Mathematical Modelling and Analysis Publisher: Taylor&Francis and VGTU
Volume 17 Number 5, November 2012, 715–731 http://www.tandfonline.com/TMMA
http://dx.doi.org/10.3846/13926292.2012.736090 Print ISSN: 1392-6292

Online ISSN: 1648-3510

Periodic Orbits Near Equilibria via Averaging Theory of Second Order

Luis Barreira[a], Jaume Llibre[b] and Claudia Valls[a]

[a] *Instituto Superior Técnico*
Av. Rovisco Pais, 1049-001 Lisboa, Portugal
[b] *Universitat Autònoma de Barcelona*
08193 Bellaterra, Barcelona, Catalonia, Spain
E-mail(*corresp.*): barreira@math.ist.utl.pt
E-mail: jllibre@mat.uab.cat
E-mail: cvalls@math.ist.utl.pt

Received February 27, 2012; revised August 20, 2012; published online November 1, 2012

Abstract. Lyapunov, Weinstein and Moser obtained remarkable theorems giving sufficient conditions for the existence of periodic orbits emanating from an equilibrium point of a differential system with a first integral. Using averaging theory of first order we established in [1] a similar result for a differential system *without* assuming the existence of a first integral. Now, using averaging theory of the second order, we extend our result to the case when the first order average is identically zero. Our result can be interpreted as a kind of degenerated Hopf bifurcation.

Keywords: differential equation, nonlinear differential equation, perturbation method.

AMS Subject Classification: 37G15.

1 Introduction and Statement of the Main Results

Consider a system of ordinary differential equations

$$\dot{x} = f(x), \quad x = (x_1, \ldots, x_m) \tag{1.1}$$

near an equilibrium point, which we assume to be the origin $x = 0$. The variables x_k, for $k = 1, \ldots, m$, are real, and the dot refers to differentiation with respect to the independent variable t. For the equilibrium point $x = 0$, we consider the linear variational equation

$$\dot{x} = Ax, \quad A = f_x(0), \tag{1.2}$$

where $f_x(0)$ denotes the Jacobian matrix of the function f evaluated at $x = 0$. Clearly, every pair of conjugated purely imaginary eigenvalues of A gives rise to periodic solutions of (1.2). We consider the classical problem of finding periodic solutions near $x = 0$ for the nonlinear system (1.1). As it is well known, for this purpose the presence of purely imaginary eigenvalues is necessary but not sufficient. In 1907 Lyapunov [4] established the existence of a one-parameter family of periodic solutions under two assumptions. Namely, he assumed the existence of a first integral and a nonresonance condition on the purely imaginary eigenvalues of A (see Theorem 9.2.1 of [6]).

A special role in the theory is played by the Hamiltonian systems

$$\dot{x}_k = H_{x_{n+k}}, \qquad \dot{x}_{n+k} = -H_{x_k}, \quad k = 1, \dots, n,$$

where H_{x_l} denotes the partial derivative of the Hamiltonian $H(x_1, \dots, x_{2n})$ with respect to the variable x_l. In 1973 Weinstein [10, 11] showed that the additional nonresonance condition is not necessary for Hamiltonian systems. In 1976 Moser [7] established a similar result for system (1.1) assuming the existence of a first integral $H(x)$ with $H_x(0) = 0$ and positive definite Hessian $H_{xx}(0)$ without requiring the system to be Hamiltonian.

Our goal is to obtain similar results to those obtained by Lyapunov, Weinstein and Moser for system (1.1) but now *without* assuming the existence of a first integral. We consider vector fields $f = f_\varepsilon$ depending on a real parameter ε such that when $\varepsilon = 0$ the origin is an equilibrium point of system (1.1) with eigenvalues $\pm\omega i \neq 0$ and 0 with multiplicity $m - 2$. For such systems we provide sufficient conditions so that periodic orbits bifurcate from the origin when $\varepsilon \neq 0$ is sufficiently small. In [1] we used averaging theory of first order to establish a similar result for a differential system without assuming the existence of a first integral. In the present paper, using averaging theory of the second order, we extend the results of [1] to the case when the first order average is identically zero.

