageable function) is not a σ_c-function. In fact, the notion of σ_c-function is completely independent, from the three notions mentioned in Definitions 1, 2 and 3 which can be seen by the following examples.

Example 2. Let $\gamma\colon [0,\infty) \times [0,\infty) \to \mathbb{R}$ be the function defined for all $t, s \in [0,\infty)$ by

$$\gamma(t,s) = \begin{cases} s/2 - 3t/2, & \text{when } t < s; \\ 0, & \text{when } t \geq s. \end{cases}$$

Then it can be seen that $\gamma \in \Sigma_c$ for any fixed $c \in [1,3)$, which satisfies the condition (S).

But, γ is not a simulation function (or manageable function). For instance, take $a_n = b_n = 1$ for all $n \in \mathbb{N}$, then $\limsup_{n\to\infty} \gamma(a_n, b_n) = 0$, hence the condition ($\zeta$3) (or ($\eta$1)) is violated.

Example 3. Let $\beta\colon [0,\infty) \times [0,\infty) \to \mathbb{R}$ be the function defined for all $t, s \in [0,\infty)$, by $\beta(t,s) = s/2 - t$. Then clearly β is a simulation function. In fact, it is both manageable and R-function. But β does not satisfy the property (σ1). Because, if we take $a_n = 1 + 1/n$, and $b_n = a_n + a_{n-1}$, for all $n \in \mathbb{N}$, then $\beta(a_n, b_n) = (a_{n-1} - a_n)/2 > 0$ but $\{a_n\}$ does not converge to 0. Therefore, $\beta \notin \Sigma_c$ for any $c \geq 1$.

Example 4. Let $g\colon [0,\infty) \times [0,\infty) \to \mathbb{R}$ be the function defined for all $t, s \in [0,\infty)$ by

$$g(t,s) = \begin{cases} -1, & \text{if } t \leq s; \\ 1, & \text{if } t > s. \end{cases}$$

Then, for all $n \in \mathbb{N}$ and for every $\{a_n\} \subset (0,\infty)$, we have, $g(a_n, a_{n-1} + a_n) < 0$. Hence, condition ($\sigma$1) is trivially true. Also, if $\{a_n\}, \{b_n\} \subset (0,\infty)$ are two convergent sequences such that, $cL = \lim_{n\to\infty} b_n \geq \lim_{n\to\infty} a_n = L \geq 0$, (for $c > 1$)

satisfying that, $g(a_n, b_n) > 0$, for all $n \in \mathbb{N}$, then it implies that, $a_n > b_n$ for all $n \in \mathbb{N} \implies L \geq cL$ (for $c > 1$) $\implies L = 0$. So g satisfies ($\sigma 2$) and hence, $g \in \Sigma_c$ for every fixed $c > 1$. But g is not an R-function which can be seen by considering $a_n = n$, for all $n \in \mathbb{N}$.

Remark 4. The domain of the functions γ and g can be chosen any subset A of $\mathbb{R}$, rather than $[0, \infty)$, to get examples of σ_c-functions with different domain. Also, the above function β satisfies ($\zeta 2$) but still not a σ_c-function, which would be very important when we state our conditions for the extension of Kannan mapping to get fixed point.

We will now consider some more examples of σ_c-functions using Geraghty functions and L-functions.

Example 5. Consider a function, $\pi\colon [0, \infty) \to \mathbb{R}$, such that, $\pi(t) \leq t$, for all $t \in [0, \infty)$. Define a new function $\Theta_\pi\colon [0, \infty) \times [0, \infty) \to \mathbb{R}$ by: $\Theta_\pi(t, s) = \alpha\pi(s)t$, for all $t, s \in [0, \infty)$, with, $0 < \alpha < 1/2$. Then, Θ_π is a σ_c-function for every fixed $c \in [1, 2]$, which also satisfies the property (S), can be readily seen.

Example 6. In [2], it is highlighted about an important property of L-function l, that, $l(t) \leq t$, for all $t \in [0, \infty)$. So by previous example, for every L-function l, a function defined by: $\Theta_l(t, s) = \alpha l(s)t$, for all $t, s \in [0, \infty)$; $(0 < \alpha < 1/2)$ is a σ_c-function for every fixed $c \in [1, 2]$, which also satisfies the property (S).

Proposition 1. *Let $g\colon [0, \infty) \to [0, 1)$ be a Geraghty function. Define the function $\Theta_g\colon [0, \infty) \times [0, \infty) \to \mathbb{R}$ by: $\Theta_g(t, s) = \alpha g(s)s - t$, for all $t, s \in [0, \infty)$, with, $0 < \alpha < 1/2$, is a σ_c-function for every fixed $c \in [1, 2]$ which also satisfies the property (S).*

Proof. For $0 < \alpha < 1/2$, the proof is clear from Example 5 and the fact that, $g(s) < 1$, i.e., $g(s)s < s$. Now for $\alpha = 1/2$, we have, $\Theta_g(t, s) = g(s)s/2 - t$, for all $t, s \in [0, \infty)$.

($\sigma 1$) If $\{a_n\} \subset (0, \infty)$ is a sequence such that $\Theta_g(a_n, a_{n-1} + a_n) > 0$, then, for all $n \in \mathbb{N}$, we have,

$$\frac{1}{2}g(a_{n-1}+a_n)(a_{n-1} + a_n)-a_n > 0 \implies 0 < a_n < \frac{1}{2}g(a_{n-1}+a_n)(a_{n-1}+a_n) < \frac{1}{2}(a_{n-1} + a_n)] \implies a_n < \frac{1}{2}(a_{n-1} + a_n) \implies a_n < a_{n-1}.$$

So, $\{a_n\}$ is strictly monotone decreasing sequence of positive reals, hence convergent to L (say). Hence, $L \leq \lim g(a_{n-1} + a_n) \cdot 2L/2 = L \implies \lim g(a_{n-1} + a_n) = 1$, and so by the property of Geraghty function we have that, $(a_{n-1} + a_n) \to 0$ as $n \to \infty$; which implies $a_n \to 0$ as $n \to \infty$.

($\sigma 2$) If $\{a_n\}, \{b_n\} \subset (0, \infty)$ are two convergent sequences such that, $cL = \lim b_n \geq \lim a_n = L \geq 0$, satisfying that, $\Theta_g(a_n, a_{n-1} + a_n) > 0$, then, we have $g(b_n)b_n/2 - a_n > 0 \implies a_n \leq g(b_n)b_n/2 < b_n/2 \implies L \leq \lim g(b_n)cL/2 < cL/2 \implies \lim g(b_n) = 1$, and so, $b_n \to 0$ as $n \to \infty$. This shows that $a_n \to 0$ as $n \to \infty$.

By similar arguments one can check the property (S) and this completes the proof of the proposition. □

Definition 6. Let $T\colon X \to X$ be an operator. A sequence $\{x_n\}_{n\geq 0}$ in X is said to satisfy the asymptotic regularity property with respect to T, if $\lim_{n\to\infty} d(Tx_{n+1}, Tx_n) = 0$. Now, if $\{x_n\}_{n\geq 0}$ is formed by the Picard's interation, i.e., $x_n = T^n x_0$, then T is simply turns out to be the asymptotically regular at the base point x_0.

