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Nonlinear Problems with Asymmetric Principal Part

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Abstract. The boundary value problem

$$x'' = -\lambda f(x^+) + \mu f(x^-) + h(t,x,x'), \quad x(0) = 0 = x(1)$$

is considered provided that $f : [0,+\infty) \to [0,+\infty)$ is Lipschitzian and $h : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and Lipschitzian in x and x'. We assume that f is bounded by two linear functions kx and lx, where $k > l > 0$, and h is bounded. We find the conditions on (λ, μ) which guarantee the existence of a solution to the problem. These conditions are of geometrical nature.

Keywords: nonlinear spectra, Fučík spectrum, comparison, angular functions, Dirichlet boundary value problem.

AMS Subject Classification: 34B08; 34B15.

1 Introduction

There is intensive literature on boundary value problems for the second order ordinary differential equations which depend on two parameters, for example [1, 2, 3, 5, 7, 10, 11, 12]. A special class of problems deals with the so called asymmetric equations. The classical representative of such equations is the Fučík equation

$$x'' = -\lambda x^+ + \mu x^-, \quad x^+ = \max\{x, 0\},\ x^- = \max\{-x, 0\},\ \lambda > 0,\ \mu > 0,$$

which is usually considered together with some boundary conditions, for instance, the Dirichlet ones $x(0) = 0$, $x(1) = 0$.

The results on Fučík problem can be used for investigation of essentially nonlinear problems of the type

$$x'' + g(x) = h(t, x, x'), \quad x(0) = 0, \quad x(1) = 0,$$

where the ratio $g(x)/x$ tends to finite limits as $x \to \pm\infty$ and h is bounded. These limits (as points in $\mathbb{R}^2$) have to be separated from the Fučík spectrum, so that the problem had a solution.

There were attempts [4, 5, 6] to consider Fučík type equations of the form

$$x'' = -\lambda f(x^+) + \mu g(x^-),$$

where f and g are (nonlinear) positively valued functions.

In this paper we consider the boundary value problem

$$x'' = -\lambda f(x^+) + \mu f(x^-) + h(t, x, x'), \quad x(0) = 0, \quad x(1) = 0, \tag{1.1}$$

where f is a positive valued function such that $f(0) = 0$ and h is a bounded nonlinearity. Functions f and h are such that there is a unique solvability of the Cauchy problems and continuous dependence of solutions on the initial data.

We suppose that

$$lx < f(x) < kx, \quad \forall x > 0, \ 0 < l < k. \tag{1.2}$$

If $f(x)$ is a linear function (i.e., $f(x) = kx$) then the two-parameter problem is given as

$$x'' = -\lambda k x^+ + \mu k x^- + h(t, x, x'), \quad x(0) = 0, \quad x(1) = 0.$$

It is known [3] that this problem is solvable for any bounded nonlinearity h if $(\lambda k, \mu k)$ belongs to "good" regions in the first quadrant of (λ, μ)-plane. We discuss this below.

If the principal part looks like in problem

$$x'' = -\lambda f(x^+) + \mu g(x^-) + h(t, x, x'), \quad x(0) = 0, \quad x(1) = 0,$$

then analysis of it becomes more complicated. There are some results which state that the reduced problem

$$x'' = -\lambda f(x^+) + \mu g(x^-), \quad x(0) = 0, \quad x(1) = 0,$$

is non-trivially solvable if (λ, μ) belongs to solution surfaces [8, 11]. If $g(x)$ is bounded between two linear functions (like function $f(x)$), then similar results can be obtained. The aim of this paper is to present the existence results for the problem (1.1).