The classical Hopf bifurcation is a local bifurcation in which an equilibrium point of a differential system loses stability as a pair of complex conjugate eigenvalues of the linearization around the equilibrium point cross the imaginary axis of the complex plane. Under reasonably generic assumptions about the differential system, we can expect to see a small-amplitude limit cycle branching from the equilibrium point. We note that our results can be interpreted as a kind of special degenerate Hopf bifurcation because for $\varepsilon = 0$ we have a pair of complex conjugate eigenvalues of the linearization around the equilibrium point on the imaginary axis, but moving ε it is not necessary that this pair of complex eigenvalues cross the imaginary axis at $\varepsilon = 0$.

Our results are stated in subsection 1.2, and are proved in the following sections. The proofs use averaging theory of the second order. We refer to Section 1.1 for a summary of this theory.

1.1 Averaging theory of the second order

The next theorem provides a second order approximation for the limit cycles of a periodic system when its average vanishes at first order. For a statement see [5], and for a proof see Theorem 3.5.1 of Sanders and Verhulst [8], or [2].

Consider functions $f, g\colon [0, \infty) \times \Omega \to \mathbb{R}^n$ and $R\colon [0, \infty) \times \Omega \times (0, \varepsilon_0] \to \mathbb{R}^n$, where Ω is an open subset of $\mathbb{R}^n$, such that f, g and R are T-periodic in the first variable. We set

$$f^1(t,x) = \frac{\partial f}{\partial x}(t,x) y^1(t,x), \quad \text{where } y^1(t,x) = \int_0^t f(s,x)\,ds, \tag{1.3}$$

and we consider the averages of f, f^1 and g, defined respectively by

$$f^0(x) = \frac{1}{T}\int_0^T f(t,x)\,dt, \quad \left(f^1\right)^0(x) = \frac{1}{T}\int_0^T f^1(t,x)\,dt, \tag{1.4}$$
$$g^0(x) = \frac{1}{T}\int_0^T g(t,x)\,dt.$$

We also consider the two initial value problems

$$\dot{x} = \varepsilon f(t,x) + \varepsilon^2 g(t,x) + \varepsilon^3 R(t,x,\varepsilon), \quad x(0) = x_0, \tag{1.5}$$

and

$$\dot{y} = \varepsilon f^0(y) + \varepsilon^2 \left(\left(f^1\right)^0(y) + g^0(y)\right), \quad y(0) = x_0. \tag{1.6}$$

Theorem 1. *Assume that:* (i) $f^0 = 0$; (ii) $\partial f/\partial x$, g *and* R *are Lipschitz in* x, *and all these functions are continuous on their domain of definition*; (iii) $R(t,x,\varepsilon)$ *is bounded by a constant uniformly on* $[0, L/\varepsilon] \times \Omega \times (0, \varepsilon_0]$; *and* (iv) *the solution* $y(t)$ *belongs to* Ω *in the interval of time* $[0, 1/\varepsilon]$. *Then the following statements hold.*

(a) *At time scale* $1/\varepsilon$ *we have* $x(t) = y(t) + \varepsilon y^1\big(t, y(t)\big) + O\big(\varepsilon^2\big)$.

(b) *If* p *is an equilibrium point of the averaged system* (1.6) *with*

$$\det\left(\left(f^1\right)_y^0 + g_y^0\right)(p) \neq 0, \tag{1.7}$$

then there exists a limit cycle $\phi(t,\varepsilon)$ *of period* T *for system* (1.5) *that is close to* p, *such that* $\phi(0,\varepsilon) \to p$ *as* $\varepsilon \to 0$.

(c) *The stability or instability of the limit cycle* $\phi(t,\varepsilon)$ *is given respectively by the stability or instability of the equilibrium point* p *of system* (1.6).

Of course, there are other tools different to the averaging theory for studying the existence of periodic solutions, one of them is the degree theory, see for instance [3]. But both tools and almost all the tools for finding periodic solutions are based on the Poincaré return map.