Definition 7. (see, [12]) Given two self-mappings $T, S\colon X \to X$ and a sequence $\{x_n\}_{n\geq 0} \subseteq X$, we say that $\{x_n\}_{n\geq 0}$ is a Picard sequence of the pair (T, S) (based on x_0) if $Sx_{n+1} = Tx_n$ for all $n \geq 0$. We say that X verifies the CLR(T, S)-property, if there exists in X a Picard sequence of (T, S) based on some point x_0.

Definition 8. Let $T\colon X \to X$ be an operator. A sequence $\{x_n\}$ is said to be S-asymptotically similar with respect to T, if $\lim_{n\to\infty} d(TSx_n, SSx_n) = 0$.

Definition 9. Let (X, d) be a metric space, and $S\colon X \to X$ be a function. A mapping $T\colon X \to X$ is called a $\varSigma_c$-S-Kannan with respect to some $\sigma_c \in \varSigma_c$, if, it satisfies the condition that

$$\sigma_c(d(Tx, Ty), d(Tx, Sx) + d(Ty, Sy)) > 0 \text{ for all } x, y \in X. \tag{3.1}$$

The mapping T is called, $\varSigma_c$-Kannan with respect to some $\sigma_c \in \varSigma_c$, if, it satisfies the condition that

$$\sigma_c(d(Tx, Ty), d(Tx, x) + d(Ty, y)) > 0 \text{ for all } x, y \in X.$$

Lemma 1. *Let (X, d) be a metric space verifies the CLR(T, S)-property and let $T\colon X \to X$ be a $\varSigma_c$-S-Kannan, with respect to some $\sigma_c \in \varSigma_c$, then the Picard sequence of (T, S) (based on x_0) satisfies, either the coincidence property with respect to the pair (T, S); or, the asymptotically regularity property with respect to the operator T.*

Proof. Given that, (X, d) verifies the CLR(T, S)-property. So there exists in X a Picard sequence $\{x_n\}_{n\geq 0}$ of (T, S) based on some point x_0 of X which satisfying the condition that, $Sx_{n+1} = Tx_n$ for all $n \geq 0$.

Now we have the following two cases:

- **Case I:** We assume that, $Tx_p = Tx_{p-1}$, for some $p \in \mathbb{N}$. Then $Tx_p = Tx_{p-1} = Sx_p$ and X satisfies the coincidence property with respect to the pair (T, S).
- **Case II:** Now we assume, $Tx_n \neq Tx_{n-1}$ for all $n \in \mathbb{N}$.

Now as $T\colon X \to X$ be a $\varSigma_c$-S-Kannan, with respect to some $\sigma_c \in \varSigma_c$, we have that,

$$\sigma_c(d(Tx, Ty), d(Tx, Sx) + d(Ty, Sy)) > 0 \text{ for all } x, y \in X.$$

So, $\sigma_c(d(Tx_{n+1}, Tx_n), d(Tx_{n+1}, Sx_{n+1}) + d(Tx_n, Sx_n)) > 0$ for all $n \in \mathbb{N}$. Now as, $Sx_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. We have,

$$\sigma_c(d(Tx_{n+1}, Tx_n), d(Tx_{n+1}, Tx_n) + d(Tx_n, Tx_{n-1})) > 0.$$

Choose $a_n = d(Tx_{n+1}, Tx_n)$, then, $a_{n-1} = d(Tx_n, Tx_{n-1})$; and so, $a_n > 0$ with $\sigma_c(a_n, a_{n-1} + a_n) > 0$, for all $n \in \mathbb{N}$. Then by $(\sigma 1)$ we get, $\{a_n\} \to 0$. This clearly shows that, the Picard sequence of (T, S) (based on x_0) satisfies the asymptotically regularity property with respect to the operator T. $\square$

Lemma 2. *Let (X, d) be a metric space verifies the CLR(T, S)-property; and $T\colon X \to X$ be a Σ_c-S-Kannan with respect to some $\sigma_c \in \Sigma_c$, satisfying, either the condition that, $\sigma_c(t, s) < s - t$ for all $s, t > 0$, or satisfying the property (S). Then the Picard sequence $\{x_n\}$ of the pair (T, S) (based on x_0), is an S-bounded sequence.*

Proof. On contrary, assume that $\{x_n\}$ is not S-bounded. Without loss of generality we assume that $Sx_{n+p} \neq Sx_n$ for all $n, p \in \mathbb{N}$; and so, clearly, $Tx_{n+p} \neq Tx_n$ for all $n, p \in \mathbb{N}$. As $\{x_n\}$ is not S-bounded, for each k, there exists two subsequences $\{Sx_{n_k}\}$ and $\{Sx_{m_k}\}$ of $\{Sx_n\}$ with $k \leq n_k < m_k$, for each $k \in \mathbb{N}, m_k, n_k$ are the minimum integers, such that,

$$d(Sx_{n_k}, Sx_{m_k}) > 1 \quad \text{and} \quad d(Sx_{n_k}, Sx_p) \leq 1 \quad \text{for} \quad n_k \leq p \leq m_k - 1. \tag{3.2}$$

Case I: Now, if T is a Σ_c-S-Kannan with respect to some $\sigma_c \in \Sigma_c$, satisfying the condition that, $\sigma_c(t, s) < s - t$ for all $s, t > 0$. So we have that,

$$d(Tx, Ty) < d(Tx, Sx) + d(Ty, Sy), \text{ for all } x, y \in Y,$$

where Y is the set of points x, y, of which both the sides of the above inequality provides non-zero entries.

Now, we assume that $Tx_{n_k-1} = Sx_{n_k} \neq Sx_{n_k-1}$, $Tx_{m_k-1} = Sx_{m_k} \neq Sx_{m_k-1}$. Also, $Tx_{n+p} \neq Tx_n$, i.e., $x_{n_k-1}, x_{m_k-1} \in Y$, and we have,

$$d(Tx_{n_k-1}, Tx_{m_k-1}) < d(Tx_{n_k-1}, Sx_{n_k-1}) + d(Tx_{m_k-1}, Sx_{m_k-1}).$$

Now clearly,

$$\begin{aligned} 1 < d(Sx_{n_k}, Sx_{m_k}) = d(Tx_{n_k-1}, Tx_{m_k-1}) &< d(Tx_{n_k-1}, Sx_{n_k-1}) \\ &+ d(Tx_{m_k-1}, Sx_{m_k-1}) \end{aligned}$$

as T is Σ_c-S-Kannan. This implies

$$\begin{aligned} 1 < d(Tx_{n_k-1}, Sx_{n_k-1}) + d(Tx_{m_k-1}, Sx_{m_k-1}) &= d(Tx_{n_k-1}, Tx_{n_k-2}) \\ &+ d(Tx_{m_k-1}, Tx_{m_k-2}). \end{aligned}$$

Now we see that both the entries on the right hand side are the subsequence of $d(Tx_n, Sx_n) = d(Tx_n, Tx_{n-1})$, such that, $Tx_n \neq Tx_{n-1}$ for all $n \in \mathbb{N}$. Then by Case II of Lemma 1, then Picard sequence of (T, S) (based on x_0) satisfies the asymptotically regularity property with respect to the operator T. So, taking limit on both sides as $k \to \infty$, we get, $1 \leq 0$, which is a contradiction. Hence, $\{x_n\}$ is S-bounded.