2 Quasi-Linear Fučík Problem

Consider the problem

$$x'' = -\lambda x^+ + \mu x^- + h(t, x, x'), \quad x(0) = 0, \quad x(1) = 0. \tag{2.1}$$

In order to formulate the existence conditions we need first to consider the Fučík spectrum. The Fučík spectrum Σ_F is a set of points (λ, μ) such that the problem

$$x'' = -\lambda x^+ + \mu x^-, \quad x(0) = 0, \quad x(1) = 0 \tag{2.2}$$

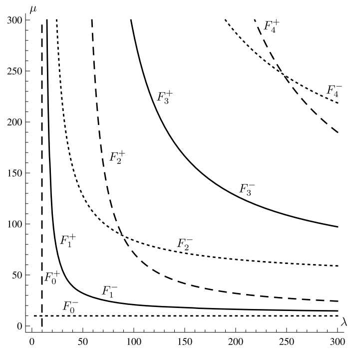

Figure 1. The Fučík spectrum.

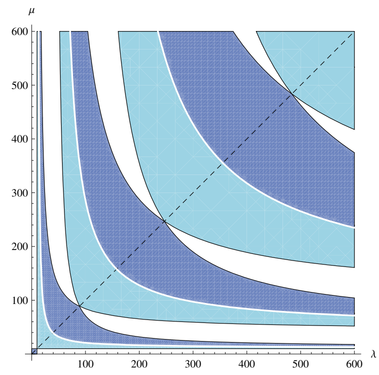

Figure 2. "Good" regions shaded; light shaded for solutions with $2n-1$ zeros in $(0,1)$, dark shaded for solutions with $2n$ zeros in $(0,1)$, $n = 1, 2, \ldots$; "bad" regions white.

has a nontrivial solution $x(t; \lambda, \mu)$. The Fučík spectrum Σ_F consists of a set of curves located in the first quadrant of (λ, μ)-plane [3] (see, Fig. 1).

If $(\lambda, \mu) \notin \Sigma_F$ then the problem (2.2) has only the trivial solution. This is insufficient for solvability of the problem (2.1) for any bounded h. The solvability, however, can be guaranteed for "good" regions.

Since it is essential for proving of the main result, we discuss solvability of the problem (2.1). Consider the Cauchy problem

$$x'' = -\lambda x^+ + \mu x^- + h(t, x, x'), \quad x(0) = 0, \quad x'(0) = \alpha, \tag{2.3}$$

where h is bounded. Introduce the functions $u(t)$ and $v(t)$ as solutions of the Cauchy problems

$$u'' = -\lambda u^+ + \mu u^-, \quad u(0) = 0, \quad u'(0) = 1,$$
$$v'' = -\lambda v^+ + \mu v^-, \quad v(0) = 0, \quad v'(0) = -1.$$

Let $x(t; \alpha)$ be a solution of (2.3). The normalized functions $y(t; \alpha) = x(t; \alpha)/\alpha$ tend respectively to the functions $u(t)$ and $v(t)$ as $\alpha \to \pm\infty$. Notice that $y(t; \alpha)$ satisfies also the equation

$$y'' = -\lambda y^+ + \mu y^- + h(t, x, x')/\alpha,$$

where $h(t, x, x')/\alpha$ tends to zero uniformly in t, x, x' as $\alpha \to \infty$. If the condition $y(1; +\infty)y(1; -\infty) < 0$ is satisfied, which is equivalent to

$$u(1)v(1) < 0, \tag{2.4}$$

then the existence of $x(t; \alpha_0)$ which solves the problem (2.1) can be concluded. Therefore problem (2.1) is solvable if (λ, μ) is not in Σ_F but condition (2.4) holds. The regions of (λ, μ)-plane where $u(1)v(1) < 0$ are shaded in Fig. 2.

So if (λ, μ) are in the shaded region but not in the Fučík spectrum then the problem (2.1) is solvable for any bounded $h(t, x, x')$. If $h(t, 0, 0) \not\equiv 0$ then there exists a non-trivial solution.

If precise location of a point (λ, μ) between definite branches of the Fučík spectrum is given, then we can state the existence of a solution with definite nodal structure.