1.2 Statement of the results

We formulate in this section our results for equation (1.1). Let us assume throughout the paper that the function f is of class $\mathcal{C}^3$ near $x = 0$, with

$$f(0) = \left(0, 0, \varepsilon^2\lambda_3 + \varepsilon^3\mu_3, \ldots, \varepsilon^2\lambda_m + \varepsilon^3\mu_m\right),$$

$$f_x(0) = \begin{pmatrix} 0 & -\omega & 0 & \cdots & 0 \\ \omega & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

and that the second and third derivatives $f_{xx}(0)$ and $f_{xxx}(0)$ are independent of ε, where $\omega, \lambda_3, \mu_3, \ldots, \lambda_m, \mu_m$ are real constants such that $\omega \neq 0$ and $\mu_3^2 + \cdots + \mu_m^2 \neq 0$. Under these standing assumptions, and introducing the notation

$$f_{ij}^l = f_{x_i x_j}^l(0) \quad \text{and} \quad f_{ijk}^l = f_{x_i x_j x_k}^l(0) \quad \text{for } l = 1, 2, \ldots, m$$

(which should not be confused with the notation in (1.3) and (1.4)), we can rewrite system (1.1) in the form

$$\begin{aligned} \dot{x}_1 &= -\omega x_2 + \sum_{i=1}^m \sum_{j=i}^m \delta_{ij} f_{ij}^1 x_i x_j + \sum_{i=1}^m \sum_{j=i}^m \sum_{k=j}^m \delta_{ijk} f_{ijk}^1 x_i x_j x_k \\ &\quad + O_4(x_1, \ldots, x_m, \varepsilon), \\ \dot{x}_2 &= \omega x_1 + \sum_{i=1}^m \sum_{j=i}^m \delta_{ij} f_{ij}^2 x_i x_j + \sum_{i=1}^m \sum_{j=i}^m \sum_{k=j}^m \delta_{ijk} f_{ijk}^2 x_i x_j x_k \\ &\quad + O_4(x_1, \ldots, x_m, \varepsilon), \\ \dot{x}_k &= \varepsilon^2 \lambda_k + \varepsilon^3 \mu_k + \sum_{i=1}^m \sum_{j=i}^m \delta_{ij} f_{ij}^k x_i x_j + \sum_{i=1}^m \sum_{j=i}^m \sum_{k=j}^m \delta_{ijk} f_{ijk}^k x_i x_j x_k \\ &\quad + O_4(x_1, \ldots, x_m, \varepsilon), \end{aligned} \tag{1.8}$$

for $k = 3, \ldots, m$, where $\delta_{ij} = 1$ for $i \neq j$, $\delta_{ii} = 1/2$, $\delta_{iii} = 1/6$, $\delta_{ijk} = 1/2$ for $i = j < k$ or $i < j = k$, and $\delta_{ijk} = 1$ for $i < j < k$. Moreover, each $O_4(x_1, \ldots, x_m, \varepsilon)$ denotes a term of order 4 in $x_1, \ldots, x_m$ and ε.

We introduce coordinates $(\rho, \theta, y_3, \ldots, y_m)$ in $\mathbb{R}^m$ satisfying $x_1 = \varepsilon\rho\cos\theta$, $x_2 = \varepsilon\rho\sin\theta$, and $x_k = \varepsilon y_k$ for $k = 3, \ldots, m$. Our main aim is to apply the averaging theory of the second order when the averaged system vanishes at the first order (see Section 1.1). In this case one is not able to apply the averaging theory of the first order to the system, or more precisely to the reduced system in $\mathbb{R}^{m-1}$ using the coordinates $\rho, y_3, \ldots, y_m$ in the region $\dot{\theta} \neq 0$, taking the variable θ as the new time. It is shown in [1] that the averaged system vanishes at the first order if and only if the following conditions hold:

(H1) $\lambda_j = 0$ for $j = 3, \ldots, m$;

(H2) $f_{1j}^1 = -f_{2j}^2$ for $j = 3, \ldots, m$;

(H3) $f_{11}^k = -f_{22}^k$ for $j = 3, \ldots, m$;

(H4) $f_{jl}^k = 0$ for $j = 3, \ldots, m$ and $l = j, \ldots, m$.

Thus, these will be standing assumptions in the paper.

The following result gives explicitly the averaged system of the second order for an arbitrary dimension m.