Case II: Now suppose, σ_c satisfies the property (S). Then, we choose, $a_k = d(Sx_{n_k}, Sx_{m_k})$ and

$b_k = d(Tx_{n_k-1}, Sx_{n_k-1}) + d(Tx_{m_k-1}, Sx_{m_k-1})$. Then, by the given condition, we have, $a_k, b_k > 0$ satisfying $\sigma_c(a_k, b_k) > 0$ with $b_k \to 0$. So by the property (S) we have $a_k \to 0$, which is again a contradiction to equation (3.2). This proves the lemma. □

We now use a similar type of idea as given in [9] to prove the next lemma.

Lemma 3. *Let (X, d) be a metric space verifies the CLR(T, S)-property; and $T\colon X \to X$ be a Σ_c-S-Kannan with respect to some $\sigma_c \in \Sigma_c$, satisfying, either the condition that, $\sigma_c(t, s) < s - t$ for all $s, t > 0$, or, satisfying the property (S) . Then the Picard sequence $\{x_n\}$ of the pair (T, S) (based on x_0), is an S-Cauchy sequence.*

Proof. Consider $C_n = \sup\{d(Sx_i, Sx_j)\colon i, j \geq n\}$. Note that the sequence $\{C_n\}$ is a monotonically decreasing sequence of positive reals and by Lemma 2, the sequence $\{x_n\}$ is S-bounded, therefore $C_n < \infty$ for all $n \in \mathbb{N}$. Thus $\{C_n\}$ is monotone, bounded sequence, hence convergent. So there exists $C \geq 0$ such that $\lim_{n\to\infty} C_n = C$.

Now, if $C > 0$, then by the definition of C_n, for every $k \in \mathbb{N}$ there exists n_k, m_k such that $m_k > n_k \geq k$ and

$$C - 1/k < d(Sx_{m_k}, Sx_{n_k}) \leq C.$$

Hence, $\lim_{k\to\infty} d(Sx_{m_k}, Sx_{n_k}) = C$.

Case I: Now, suppose T is a Σ-S-Kannan with respect to some $\sigma_c \in \Sigma_c$, satisfying the condition that, $\sigma_c(t, s) < s - t$ for all $s, t > 0$, so we have,

$$\begin{aligned} d(Sx_{n_k}, Sx_{m_k}) &= d(Tx_{n_k-1}, Tx_{m_k-1}) \\ &< d(Tx_{n_k-1}, Sx_{n_k-1}) + d(Tx_{m_k-1}, Sx_{m_k-1}) \\ \implies d(Sx_{n_k}, Sx_{m_k}) &< d(Tx_{n_k-1}, Tx_{n_k-2}) + d(Tx_{m_k-1}, Tx_{m_k-2}). \end{aligned}$$

So, by previous argument, taking limit on both sides as $k \to \infty$, we get,

$$\lim_{k\to\infty} d(Sx_{m_k}, Sx_{n_k}) = C \leq 0.$$

This is a contradiction to the assumption that $C > 0$. Hence $C = 0$.

Case II: Suppose, σ_c satisfies the property (S). Then, we choose:

$$a_k = d(Sx_{n_k}, Sx_{m_k}) \text{ and } b_k = d(Tx_{n_k-1}, Sx_{n_k-1}) + d(Tx_{m_k-1}, Sx_{m_k-1}).$$

Then, by the given condition, we have, $a_k, b_k > 0$ satisfying $\sigma_c(a_k, b_k) > 0$ with $b_k \to 0$. So by the property (S) we have $a_k \to 0$. Hence, $C = 0$, and this completes the proof of lemma. □

We now state one of our main theorem for σ_c-function, which is a generalization of the Theorem 2 (Kannan, [8]).

Theorem 3. *Let (X, d) be a metric space verifies the CLR(T, S)-property and let $\{x_n\}_{n\geq 0}$ be a Picard sequence of the pair (T, S) (based on x_0). Let T be a map of X into itself such that:*

(i) $T\colon X \to X$ be a Σ_c-S-Kannan with respect to some $\sigma_c \in \Sigma_c$ (for $c = 1$), satisfying either of the condition that

(a) $\sigma_c(t, s) < s - t$ for all $s, t > 0$; or (b) σ_c satisfying the property (S).

(ii) T and S both are continuous at a point $Sq \in X$; and

(iii) the Picard sequence Tx_n has a subsequence $\{Tx_{n_k}\}$ converging to Sq.

Then, either the pair (T, S) has a coincidence point, or the following two conditions hold:

(B_1) $\{x_n\}_{n\geq 0}$ is S-asymptotically similar, implies, Sq is a fixed point of T;

(B_2) Sq is a fixed point of S, implies, Sq is a fixed point of T.

Proof. If at least two consecutive terms of the Picard sequence of pair (T, S) are equal, then by the Case I of the Lemma 1, the pair (T, S) has a coincidence point.

We now assume that no terms of the Picard sequence of the pair (T, S) are equal, and we show that both the conditions (B_1) and (B_2) hold.

$(\mathbf{B_1})$ Given that, T is continuous at $Sq \in X$, $\{TTx_{n_k}\}$ converging to TSq. We assume, $TSq \neq Sq$, and will arrive at a contradiction. As, $TSq \neq Sq$, we consider two disjoint open balls, say $B(TSq, r_1)$ and $B(Sq, r_2)$, with centres at TSq, Sq, and radii r_1, r_2 respectively.

We choose, $r = \min\{r_1, r_2, d(TSq, Sq)/4\} > 0$. Now, as the subsequence $\{Tx_{n_k}\}$ converging to Sq, and $\{TTx_{n_k}\}$ converges to TSq; there exists a positive integer M, such that, for all $k > M$, we have that,

$$Tx_{n_k} \in B(Sq, r) \quad \text{and} \quad TTx_{n_k} \in B(TSq, r)$$

and so, clearly, for each $k > M$, we have that

$$\begin{aligned} 0 < 4r < d(TSq, Sq) &\leq d(TSq, TTx_{n_k}) + d(Tx_{n_k}, TTx_{n_k}) + d(Tx_{n_k}, Sq) \\ \implies 4r < 2r + d(Tx_{n_k}, TTx_{n_k}). \end{aligned}$$

That is, we have,

$$d(Tx_{n_k}, TTx_{n_k}) > 2r > 0. \tag{3.3}$$

Case I: If the condition (i)-(a) is satisfied, then we have that

$$\begin{aligned} 0 < d(Tx_{n_k}, TTx_{n_k}) &< d(Tx_{n_k}, Sx_{n_k}) + d(TTx_{n_k}, STx_{n_k}) \\ &= d(Tx_{n_k}, Tx_{n_k-1}) + d(TSx_{n_k+1}, SSx_{n_k+1}). \end{aligned}$$

Now, as assumed in the condition (B_1), $\{x_n\}_{n\geq 0}$ is S-asymptotically similar; and using Lemma 1, we see that, the right hand side tends to 0. So, we get, $\lim_{k\to\infty} d(Tx_{n_k}, TTx_{n_k}) = 0$, which is a contradiction to (3.3). Hence, $TSq = Sq$, i.e., Sq is a fixed point of T.