3 The Problem

Consider the problem (1.1), where $f(x)$ is such that (1.2) fulfils. To formulate the existence result we need to consider two auxiliary problems

$$x'' = -\lambda k x^+ + \mu k x^-, \quad x(0) = 0, \quad x(1) = 0, \tag{3.1}$$

$$x'' = -\lambda l x^+ + \mu l x^-, \quad x(0) = 0, \quad x(1) = 0. \tag{3.2}$$

Denote the spectra of these problems $\Sigma_F(k)$ and $\Sigma_F(l)$ respectively. Both spectra have "good" regions. Let $D(k)_i$ be a part of "good" region where solutions of the IVPs (3.1), $x(0) = 0$, $x'(0) = \pm 1$ have exactly i zeros in $(0, 1)$. In "good" regions these two solutions also are of opposite signs at $t = 1$ and this is important.

Similarly regions $D(l)_i$ are introduced.

Notice that the spectrum $\Sigma_F(k)$ (and $\Sigma_F(l)$) can be obtained from the Fučík spectrum Σ_F by compression (if $k > 1$) or by extension (if $0 < k < 1$).

Theorem 1. *Suppose that $f : [0, +\infty) \to [0, +\infty)$ is Lipschitzian and $lx < f(x) < kx$ for $x > 0$, $0 < l < k$. Assume that $h : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfy the Lipschitz condition in x and x'. Let (λ, μ) be in $D_i = D(k)_i \cap D(l)_i$ for some $i \in \{0, 1, \ldots\}$. Then provided that h is bounded the problem* (1.1) *has a solution.*

To prove the theorem, we need some comparison results which we consider in separate subsections.

3.1 Differential inequality

The following assertion is a slight modification of Theorem 14.1 in [9].

Theorem 2. *Let $\varphi(t)$ and $\psi(t)$ be $C^1([a, b])$ functions which satisfy*

$$\frac{d\varphi}{dt} > F\big(t, \varphi(t)\big), \quad \frac{d\psi}{dt} = F\big(t, \psi(t)\big), \quad a \le t \le b,$$

and $\varphi(a) = \psi(a)$, where $F \in C([a, b], R)$. Then $\varphi(t) > \psi(t)$ for $a < t \le b$. If

$$\frac{d\varphi}{dt} < F\big(t, \varphi(t)\big) \quad \text{and} \quad \varphi(a) = \psi(a),$$

then $\varphi(t) < \psi(t)$ for $a < t \le b$.

Proof. Evidently $\frac{d\varphi}{dt}(a) > \frac{d\psi}{dt}(a)$. Therefore $\varphi(t) > \psi(t)$ for $t \in (a, a + \varepsilon)$ for some positive ε. The graph of $\varphi(t)$ cannot cross the graph of $\psi(t)$ downwards. Therefore $\varphi(t) > \psi(t)$ for $t \in (a, b]$. □

3.2 Angular functions

In this subsection we follow the comparison results of [9, Ch. 15] adapting them to our cases.

Consider two second order equations written in a form of systems of two first order equations:

$$\begin{cases} \dfrac{dx}{dt} = y, \quad \dfrac{dy}{dt} = -q(x) \end{cases} \tag{3.3}$$

and

$$\begin{cases} \dfrac{dx}{dt} = y, \quad \dfrac{dy}{dt} = -\tilde{q}(x), \end{cases} \tag{3.4}$$

where $\tilde{q}(x)$ possesses the property of positive homogeneity, that is, $\tilde{q}(cx) = c\tilde{q}(x)$ for $c \geq 0$ (in fact $\tilde{q}(x)$ is a piece-wise linear function defined separately for $x \geq 0$ and $x < 0$). Suppose that

$$xq(x) > x\tilde{q}(x), \quad x \neq 0. \tag{3.5}$$

Introduce the polar coordinates $(x, y) = (x, x')$ as

$$x(t) = r(t)\sin\varphi(t), \quad x'(t) = r(t)\cos\varphi(t)$$

and let $(r(t), \varphi(t))$, $(\tilde{r}(t), \tilde{\varphi}(t))$ be coordinates for (3.3), (3.4) respectively.