Theorem 2. *In coordinates* $(\rho, y_3, \ldots, y_m)$ *and in the region* $\dot{\theta} \neq 0$, *the averaged system of the second order of the differential system* (1.8) *is given by*

$$\begin{aligned}
\frac{d\rho}{d\theta} = \varepsilon^2 \Bigg[& \frac{\rho^3}{16\omega^2} \Bigg(f_{11}^1 f_{12}^1 + f_{12}^1 f_{22}^1 - f_{11}^1 f_{11}^2 - f_{11}^2 f_{12}^2 + f_{22}^1 f_{22}^2 - f_{12}^2 f_{22}^2 \\
& - \sum_{j=3}^{m} f_{2j}^1 f_{22}^j - \sum_{j=3}^{m} f_{1j}^2 f_{22}^j + 2 \sum_{j=3}^{m} f_{2j}^2 f_{12}^j \Bigg) \\
& + \frac{\rho}{2\omega^2} \sum_{i=3}^{m} \sum_{j=i}^{m} \delta_{ij} \left(f_{ij}^1 f_{12}^1 + f_{ij}^1 f_{22}^2 - f_{ij}^2 f_{11}^1 - f_{ij}^2 f_{12}^2 \right) y_i y_j \\
& + \frac{\rho}{\omega^2} \sum_{j=3}^{m} \sum_{i=j}^{m} \sum_{l=3}^{m} \delta_{ij} \left(f_{ij}^2 f_{1l}^j - f_{ij}^1 f_{2l}^j \right) y_i y_l \\
& + \frac{\rho^3}{16\omega} \left(f_{111}^1 + f_{122}^1 + f_{112}^2 + f_{222}^2 \right) \\
& + \frac{\rho}{2\omega} \sum_{i=3}^{m} \sum_{j=i}^{m} \left(\delta_{1ij} f_{1ij}^1 + \delta_{2ij} f_{2ij}^2 \right) y_i y_j \Bigg], \\
\frac{dy_k}{d\theta} = \varepsilon^2 \Bigg[& \mu_k - \frac{1}{\omega^2} \sum_{i=3}^{m} \sum_{j=i}^{m} \sum_{l=3}^{m} \delta_{ij} \left(f_{ij}^2 f_{1l}^k - f_{ij}^1 f_{2l}^k \right) y_i y_j y_l \\
& + \frac{\rho^2}{4\omega^2} \sum_{j=3}^{m} \big(f_{2j}^1 f_{22}^k + f_{1j}^2 f_{22}^k - 2 f_{2j}^2 f_{12}^k \\
& - f_{11}^2 f_{1j}^k - f_{22}^2 f_{1j}^k + f_{22}^k f_{2j}^k + f_{22}^1 f_{2j}^k \big) y_j \\
& + \frac{\rho^2}{4\omega} \sum_{j=3}^{m} \left(f_{11j}^k + f_{22j}^k \right) y_j + \frac{1}{\omega} \sum_{i=3}^{m} \sum_{j=i}^{m} \sum_{l=j}^{m} \delta_{ijl} f_{ijl}^k y_i y_j y_l \Bigg],
\end{aligned} \tag{1.9}$$

for $k = 3, \ldots, m$.

Theorem 2 is proved in Section 2.

By Theorem 1, looking for the equilibrium points of system (1.9) satisfying condition (1.7) we obtain periodic orbits of system (1.8). Note that due to the change of coordinates $x_1 = \varepsilon\rho\cos\theta$, $x_2 = \varepsilon\rho\sin\theta$, and $x_k = \varepsilon y_k$ for $k = 3, \ldots, m$, those periodic orbits tend to the equilibrium point located at the origin of coordinates of system (1.8) when $\varepsilon \to 0$. This also justifies in some sense that we are studying a class of degenerate Hopf bifurcation. Moreover, taking into account that the radial polar coordinate ρ is only well defined when $\rho > 0$, we are only interested in the equilibrium points $(\rho, y_3, \ldots, y_m)$ of system (1.9) with $\rho > 0$.

Corollary 1. *Assume that the differential system* (1.9) *satisfying* (H1)–(H4) *has finitely many equilibrium points. Then for any sufficiently small* $\varepsilon \neq 0$ *there* *are at most* 3^{m-1} *periodic orbits bifurcating from the origin of system* (1.8) *using the averaging theory of the second order.*

The proof of Corollary 1 follows directly from the fact that every equilibrium point of system (1.9) satisfying condition (1.7) corresponds to a periodic orbit of system (1.8) which tends to the origin of this system when $\varepsilon \to 0$, and from the Bézout's Theorem (see for instance [9]) applied to system (1.9).