Case II: If the condition (i)-(b) is satisfied, then we have,

$$\sigma_c(d(Tx_{n_k}, TTx_{n_k}), d(Tx_{n_k}, Sx_{n_k}) + d(TTx_{n_k}, STx_{n_k})) > 0.$$

We assume,

$$a_k = d(Tx_{n_k}, TTx_{n_k}) \text{ and } b_k = d(Tx_{n_k}, Sx_{n_k}) + d(TTx_{n_k}, STx_{n_k}).$$

Then, since $\lim_{k\to\infty} b_k = 0$, we have $\lim_{k\to\infty} a_k = 0$. This is a contradiction to (3.3). Hence, $TSq = Sq$, i.e., Sq is a fixed point of T.

($\mathbf{B_2}$) If the condition (i)-(a) is satisfied by σ_c, i.e, $\sigma_c(t,s) < s - t$, for all $s, t > 0$. Then, we have that,

$$\sigma_c(d(Tx_{n_k}, TTx_{n_k}), d(Tx_{n_k}, Sx_{n_k}) + d(TTx_{n_k}, STx_{n_k})) > 0.$$

Choose, $a_k = d(Tx_{n_k}, TTx_{n_k})$ and $b_k = d(Tx_{n_k}, Sx_{n_k}) + d(TTx_{n_k}, STx_{n_k})$. Now, we see that $a_k \to d(Sq, TSq) = L$, $b_k \to d(TSq, SSq) = d(TSq, Sq) = L$, (by the assumption made in (c)). Since $T\colon X \to X$ be a σ_c-Kannan with respect to some $\sigma_c \in \Sigma_c$ (for $c = 1$), and $L = \lim a_k \leq \lim b_k = L$; it then implies that $L = 0$ and hence, $d(Sq, TSq) = 0$. So, $TSq = Sq$ and Sq is a fixed point of T.

Assuming the condition (i)-(b) to be satisfied by σ_c, i.e, if σ_c satisfies the condition (S), then similarly one can obtain the results, by using Lemma 1 and Lemma 3. This completes the proof. □

Next, we state one of our main theorem for σ_c-function, which is the generalization of the Theorem 1 (Kannan, [7]).

Theorem 4. *Let (X, d) be a metric space verifies the CLR(T,S)-property and let T be a map of X into itself such that:*

(i) $(S(X), d)$ is complete, (or $(T(X), d)$ is complete);

(ii) T is a Σ_c-S-Kannan with respect to some $\sigma_c \in \Sigma_c$ (for $c = 1$) satisfying either of the condition that;

(a) $\sigma_c(t,s) < s - t$ for all $s, t > 0$; or (b) satisfying the property (S).

Then, X satisfies the coincidence property with respect to the pair (T, S).

Proof. Suppose, T is a Σ_c-S-Kannan with respect to some $\sigma_c \in \Sigma_c$ (for $c = 1$), satisfying the condition

$$\sigma_c(t,s) < s - t \quad \text{for all} \quad s, t > 0.$$

Hence, by Lemma 3, the Picard sequence $\{x_n\}$ of the pair (T, S) (based on x_0), is an S-Cauchy sequence.

Now, suppose $(S(X), d)$ is complete (similar process will work if $(T(X), d)$ is complete). Then, $\{Sx_n\}$ must be convergent and converges to a point z (say) in $S(X)$. Since, $Sx_{n+1} = Tx_n$ for all $n \geq 0$, the sequence $\{Tx_n\}$ is also convergent and converges to the same point z. As, $z \in S(X)$, there is at least one point w (say) in X such that $Sw = z$. Also, if T satisfy the condition (ii)-(a), i.e, $\sigma_c(t,s) < t - s$ for all $s, t > 0$, then

$$\sigma_c(d(Tx_n, Tw), d(Tx_n, Sx_n) + d(Tw, Sw)) > 0.$$

Now choose $a_n = d(Tx_n, Tw)$ and $b_n = d(Tx_n, Sx_n) + d(Tw, Sw)$ for all $n \in \mathbb{N}$. Then, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} d(Tx_n, Tw) = \lim_{n\to\infty} d(Sx_{n+1}, Tw) = d(z, Tw) = d(Sw, Tw)$ and

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} [d(Tx_n, Sx_n) + d(Tw, Sw)]$$
$$= \lim_{n\to\infty} [d(Sx_{n+1}, Sx_n) + d(Tw, Sw)] = d(Tw, Sw).$$

Hence,

$$d(Tw, Sw) = \lim_{n\to\infty} b_n \geq \lim_{n\to\infty} a_n = d(Tw, Sw) = L \geq 0$$

and so, $L = 0$ (by ($\sigma 2$)). Hence X satisfies the coincidence property with respect to the pair (T, S).

Assuming that the σ_c-function satisfies the condition (ii)-(b), i.e, the property (S). Then, using Lemma 1 and Lemma 3 and a similar process, one can obtain the result. This completes the proof. □

Corollary 1. Let (X, d) be a complete metric space, and let T be a map of X into itself such that T is a Σ_c-Kannan with respect to some $\sigma_c \in \Sigma_c$ for $c = 1$. Suppose that σ_c satisfies either, the condition (S) or, that, $\sigma_c(t, s) < s - t$ for all $s, t > 0$. Then T has unique fixed point in X.

Proof. Putting, $S = Id_X$, i.e, $S(x) = x$ for all $x \in X$ in Theorem 4 we obtain the existence of the fixed point of T. Only to prove the uniqueness. Note that, for $S = Id_X$ the Picard sequence of the pair (T, S) based on some point $x_0 \in X$, now reduces into the Picard sequence $\{x_n\}$ of T based on x_0 for arbitrary $x_0 \in X$, where $x_n = T^n x_0$; and $\{x_n\}$ converges to u such that u is a fixed point of T. If possible, assume $Tv = v$ with $u \neq v$, for some $v \in X$. Then, we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \neq x_{n_k+1}$ and, $Tx_{n_k} \neq Tv$ for all $k \in \mathbb{N}$ (because, if not so, then taking limits $u = Tv = v$).

Now, if $\sigma_c(t, s) < s - t$, for all $s, t > 0$ we have that, $d(Tx_{n_k}, Tv) < d(Tx_{n_k}, x_{n_k}) + d(Tv, v)$, and taking limits we get, $d(u, Tv) = 0$, i.e., $u = Tv = v$. Also if, the condition (S) is satisfied, then we choose,

$$a_k = d(Tx_{n_k}, Tv) \text{ and } b_k = d(Tx_{n_k}, x_{n_k}) + d(Tv, v)$$

and $b_k \to 0$ as $k \to \infty$, we get $a_k \to 0$ as $k \to \infty$. So, $d(u, Tv) = 0$, i.e., $u = Tv = v$. This completes the proof of the corollary. □

Corollary 2. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a function such that $T(\mathbb{R}^n)$ is closed and for each $x \in \mathbb{R}^n$ there is $y \in \mathbb{R}^n$ $Tx = -Ty$. Suppose that the condition: $\|Tx - Ty\| \leq 2\alpha(\|Tx\| + \|Ty\|)$, for a fixed $0 < \alpha < 1/2$; is satisfied. Then, there is root of the function in $\mathbb{R}^n$, i.e., there is a point a in $\mathbb{R}^n$ for which $Ta = 0$.