One gets for $\varphi(t)$ and $\tilde{\varphi}(t)$ that

$$\frac{d\varphi}{dt} = \frac{1}{r}\big[r\cos^2\varphi + q(r\sin\varphi)\sin\varphi\big]. \tag{3.6}$$

On the other hand,

$$\begin{aligned} \frac{d\tilde{\varphi}}{dt} &= \frac{1}{\tilde{r}}\big[\tilde{r}\cos^2\tilde{\varphi} + \tilde{q}(\tilde{r}\sin\tilde{\varphi})\sin\tilde{\varphi}\big] \\ &= \cos^2\tilde{\varphi} + \tilde{q}(\sin\tilde{\varphi})\sin\tilde{\varphi} := F(\tilde{\varphi}). \end{aligned}$$

It follows from (3.5) that $q(r\sin\varphi)\sin\varphi > \tilde{q}(r\sin\varphi)\sin\varphi$ if $\varphi \neq \bmod (\pi)$ and therefore

$$\frac{d\varphi(t)}{dt} > F\big(\varphi(t)\big)$$

and, if $\varphi(a) = \tilde{\varphi}(a)$, then, by Theorem 2, $\varphi(t) > \tilde{\varphi}(t)$ for any $t \in [a, b]$.

If inequality (3.5) is changed to the opposite then

$$\frac{d\varphi(t)}{dt} < F\big(\varphi(t)\big)$$

and, if $\varphi(a) = \tilde{\varphi}(a)$, then, by Theorem 2, $\varphi(t) < \tilde{\varphi}(t)$ for any $t \in [a, b]$.

3.3 Comparison of angular functions

Consider shortened equation

$$x'' = -\lambda f(x^+) + \mu f(x^-) \tag{3.7}$$

and compare it to equations (3.1) and (3.2) having in mind the relations (1.2).

Notice that for $\lambda > 0$, $\mu > 0$

$$x(\lambda k x^+ - \mu k x^-) > x\big(\lambda f(x^+) - \mu f(x^-)\big) > x(\lambda l x^+ - \mu l x^-), \quad x \neq 0. \quad (3.8)$$

The right-hand sides of equations (3.1) and (3.2) are positive homogeneous functions, therefore the arguments of preceding subsection are applicable.

If $\varphi_k(t)$, $\varphi(t)$ and $\varphi_l(t)$ are the angular functions for equations (3.1), (3.8), (3.2) respectively, one has that

$$\varphi_k(t) > \varphi(t) > \varphi_l(t), \quad t \in (0, 1] \quad (3.9)$$

if $\varphi_k(0) = \varphi(0) = \varphi_l(0)$. Thus we have arrived to the following result.

Lemma 1. *Let* (λ, μ) *be in* $D(k)_i \cap D(l)_i$ *for some* $i \in \{0, 1, \dots\}$. *Then the angular functions for equations* (3.1), (3.7), (3.2), *which satisfy*

$$\varphi_k(0) = \varphi(0) = \varphi_l(0) = \varphi_0, \quad \varphi_0 = 0 \quad or \quad \varphi_0 = \pi$$

satisfy also the inequalities (3.9).

Remark 1. The above lemma means that for $(\lambda, \mu) \in D(k)_i \cap D(l)_i$ any solution of equation (3.7) with the initial conditions $x(0) = 0$, $x'(0) > 0$ has exactly i zeros in $(0, 1)$ and $x(1) \neq 0$. The same is true for solutions of equation (3.7) with the initial conditions $x(0) = 0$, $x'(0) < 0$.

3.4 Result

Consider equation

$$x'' = -\lambda f(x^+) + \mu f(x^-) + h(t, x, x') \quad (3.10)$$

and the equivalent system

$$\begin{cases} \dfrac{dx}{dt} = y, \quad \dfrac{dy}{dt} = -q(x) + h(t, x, y), \end{cases}$$

where $q(x) = \lambda f(x^+) - \mu f(x^-)$. Suppose polar coordinates $(\rho(t), \theta(t))$ are introduced as $x(t) = \rho(t)\sin\theta(t)$, $x'(t) = \rho(t)\cos\theta(t)$. The expression for $\theta(t)$ is given as

$$\frac{d\theta}{dt} = \big[\rho\cos^2\theta + q(\rho\sin\theta)\sin\theta - h(t, \rho\sin\theta, \rho\cos\theta)\sin\theta\big]/\rho. \quad (3.11)$$