In dimension 3 we can be more precise. Set

$$\begin{aligned}
\Delta_1 &= \frac{1}{16\omega^2}\big[f^1_{11}f^1_{12} + f^1_{12}f^1_{22} - f^1_{11}f^2_{11} - f^2_{11}f^2_{12} + f^1_{22}f^2_{22} - f^2_{12}f^2_{22}\\
&\quad - f^1_{23}f^3_{22} - f^2_{13}f^3_{22} + 2f^2_{23}f^3_{12} + \omega\big(f^1_{111} + f^1_{122} + f^2_{112} + f^2_{222}\big)\big],\\
\Delta_2 &= \frac{\rho}{4\omega^2}\big(f^1_{33}f^1_{12} + f^1_{33}f^2_{22} - f^2_{33}f^1_{11} - f^2_{33}f^2_{12}\big) + \frac{1}{2}\big(f^2_{33}f^3_{13} - f^1_{33}f^3_{23}\big)\\
&\quad + 16\omega\big(f^1_{133} + f^2_{233}\big),\\
\Delta_3 &= \frac{1}{4\omega^2}\big(f^1_{23}f^3_{22} + f^2_{13}f^3_{22} - 2f^2_{23}f^3_{12} - f^2_{11}f^3_{13} - f^2_{22}f^3_{13} + f^3_{22}f^3_{23} + f^1_{22}f^3_{23}\big)\\
&\quad + \frac{4}{\omega}f^3_{113},\\
\Delta_4 &= -\frac{1}{2\omega^2}\big(f^2_{33}f^3_{13} - f^1_{33}f^3_{23}\big) + \frac{1}{6\omega}f^3_{333}.
\end{aligned}$$

Then we have the following statement.

Theorem 3. *Under the assumptions of Section* 1.2*, if* $\mu_3 \neq 0$*,* $\Delta_1\Delta_2 < 0$ *and* $\Delta_2\Delta_3 - \Delta_1\Delta_4 \neq 0$*, then for any sufficiently small* $\varepsilon \neq 0$ *system* (1.8) *has a periodic solution which is close to the circle of radius* $\varepsilon\big(|\omega\mu_3|^{1/3}\sqrt{|\Delta_2|}\big)/\big(|\Delta_1|^{3/2}|\Delta_2\Delta_3 - \Delta_1\Delta_4|\big)$.

Theorem 3 is proved in Section 3.

The following system in $\mathbb{R}^4$ satisfies the assumptions of subsection 1.2, and has three periodic orbits.

Example 1. The system

$$\begin{aligned}
\dot{x}_1 &= -\omega x_2 + x_1x_2 + x_4^2 + \frac{1}{\omega}x_1x_3^2 + \frac{1}{\omega}x_1x_3x_4 - \frac{4}{3\omega}x_1^3,\\
\dot{x}_2 &= \omega x_1,\\
\dot{x}_3 &= \varepsilon^3\mu_3 + x_4x_3^2\mu_3 - 2\mu_3x_1^2x_3 - 2\mu_3x_4x_1^2,\\
\dot{x}_4 &= \varepsilon^3\mu_4 + x_2x_3 - 2\mu_4x_1^2x_3 - 2\mu_4x_4x_1^2 + \frac{1}{\omega}(-1 + \mu_4\omega)x_4x_3^2,
\end{aligned} \tag{1.10}$$

has three limit cycles bifurcating from the origin for $\varepsilon \neq 0$ sufficiently small.

The details of the example are given in Section 4.

2 Proof of Theorem 2

Equivalent form of system (1.8)

Under the standing assumptions in Section 1.2 we can write system (1.1) as in (1.8). Furthermore, using the conditions (H1)–(H4) we can rewrite system (1.8) in the form