Proof. Put $Sx = -Tx$ in the Theorem 4 and consider $\sigma_c(t, s) = \alpha s - t$ for all $t, s \in [0, \infty)$, $0 < \alpha < 1/2$. (For the proof of being σ_c-function, see Remark 5, below). □

Remark 5. Theorems 3 and 4 are the generalizations of the Theorem 2 (Kannan; [8]) and Theorem 1 (Kannan; [7]) respectively. Because for a fixed $0 < \alpha < 1/2$ if we choose a function $\chi_\alpha \colon [0,\infty) \times [0,\infty) \to \mathbb{R}$ defined by: $\chi_\alpha(t,s) = \alpha s - t$ for all $t,s \in [0,\infty)$, then χ_α is σ_c-function (for every fixed $c \in [1,2)$), for every $0 < \alpha < 1/2$.

Proof. ($\sigma 1$) If $\{a_n\} \subset (0,\infty)$ is a sequence such that $\chi_\alpha(a_n, a_{n-1}+a_n) > 0$, for all $n \in \mathbb{N}$, then we have

$$0 < \chi_\alpha(a_n, a_{n-1}+a_n) = \alpha(a_{n-1}+a_n) - a_n = \alpha a_{n-1} - (1-\alpha)a_n \implies a_n < \frac{\alpha a_{n-1}}{1-\alpha}.$$

So, as $0 < \alpha < 1/2$, the quantity $\frac{\alpha}{1-\alpha} < 1$, and hence $\{a_n\} \to 0$.

($\sigma 2$) If $\{a_n\}, \{b_n\} \subset (0,\infty)$ are two convergent sequences such that, $cL = \lim b_n \geq \lim a_n = L \geq 0$, satisfying that, $\chi_\alpha(a_n, b_n) > 0$, for all $n \in \mathbb{N}$, then

$$\begin{aligned} 0 < \chi_\alpha(a_n, b_n) = \alpha b_n - a_n \implies 0 < a_n < \alpha b_n \implies 0 \leq L \\ \leq c\alpha L < cL/2 < L \implies L = 0 \text{ as } c < 2. \end{aligned}$$

This proves that χ_α is σ_c-function (for every fixed $c < 2$), for every $0 < \alpha < 1/2$.

Now, if an operator satisfies the condition (3.1) of Definition 9; then for $\sigma_c = \chi_\alpha$ and $S = Id_X$ we have that,

$$\begin{aligned} & \chi_\alpha(d(Tx,Ty), d(Tx,x) + d(Ty,y)) > 0, \text{ for all } x,y \in X \\ \implies \quad & d(Tx,Ty) \leq \alpha[d(Tx,x) + d(Ty,y)], \text{ for all } x,y \in X, (0 < \alpha < 1/2). \end{aligned}$$

Which is the Kannan's contraction condition. All the other conditions of Theorem 3 and Theorem 4 can be reduced into desired form, easily. □

Remark 6. The identity function is the simplest operator having fixed point, without satisfying the Kannan's contraction (or not even Banach's contraction condition). But it does satisfy our Σ_c-S-Kannan contraction condition. Also our Σ_c-S-Kannan contraction condition is a proper extension of the Kannan's contraction condition even when we take S to be the identity operator. Also Theorem 3 and Theorem 4 are the two proper generalizations of Kannan's theorem. These all facts follows from the next few examples.

Example 7. First let us consider Example 4, with a slight modification, that: consider $\omega \colon [0,\infty) \times [0,\infty) \to \mathbb{R}$ be the function defined, for all $t,s \in [0,\infty)$ by

$$\omega(t,s) = \begin{cases} -1, & \text{if } t < s; \\ 1, & \text{if } s \leq t. \end{cases}$$

Then, in view of Example 4 we have $\omega \in \Sigma_c$ with every fixed $c > 1$. Now, if we take the operator T to be the identity function then it is a Σ_c-Kannan mapping. Since for any $x,y \in X$, we have that,

$$\omega(d(Tx,Ty), d(Tx,x) + d(Ty,y)) = \omega(d(x,y), 0) = 1 > 0.$$

But T not a Kannan's mapping; because if so, then for $x \neq y$, we would have that,

$$d(x,y) = d(Tx,Ty) \leq \alpha[d(Tx,x) + d(Ty,y)] = 0,$$

which is a contradiction. Also if we take X an arbitrary metric space, and $T\colon X \to X$ be any function with at least two different images, and we choose S to be any constant function, then X satisfy the CLR(T, S) property and T satisfying the Σ_c-S-Kannan contraction condition with respect to the above function ω. Also if would have chosen the last considered T, pre-assuming that there exists a point a such that $Ta = a$ then T satisfies all the conditions above without the condition (ii) of Theorem 4; and this shows that the condition (ii) of Theorem 4, is sufficient but not necessary to have fixed point.

Example 8. To prove our Σ_c-S-Kannan contraction condition is a proper extension of the Kannan's contraction condition, we consider $X = \{1, 2, 3, 4, 5\}$ and define $T\colon X \to X$ as $Tx = 3$ if $x \neq 4$ and $T4 = 2$; and define $S\colon X \to X$ as, $Sx = 3$, $x \neq 4$ and $S4 = 5$. Then clearly T does not satisfy the Kannan's contractive condition which can be seen by considering two points 3 and 4. But this T is satisfying the Σ_c-S-Kannan contraction condition with respect to the $\sigma_c \in \Sigma_c$, defined by $\sigma_c(t, s) = 3s/7 - t$ (for any fixed $c \in [1, 2)$) (by Remark 5). In fact X satisfies the CLR(T, S) property and $S(X)$ is complete as well. So it satisfy all the properties of Theorem 4, and hence X satisfies the coincidence property with respect to the pair (T, S) and $x = 1, 2, 3$ are the coincidence points of it.

Remark 7. In [2], it has been shown that every simulation function and every manageable function is an R-function, as well as, both types of functions satisfy the property (S). Notice that, the property $(\sigma 1)$ of σ_c-functions is quite different from the property $(\rho 1)$ of manageable functions. This property is a major difference between the manageable functions and the σ_c-functions, and it plays an important role in the existence of fixed point of the class of Σ_c-S-Kannan and Σ_c-Kannan operators. The following example shows that one cannot replace the property $(\sigma 1)$ by $(\rho 1)$ even when the property (S) is satisfied.

Example 9. Let $X = \{1, 2, 3\}$ and consider the metric $d(x, y) = |x - y|$ for all $x, y \in X$. Clearly (X, d) is complete. Now, consider a function $\tau\colon [0, \infty) \times [0, \infty) \to \mathbb{R}$ be the function defined by $\tau(t, s) = 2s/3 - t$, for all $t, s \in [0, \infty)$. Clearly τ is a manageable function which satisfies the property (S).