The right hand sides of equations (3.11) and (3.6) differ only by the term $\frac{1}{\rho}h(t, \rho\sin\theta, \rho\cos\theta)\sin\theta$, which is negligibly small if $\rho(t)$ stays in a complement of the circle of sufficiently large radius for any $t \in [0, 1]$ (recall that h is bounded). This is the case for the solutions of equation (3.10) which satisfy the initial conditions

$$x(0) = 0, \quad x'(0) = \pm\Delta, \quad (3.12)$$

if $\Delta \to +\infty$. For this, let us mention the following result.

Lemma 2. *For solutions of the problems* (3.10), (3.12) *a function* $m(\Delta)$ *exists such that* $m(\Delta) \to +\infty$ *as* $\Delta \to +\infty$ *and* $\rho(t) \geq m(\Delta)$ *for any* $t \in [0,1]$.

Lemma follows from Lemma 15.1 in [9] since all solutions of equation (3.10) are extendable to the interval $[0,1]$. The latter follows from the assumptions on f (1.2) and boundedness of h. It follows from the above arguments that

$$\varphi_k(t) \geq \theta(t) \geq \varphi_l(t), \quad t \in [0,1]$$

if $\varphi_k(0) = \varphi(0) = \varphi_l(0)$, where $\theta(t)$ is the angular function for solutions of (3.10), (3.12) with sufficiently large Δ.

In other words, in conditions of Theorem 1, a solution $\bar{x}(t)$ of equation (3.10) with the initial conditions $x(0) = 0$, $x'(0) = \Delta$ has exactly i zeros in $(0,1)$ and $\bar{x}(1) \neq 0$. A solution $\underline{x}(t)$ of equation (3.10) with the initial conditions

$$x(0) = 0, \quad x'(0) = -\Delta$$

also has exactly i zeros in $(0,1)$ and $\underline{x}(1) \neq 0$. What is important, one has also $\bar{x}(1)\underline{x}(1) < 0$. Then one concludes, considering the Cauchy problem (3.10),

$$x(0) = 0, \quad x'(0) = \delta, \quad \delta \in (-\Delta, \Delta)$$

and employing the continuous dependence of solutions on the initial data, that for some δ a solution $x(t)$ vanishes at $t = 1$. This completes the proof of Theorem 1.

4 Elementary Analysis of Regions D_i

In order to analyze the regions $D(k)_i$ for equation $x'' = -\lambda k x^+ + \mu k x^-$ recall that branches of the Fučík spectrum are given by

$$F_0^+ = \left\{(\lambda,\mu)\colon \frac{\pi}{\sqrt{\lambda k}} = 1,\ \mu \geq 0\right\}, \quad F_0^- = \left\{(\lambda,\mu)\colon \lambda \geq 0,\ \frac{\pi}{\sqrt{\mu k}} = 1\right\},$$

$$F_{2i-1}^+ = \left\{(\lambda;\mu)\colon i\frac{\pi}{\sqrt{\lambda k}} + i\frac{\pi}{\sqrt{\mu k}} = 1\right\},\ F_{2i-1}^- = \left\{(\lambda;\mu)\colon i\frac{\pi}{\sqrt{\mu k}} + i\frac{\pi}{\sqrt{\lambda k}} = 1\right\},$$

$$F_{2i}^+ = \left\{(\lambda;\mu)\colon (i+1)\frac{\pi}{\sqrt{\lambda k}} + i\frac{\pi}{\sqrt{\mu k}} = 1\right\},$$

$$F_{2i}^- = \left\{(\lambda;\mu)\colon (i+1)\frac{\pi}{\sqrt{\mu k}} + i\frac{\pi}{\sqrt{\lambda k}} = 1\right\}.$$

Similar formulas are true for equation $x'' = -\lambda l x^+ + \mu l x^-$.