$$\begin{aligned}\dot{x}_l &= (-1)^l \omega x_{3-l} + \frac{1}{2} f^l_{11} x_1^2 + f^l_{12} x_1 x_2 + (l-2) \sum_{j=3}^{m} f^2_{2j} x_1 x_j \\ &+ (l-1) \sum_{j=3}^{m} f^2_{1j} x_1 x_j + \frac{1}{2} f^l_{22} x_2^2 + \sum_{j=3}^{m} f^l_{2j} x_2 x_j + \sum_{i=3}^{m} \sum_{j=i}^{m} \delta_{ij} f^l_{ij} x_i x_j \\ &+ \frac{1}{6} f^l_{111} x_1^3 + \frac{1}{2} f^l_{112} x_1^2 x_2 + \frac{1}{2} \sum_{j=3}^{m} f^l_{11j} x_1^2 x_j \\ &+ \frac{1}{2} f^l_{122} x_1 x_2^2 + \sum_{j=3}^{m} f^l_{12j} x_1 x_2 x_j + \sum_{j=3}^{m} \sum_{k=j}^{m} \delta_{1jk} f^l_{1jk} x_1 x_j x_k \\ &+ \frac{1}{6} f^l_{222} x_2^3 + \frac{1}{2} \sum_{j=3}^{m} f^l_{22j} x_2^2 x_j + \sum_{j=3}^{m} \sum_{k=j}^{m} \delta_{2jk} f^l_{2jk} x_2 x_j x_k \\ &+ \sum_{i=3}^{m} \sum_{j=3}^{m} \sum_{k=j}^{m} \delta_{ijk} f^l_{ijk} x_i x_j x_k + O_4(x_1, \dots, x_m, \varepsilon),\end{aligned}$$

for $l = 1, 2$, and

$$\begin{aligned}\dot{x}_k &= \varepsilon^3 \mu_k - \frac{1}{2} f^k_{22} \big(x_1^2 - x_2^2\big) + f^k_{12} x_1 x_2 + \sum_{j=3}^{m} f^k_{1j} x_1 x_j \\ &+ \sum_{j=3}^{m} f^k_{2j} x_2 x_j + \frac{1}{6} f^k_{111} x_1^3 + \frac{1}{2} f^k_{112} x_1^2 x_2 \\ &+ \frac{1}{2} \sum_{j=3}^{m} f^k_{11j} x_1^2 x_j + \frac{1}{2} f^k_{122} x_1 x_2^2 + \sum_{j=3}^{m} f^k_{12j} x_1 x_2 x_j \\ &+ \sum_{j=3}^{m} \sum_{k=j}^{m} \delta_{1jk} f^k_{1jk} x_1 x_j x_k + \frac{1}{6} f^k_{222} x_2^3 + \frac{1}{2} \sum_{j=3}^{m} f^k_{22j} x_2^2 x_j \\ &+ \sum_{j=3}^{m} \sum_{k=j}^{m} \delta_{2jk} f^k_{2jk} x_2 x_j x_k + \sum_{i=3}^{m} \sum_{j=3}^{m} \sum_{k=j}^{m} \delta_{ijk} f^k_{ijk} x_i x_j x_k \\ &+ O_4(x_1, \dots, x_m, \varepsilon),\end{aligned}$$

for $k = 3, \dots, m$.

Introduction of new variables

Since we want to apply averaging theory, we introduce the change of coordinates

$$x_1 = r\cos\theta, \qquad x_2 = r\sin\theta, \qquad x_k = x_k, \quad k = 3, \dots, m,$$

after which we can rewrite system (1.8) in the form

$$\begin{aligned}\dot r &= T_{11}(\theta, r, x_3, \ldots, x_m) + T_{12}(\theta, r, x_3, \ldots, x_m)\\ &\quad + H_1(\theta, r, x_3, \ldots, x_m) + O_4(r, x_3, \ldots, x_m),\\ \dot\theta &= \omega + T_2(\theta, r, x_3, \ldots, x_m) + H_2(\theta, r, x_3, \ldots, x_m)/r,\\ \dot x_k &= \varepsilon^3\mu_k + T_{k1}(\theta, r, x_3, \ldots, x_m) + T_{k2}(\theta, r, x_3, \ldots, x_m)\\ &\quad + H_k(\theta, r, x_3, \ldots, x_m) + O_4(r, x_3, \ldots, x_m),\end{aligned} \tag{2.1}$$