We now define an operator $T\colon X \to X$ by $T1 = T2 = 3$ and $T3 = 2$. To show

$$\tau(d(Tx, Ty), d(Tx, x) + d(Ty, y)) > 0$$

we show that $d(Tx, Ty) \leq 2[d(Tx, x) + d(Ty, y)]/3$. Then, τ satisfies the property $(\sigma 2)$ with every fixed $c \in [1, 3/2)$. Thus, all the conditions of Theorem 4, except the condition $(\sigma 1)$, are satisfied, and T has no fixed point.

We now move to find another different types of generalizations of Kannan's theorems, starting with the following definition.

Definition 10. Let (X, d) be a metric space and $T\colon X \to X$ and $S\colon X \to X$ be two operators. Then T is said to be S-dominated Σ_c-Kannan mapping of degree w, if for some $\sigma_c \in \Sigma_c$, the following holds:

$$\sigma_c(d^w(STx, STy), d^w(Sx, STx) + d^w(Sy, STy)) > 0 \tag{3.4}$$

for all $x, y \in X$, for any fixed $w \in \mathbb{N}$.

In [12], Malceski proved the following generalization of Kannan's theorem:

Theorem 5. *Let (X, d) be a complete metric space, $T\colon X \to X$ and $S\colon X \to X$ be two mappings such that T it is continuous, injection and sequentially convergent. If $\alpha > 0, \gamma \geq 0$ and $2\alpha + \gamma < 1$, and, satisfies the condition that,*

$$d(STx, STy) \leq \alpha[d(Sx, STx) + d(Sy, STy)] + \gamma d(Sx, Sy)$$

for all $x, y \in X$, then there is a unique fixed point of T.

Now we find another different type of generalization of Theorem 1 (Kannan, [7]), by using σ_c-function, which is quite analogous to the above Theorem 5 and extends the theorems given in [10] and [13].

Theorem 6. *Let (X, d) is a metric space, and $T\colon X \to X$ and $S\colon X \to X$ be two operators, such that, T is S-dominated Σ_c-Kannan mapping of degree w, with respect to some $\sigma_c \in \Sigma_c$ $(c = 1)$, with the following conditions hold: (a) $S(X)$ is complete; (b) S is injective; (c) either $\sigma_c(t, s) < s - t$; or, σ_c satisfy the condition (S). Then, there exists a unique fixed point of the operator T in X.*

Proof. We first show the existence of fixed point. Let the sequence $\{x_n\}_{n \geq 0}$ is formed by the Picard's interation, i.e., $x_n = T^n x_0$, with initial base point x_0 we assume, $x_{n+1} \neq x_n$ for every $n \in \mathbb{N}$, because if not so for some index k, then $Tx_k = x_{k+1} = x_k$ and we would get fixed point and our proof would be over. So, assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Now, the injectivity of S implies that, $Sx_{n+1} \neq Sx_n$ for all $n \in \mathbb{N}$.

Given, $\sigma_c(d^w(STx, STy), d^w(Sx, STx) + d^w(Sy, STy)) > 0$ for all $x, y \in X$. So, $\sigma_c(d^w(STx_{n+1}, STx_n),\ d^w(STx_{n+1}, Sx_{n+1}) + d^w(STx_n, Sx_n)) > 0$ for all $n \in \mathbb{N}$, which implies

$$\sigma_c(d^w(Sx_{n+2}, Sx_{n+1}), d^w(Sx_{n+2}, Sx_{n+1}) + d^w(Sx_{n+1}, Sx_n)) > 0$$

as, $x_{n+1} = Tx_n$ for all $n \geq 0$. Choose $a_n = d^w(Sx_{n+2}, Sx_{n+1}) (> 0)$, then, $a_{n-1} = d^w(Sx_{n+1}, Sx_n)$ and so $\sigma_c(a_n, b_n) > 0$, where $b_n = a_{n-1} + a_n$ for all $n \in \mathbb{N}$. So by $(\sigma 1)$ we have $d^w(Sx_{n+1}, Sx_n) = a_{n-1} \to 0$, as $n \to \infty$. That is

$$d(Sx_{n+1}, Sx_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{3.5}$$

Claim: The sequence $\{x_n\}$ is S-bounded.

Proof of Claim: On contrary, we assume, that $\{x_n\}$ is not S-bounded. Without loss of generality we assume that $Sx_{n+p} \neq Sx_n$ for all $n, p \in \mathbb{N}$; and so clearly, $Tx_{n+p} \neq Tx_n$ for all $n, p \in \mathbb{N}$. Now as $\{x_n\}$ is not S-bounded, for each $k \in \mathbb{N}$, there exist two subsequences $\{Sx_{n_k}\}$ and $\{Sx_{m_k}\}$ of $\{Sx_n\}$ with $k \leq n_k < m_k$ such that, m_k is the minimum integer and for $n_k \leq p \leq m_k$ the following is satisfied:

$$d(STx_{n_k-1}, STx_{m_k-1}) = d(Sx_{n_k}, Sx_{m_k}) > 1 \text{ and } d(Sx_{n_k}, Sx_p) \leq 1. \tag{3.6}$$

Subcase I: If $\sigma_c(t, s) < s - t$, then, by equation (3.6), we have that

$$\begin{aligned} 1 < d^w(Sx_{n_k}, Sx_{m_k}) &= d^w(STx_{n_k-1}, STx_{m_k-1}) < d^w(STx_{n_k-1}, Sx_{n_k-1}) \\ &+ d^w(STx_{m_k-1}, Sx_{m_k-1}) = d^w(Sx_{n_k}, Sx_{n_k-1}) + d^w(Sx_{m_k}, Sx_{m_k-1}). \end{aligned}$$

Then, by (3.6), taking limit on both sides as $k \to \infty$, we arrive at a contradiction.

Subcase II: If σ_c satisfy the condition (S), then we assume,

$$a_k = d^w(Sx_{n_k}, Sx_{m_k}), \quad b_k = d^w(Sx_{n_k}, Sx_{n_k-1}) + d^w(Sx_{m_k}, Sx_{m_k-1}).$$

Then by the given condition, we have, $\sigma_c(a_k, b_k) > 0$ with $b_k \to 0$ so by the property (S) we have $a_k \to 0$.

Claim: The sequence $\{x_n\}$ is S-Cauchy.

Proof of Claim: Consider $C_n = \sup\{d(Sx_i, Sx_j)\colon i, j \geq n\}$. Then, by the S-boundedness and monotonicity of C_n, we have $\lim_{n\to\infty} C_n = C$ (for some C). If $C > 0$, then there exist q_k, p_k with $p_k > q_k \geq k$ such that $\lim_{k\to\infty} d(Sx_{p_k}, Sx_{q_k}) = C$. Then, similar to the previous cases (or by Subcase I, Subcase II) we arrive at a contradiction.

Now by condition (a) there exists a point z (say) in $S(X)$, for which $Sx_n \to z$. Also, as $z \in S(X)$, there is a point say $a \in X$ such that $Sa = z$. Now as the previous one, we consider $a_n = d^w(STx_n, STa)$ and $b_n = d^w(STx_n, Sx_n) + d^w(STa, Sa)$, for all $n \in \mathbb{N}$ such that, $a_n \to d^w(z, STa)$ and $b_n \to d^w(Sa, STa)$. So by (σ2) we have, $d^w(z, STa) = 0$, i.e., $STa = z = Sa$, and the injectivity of S shows that $Ta = a$. Uniqueness of fixed point follows from the previous arguments. □

Remark 8. Here, we only use the injectivity of S, not the continuity and sequential convergence of S, and still the theorem remains true, if we assume that image of X under S is complete.