A set $D(k)_0$ is a square below F_0^- and to the left of F_0^+. A set $D(k)_1$ is a region bounded by F_0^-, F_0^+ and $F_1^\pm$. A set $D(k)_2$ is a region bounded by $F_1^\pm$ and $\min\{F_2^+, F_2^-\}$. A union of these regions is depicted in Fig. 3.

Similarly, regions $D(l)_i$ can be described. Since $l < k$, the spectrum $\Sigma_F(l)$ can be obtained from $\Sigma_F(k)$ by extension. Under the extension process

$$D(k)_0 \cap D(l)_0 = D(k)_0 \neq \varnothing.$$

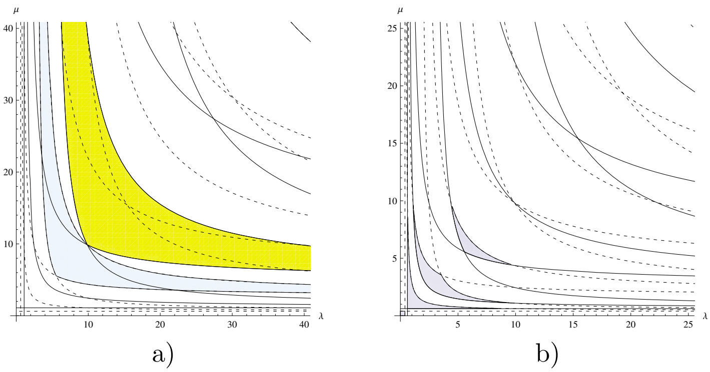

Figure 3. Several first branches of the Fučík spectra for problems (3.1) (dashed curves) and (3.2) (solid curves) are given; a) $D(k)_3$ (on the left) and $D(l)_3$ for $k/l = (4/3)^2$ are shaded, and their intersection is empty (a "common" point does not belong to $D(k)_3 \bigcap D(l)_3$ since both sets are open; b) "Good" regions $D(l)_0 \bigcap D(k)_0$, $D(l)_1 \bigcap D(k)_1$, $D(l)_2 \bigcap D(k)_2$ and $D(l)_3 \bigcap D(k)_3$ for $k/l = (5/4)^2$.

Therefore for any ratio k/l the problem (1.1) is solvable if

$$(\lambda, \mu) \in D(k)_0 \cap D(l)_0 = D(k)_0.$$

Not the case for $i > 0$. Generally, if $\frac{k}{l}$ is too large, the intersection of $D(k)_i$ and $D(l)_i$ is empty. The precise values of k/l for any $i = 1, 2, \ldots$ are given below.

Proposition 1. *If* $1 < k/l < (i + 1/i)^2$ *then* $D(k)_i \cap D(l)_i \neq \varnothing$, $i = 1, 2, \ldots$. *If* $k/l \geq (i + 1/i)^2$, *then* $D(k)_i \cap D(l)_i = \varnothing$.

The proof by elementary geometrical considerations.

Corollary 1. If $k/l \geq (i + 1/i)^2$ then $D(k)_j \cap D(l)_j = \varnothing$ for any $j \geq i$.

Therefore there exist only finite non-empty intersections $D(k)_j \cap D(l)_j$, if $k > l > 0$.

Corollary 2. If $k/l < (i + 1/i)^2$ then $D(k)_j \cap D(l)_j \neq \varnothing$ for any $j < i$.

5 Example

Consider the problem (1.1), where

$$f(x) = \frac{3}{20} x \left(8 + \frac{3x \sin(5x)}{1 + x^2} \right), \quad h(t, x, x') = \frac{1}{1 + t^2 x'^2}.$$

Then conditions (1.2) are fulfilled with $l = 0.8$ and $k = 1.5$, see Fig. 4a).