Corollary 3. (Koparde-Waghmode theorem, see [10]) If (X, d) is a complete metric space and $T\colon X \to X$ is a mapping. Suppose that, there exists $\alpha \in (0, 1/2)$ such that the following is satisfied:

$$d^2(Tx, Ty) \leq \alpha[d^2(x, Tx) + d^2(y, Ty)] \text{ for all } x, y \in X.$$

Then, T has a unique fixed point in X.

Proof. We assume σ_c function as $\sigma_c(t, s) = \alpha s - t$, with $\alpha \in (0, 1/2)$, and consider operator S as, $S(x) = x$ for all $x \in X$, and put $w = 2$, in the Theorem 6. □

Corollary 4. (Patel-Deheri theorem, see [13]) In a complete metric space, if the two operators S, T satisfy,

$$d(STx, STy) \leq \alpha[d(Sx, STx) + d(Sy, STy)],$$

where $0 < \alpha < 1/2$ and S is continuous, injection and sequentially convergent. Then, T has a unique fixed point in X.

Proof. We assume σ_c-function as $\sigma_c(t, s) = \alpha s - t$, with $\alpha \in (0, 1/2)$, and put $w = 1$, in Theorem 6. □

We now prove another generalizations of Theorem 2 (Kannan [8]).

Theorem 7. *Let (X, d) be a metric space and $T\colon X \to X$, $S\colon X \to X$ be two operators such that, T is S-dominated Σ_c-Kannan mapping of degree $w = 1$, with respect to some σ_c-function ($c = 1$). Suppose that the following conditions hold:*

(i) either $\sigma_c(t, s) < s - t$ for all $s, t > 0$; or, σ_c satisfying the condition (S);

(ii) there exists a point p such that, the Picard sequence $\{Tx_n\}$ has a subsequence $\{Tx_{n_k}\}$ converging to q;

(iii) T and S both continuous at the point $Sq \in X$; and,

(iv) S is injective and continuous at Tq.

Then, q is the unique fixed point of T.

Proof. Suppose $Tq \neq q$. Then, by (iv), $STq \neq Sq$. We consider two disjoint open balls, say $B(TSq, r_1)$ and $B(Sq, r_2)$, with centres at Tq, q, and radii r_1, r_2 respectively and choose $0 < r < \min\{r_1, r_2, d(TSq, Sq)/3\}$.

Now, as the subsequence $\{Tx_{n_k}\}$ converging to q, and S is continuous at both q and Tq. So $\{STx_{n_k}\}$ converges to Sq; and $\{STTx_{n_k}\}$ converges to STq. there exists a positive integer M, such that, for all $k > M$, we have that,

$$STx_{n_k} \in B(Sq, r) \text{ and } STTx_{n_k} \in B(STq, r),$$

and so, clearly, for each $k > M$, we have that,

$$0 < 3r < d(STq, Sq) \leq d(STq, STTx_{n_k}) + d(STTx_{n_k}, STx_{n_k}) + d(STx_{n_k}, Sq) < 2r + d(STx_{n_k}, STTx_{n_k}).$$

This implies that

$$d(STx_{n_k}, STTx_{n_k}) > r. \tag{3.7}$$

Now, T is S-dominated Σ_c-Kannan mapping of degree $w = 1$, so using (i) and, a similar idea used in Theorem 3, S is asymptotically regular and hence we have a contradiction to (3.7). This shows that $STq = Sq$ or equivalently $Tq = q$. The uniqueness follows from similar arguments as we used in previous results. □

Example 10. The S-dominated Σ_c-Kannan contraction condition is a proper extension of the Kannan's contraction condition we consider the example given in [12], [13] with a little brief.

Let $X = \{0, 1/4, 1/5, 1/6, \ldots\}$ endowed with the Euclidean metric. Define $T\colon X \to X$, by $T0 = 0$, and $T(1/n) = 1/(n+1)$. Consider $S\colon X \to X$ to be $S0 = 0$ and $S(1/n) = 1/n^n$, then T does not satisfy the Kannan's contraction condition for any constant > 0, but it is S-dominated Σ_c-Kannan mapping of degree $w = 1$ with respect to the Σ_c-function defined by $\sigma_c(t, s) = s/3 - t$ (for $0 < c < 2$). (By Remark 5). Also by defining S to be, $S0 = 0$ and $S(1/n) = 1/[e^2 n]$, T is a S-dominated Σ_c-Kannan with respect to the Σ_c-function defined by $\sigma_c(t, s) = s/6t$ (for $0 < c < 2$) (by Remark 5).

We consider one more new example of this type, but in much simpler form.

Example 11. Let (X, d) and $T\colon X \to X$ be same as we have considered in Example 8. Then, T does not satisfy the Kannan's contractive condition. But, T is a S-dominated Σ_c-Kannan, with respect to the σ_c-function defined by $\sigma_c(t, s) = \alpha s - t$ (for every fixed $c \in (0, 2)$); for all fixed $\alpha \in (0, 1/2)$.

4 Application to integral equations

In this section, we give an application of Theorem 4 in proving the existence and uniqueness of the solution of a particular type of integral equations.

Suppose, $K\colon [a, b] \times \mathbb{R} \to \mathbb{R}$ is a given function. We consider the following integral equation:

$$u(t) = \alpha \int_a^t K(t, u(s))ds, \tag{4.1}$$

where α is a nonzero real.

Let $C_{\mathbb{R}}[a, b]$ be the space of all continuous real-valued functions defined on the interval $[a, b]$ with the Bielecki's norm given by:

$$\|u\|_B = \sup_{t\in[a,b]} |u(t)|e^{-\mu t},\ u \in C_{\mathbb{R}}[a, b],$$

where $\mu > 0$ is arbitrary but fixed. The metric d induced by $\|\cdot\|_B$ is given by:

$$d(u, v) = \sup_{t\in[a,b]} |u(t) - v(t)|e^{-\mu t} \text{ for all } u, v \in C_{\mathbb{R}}[a, b].$$

Then $(C_{\mathbb{R}}[a, b], d)$ is a complete metric space.

Theorem 8. *Suppose, the function $K\colon [a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous and the following condition is satisfied: for all $t, s \in [a, b], u, v \in \mathbb{R}$ there exists a non-decreasing continuous function $\psi\colon [0, \infty) \to [0, 1/2)$ such that*

$$|K(t, u(s)) - K(t, v(s))| \le \psi\Big(D(t, u(s), v(s))e^{-\mu s}\Big)D(t, u(s), v(s)), \tag{4.2}$$

where $D(t, u(s), v(s)) = \left|u(s) - \alpha \int_a^t K(t, u(s))ds\right| + \left|v(s) - \alpha \int_a^t K(t, v(s))ds\right|$. Then the integral equation (4.1) has a unique solution.

Proof. To prove the existence and uniqueness of the solution of equation (4.1) we first convert it into a fixed point problem. Define a function $T\colon C_{\mathbb{R}}[a, b] \to C_{\mathbb{R}}[a, b]$ by

$$T(u(t)) = \alpha \int_a^t K(t, u(s))ds.$$

Then it is clear that the problem of finding the solution of (4.1) is equivalent to the problem of finding the fixed point of T. Now

$$\begin{aligned} d(Tu,Tv) &= \sup_{t\in[a,b]} |T(u(t))-T(v(t))|e^{-\mu t} \\ &= \sup_{t\in[a,b]} \left|\alpha\int_a^t K(t,u(s))ds - \alpha\int_a^t K(t,v(s))ds\right| e^{-\mu t} \\ &\le |\alpha| \sup_{t\in[a,b]} \int_a^t |K(t,u(s))-K(t,v(s))|ds\cdot e^{-\mu t} \\ &\le |\alpha| \sup_{t[a,b]} \int_a^t \psi\Big(D(t,u(s),v(s))e^{-\mu s}\Big)D(t,u(s),v(s))ds\cdot e^{-\mu t}. \end{aligned} \tag{4.3}$$

Notice that

$$\begin{aligned} &(D(t,u(s),v(s))e^{-\mu t} = \left[\left|u(s)-\alpha\int_a^t K(t,u(r))dr\right| + \left|v(s)-\alpha \right.\right. \\ &\quad \left.\left.\times\int_a^t K(t,v(r))dr\right|\right]e^{-\mu t} \le \sup_{s\in[a,b]}\left[\left|u(s)-\alpha\int_a^t K(t,u(r))dr\right|\right. \\ &\quad \left.+\left|v(s)-\alpha\int_a^t K(t,v(r))dr\right|\right]e^{-\mu t} = \sup_{s\in[a,b]}\left[\Big|u(s)-T(u(t))\Big|\right. \\ &\quad \left.+\Big|v(s)-T(v(t))\Big|\right]e^{-\mu s}e^{-\mu(t-s)} \le d\Big(u(t),T(u(t))\Big)+d\Big(v(t),T(v(t))\Big)e^{-\mu(s-t)} \end{aligned}$$

and

$$\begin{aligned} \psi((D(t,u(s),v(s))e^{-\mu s}) &= \psi\Big(\Big|u(s)-\alpha\int_a^t K\left(t,u(s)\right)ds\Big|e^{-\mu s} \\ &\quad +\Big|v(s)-\alpha\int_a^t K(t,v(s))ds\Big|e^{-\mu s}\Big) \le \psi\Big(\sup_{s\in[a,b]}\Big[\Big|u(s) \\ &\quad -\alpha\int_a^t K(t,u(r))dr\Big|e^{-\mu s}+\Big|v(s)-\alpha\int_a^t K(t,v(r))dr\Big|e^{-\mu s}\Big]\Big) \\ &\quad = \psi\Big(\sup_{s\in[a,b]}\Big[\Big|u(s)-T(u(t))\Big|e^{-\mu s}+\Big|v(s)-T(v(t))\Big|e^{-\mu s}\Big]\Big) \\ &\quad \le \psi\Big(d(u(t),T(u(t)))+d(v(t),T(v(t)))\Big). \end{aligned}$$

Therefore, it follows from (4.3) that

$$\begin{aligned} d(Tu,Tv) &\le |\alpha|\sup_{t\in[a,b]}\int_a^t \psi\Big(d(u(t),T(u(t)))+d(v(t),T(v(t)))\Big) \\ &\quad \times\Big[d(u(t),T(u(t)))+d(v(t),T(v(t)))\Big]ds\cdot e^{-\mu(t-s)}, \end{aligned}$$

i.e.

$$\begin{aligned} d(Tu,Tv) &\le |\alpha|\psi\big(d(u(t),T(u(t)))+d(v(t),T(v(t)))\big) \\ &\quad \times\big[d(u(t),T(u(t)))+d(v(t),T(v(t)))\big]\big[\big(1-e^{-\mu(b-a)}\big)/\mu\big]. \end{aligned}$$

Choose $\mu = |\alpha|$ in the above we obtain:

$$\begin{aligned} d(Tu,Tv) &\leq \psi\big(d(u(t),T(u(t))) + d(v(t),T(v(t)))\big) \\ &\quad \times \big[d(u(t),T(u(t))) + d(v(t),T(v(t)))\big]\big[1 - e^{-\mu(b-a)}\big] \\ &< \psi\big(d(u(t),T(u(t))) + d(v(t),T(v(t)))\big)\big[d(u(t),T(u(t))) + d(v(t),T(v(t)))\big]. \end{aligned} \tag{4.4}$$

Define a function $\sigma_c \colon [0,\infty) \times [0,\infty) \to \mathbb{R}$ by:

$$\sigma_c(t,s) = s\psi(s) - t \text{ for all } t,s \in [0,\infty).$$

Then it is easy to see that $\sigma_c \in \Sigma_c$ for $c = 1$. Therefore, from (4.4) we obtain that

$$\sigma_c\Big(d(Tu,Tv), d(u,Tu) + d(v,Tv)\Big) > 0 \text{ for all } u,v \in C_{\mathbb{R}}[a,b].$$

Since $\psi \colon [0,\infty) \to [0,1/2)$ we have

$$\sigma_c(t,s) = s\psi(s) - t < s - t \text{ for all } t,s \in (0,\infty).$$

Thus, all the conditions of Theorem 4 (with $S = Id_X$) are satisfied, and so, the mapping T has a unique fixed point which is the unique solution of integral equation (4.1). □

5 Conclusions

In this article, we tried to find several extension of Kannan's two different fixed point results, by introducing the concept of Σ_c-S-Kannan operator and S-dominated Σ_c-S-Kannan operator of degree w; via the new concept of σ_c-function (shown to be independent of the other three concepts of Simulation function, Manageable function and R-function). These new generalizations also extends several known theorems in this branch and the similar ideas could be profitably extended to the three dimensional case of σ_c-functions and would be helpful to find the extension of the Fisher type (see, [4]) of mapping and the similar type of operators. Now, by the above discussions, the following interesting problems arise:

Problem 1. Can the Theorem 3 (or Theorem 4) and Theorem 6 (or Theorem 7) be proved with an operator satisfying the condition (3.1) (or condition (3.4)) for a dense subset of X, instead of whole of X?

Problem 2. If we define an analogous Definition 9, and Definition 10 as follows: An operator T on a metric space is called Σ_c-S-Fisher if it satisfies, $\sigma_c(d(Tx,Ty), d(Tx,Sy) + d(Ty,Sx)) > 0$ for all $x,y \in X$ and said to be S-dominated Σ_c-Fisher mapping of degree w, if, with for some σ_c function, we have $\sigma_c(d^w(STx,STy), d^w(Sy,STx) + d^w(Sx,STy)) > 0$ for all $x,y \in X$, for any fixed $w \in \mathbb{N}$. Then, can the Theorem 3 (or Theorem 4) and Theorem 6 (or Theorem 7) be proved, with an operator satisfying the above proposed conditions for the Fisher type of operator?

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