Let $D(k)_i$ be a "good" region where the IVPs

$$x'' + \lambda k x^+ - \mu k x^- = 0, \quad x(0) = 0, \quad x'(0) = \pm 1$$

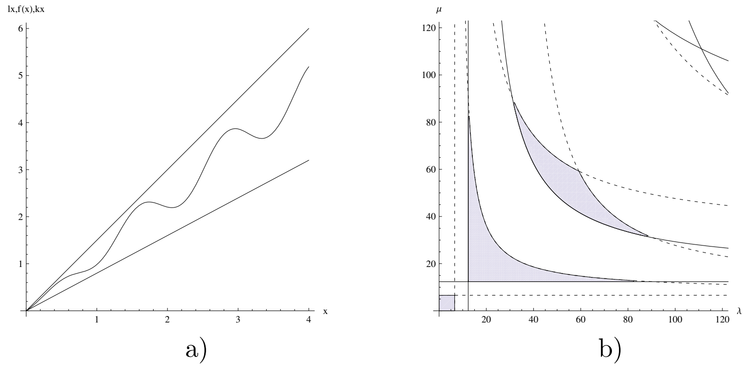

Figure 4. a) The graphs of $f(x)$ and the linear functions kx and lx, $k = 1.5$, $l = 0.8$; b) Intersections $D(l)_i \bigcap D(k)_i$, $k = 0, 1, 2$.

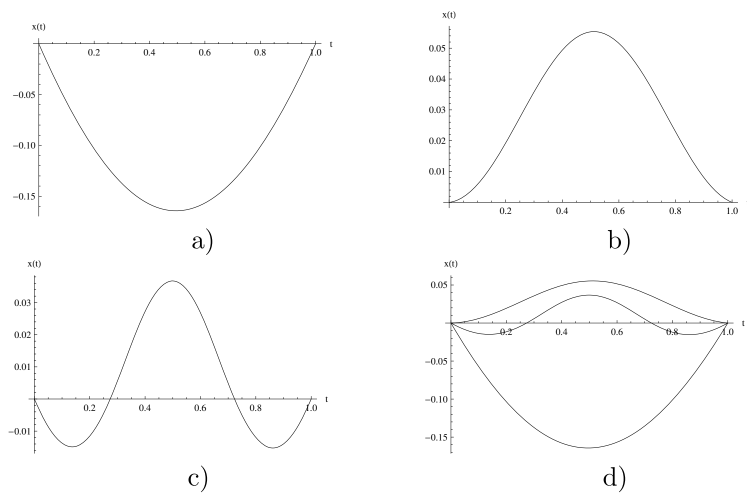

Figure 5. Solutions of BVP (1.1) where f and h are as in Example: a) $(\lambda, \mu) = (4, 2) \in D_0$, $x'(0) = -0.627431$; b) $(\lambda, \mu) = (30, 15) \in D_1$, $x'(0) = 0.0103358$; c) $(\lambda, \mu) = (70, 40) \in D_2$, $x'(0) = -0.201086$. d) Three solutions of BVP (1.1).

have solutions with exactly i zeros in $(0, 1)$ and these solutions have opposite signs at $t = 1$.

There are countably many "good" regions $D(k)_i$ and $D(l)_i$ but only three intersections $D(k)_i \cap D(l)_i$ are non-empty, namely, for $i = 0, 1, 2$, see Fig. 4b). The corresponding solutions of BVP (1.1) are depicted in Fig. 5.

6 Conclusions

The problem

$$x'' = -\lambda x^+ + \mu x^- + h(t, x, x'), \quad x(0) = 0, \quad x(1) = 0$$

is solvable if (λ, μ) is in one of "good" regions (with respect to the Fučík spectrum Σ_F) depicted in Fig. 2 and h is bounded. There are infinite number of "good" regions.

The same is true for the problem

$$x'' = -\lambda f(x^+) + \mu f(x^-) + h(t, x, x'), \quad x(0) = 0, \quad x(1) = 0,$$

where $lx < f(x) < kx$ and some technical assumptions (mentioned in Theorem 1) are in force. The essential difference is that the number of "good" regions is always finite. If k is significantly greater than l then only one "good" region D_0 exists.

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