for $k = 3, \ldots, m$, where

$$\begin{aligned}T_{11} &= r^2\left[\frac{1}{2}f_{11}^1\cos^3\theta + \left(f_{12}^1 + \frac{1}{2}f_{11}^2\right)\cos^2\theta\sin\theta\right.\\ &\quad\left. + \left(\frac{1}{2}f_{22}^1 + f_{12}^2\right)\cos\theta\sin^2\theta + \frac{1}{2}f_{22}^2\sin^3\theta\right]\\ &\quad + r\sum_{j=3}^m\left[f_{2j}^2\left(\sin^2\theta - \cos^2\theta\right) + \left(f_{2j}^1 + f_{1j}^2\right)\cos\theta\sin\theta\right]x_j\\ &\quad + \sum_{i=3}^m\sum_{j=i}^m \delta_{ij}\left(f_{ij}^1\cos\theta + f_{ij}^2\sin\theta\right)x_ix_j,\\ T_{12} &= r^3\left[\frac{1}{6}f_{111}^1\cos^4\theta + \frac{1}{2}\left(f_{122}^1 + f_{112}^2\right)\cos^2\theta\sin^2\theta + \frac{1}{6}f_{222}^2\sin^4\theta\right]\\ &\quad + r\sum_{i=3}^m\sum_{j=i}^m\left(f_{1ij}^1\cos^2\theta + f_{2ij}^1\sin^2\theta\right)x_ix_j,\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}T_2 &= r\left[\frac{1}{2}f_{11}^2\cos^3\theta + \left(f_{12}^2 - \frac{1}{2}f_{11}^1\right)\cos^2\theta\sin\theta\right.\\ &\quad\left. + \left(\frac{1}{2}f_{22}^2 - f_{12}^1\right)\cos\theta\sin^2\theta - \frac{1}{2}f_{22}^1\sin^3\theta\right]\\ &\quad + \sum_{j=3}^m\left(f_{1j}^2\cos^2\theta - f_{2j}^1\sin^2\theta + 2f_{2j}^2\cos\theta\sin\theta\right)x_j\\ &\quad + \frac{1}{r}\sum_{i=3}^m\sum_{j=i}^m\delta_{ij}\left(f_{ij}^2\cos\theta - f_{ij}^1\sin\theta\right)x_ix_j,\end{aligned}$$

and where for $k = 3, \ldots, m$,

$$\begin{aligned}T_{k1} &= r^2\left[-\frac{1}{2}f_{22}^k\left(\cos^2\theta - \sin^2\theta\right) + f_{12}^k\cos\theta\sin\theta\right]\\ &\quad + r\sum_{j=3}^m\left(f_{1j}^k\cos\theta + f_{2j}^k\sin\theta\right)x_j,\end{aligned} \tag{2.3}$$

and

$$T_{k2} = \frac{r^2}{2}\sum_{j=3}^m\left(f_{11j}^k\cos^2\theta + f_{22j}^k\sin^2\theta\right)x_j + \sum_{i=3}^m\sum_{j=i}^m\sum_{l=j}^m\delta_{ijl}f_{ijl}^kx_ix_jx_l.$$

Moreover, $H_1 = H_1(\theta, r, x_3, \ldots, x_m)$ and $H_k = H_k(\theta, r, x_3, \ldots, x_m)$ for $k = 3, \ldots, m$ are sums of terms of order 3 in $r, x_3, \ldots, x_m$, multiplied by a function of θ among the ones in

$$\begin{gathered}\cos\theta,\ \sin\theta,\ \cos\theta\sin\theta,\ \cos^3\theta,\ \sin^3\theta,\\ \cos^2\theta\sin\theta,\ \cos\theta\sin^2\theta,\ \cos^3\theta\sin\theta,\ \cos\theta\sin^3\theta.\end{gathered} \tag{2.4}$$

This readily implies that

$$\int_0^{2\pi} H_1(\theta, r, x_3, \ldots, x_m)\, d\theta = \int_0^{2\pi} H_k(\theta, r, x_3, \ldots, x_m)\, d\theta = 0,$$

for $k = 3, \ldots, m$. Finally, $H_2 = H_2(\theta, r, x_3, \ldots, x_m)$ is a function at least of order 3 in the variables $r, x_3, \ldots, x_m$.

We note that in the expression of $\dot\theta$ in (2.1) we have that T_2 is a function at least of order 1 and H_2 of order at least 3 in the variables $r, x_3, \ldots, x_m$, so sufficiently close the origin $\dot\theta \approx \omega$, and consequently our results work at least in a neighborhood of the origin.

Reduction of the system to form (1.5)

In the region $\dot\theta \neq 0$ system (2.1) yields the equations

$$\begin{aligned}\frac{dr}{d\theta} &= \frac{T_{11} + T_{12} + H_1 + O_4(r, \theta, x_3, \ldots, x_m)}{\omega + T_2 + H_2/r},\\ \frac{dx_k}{d\theta} &= \frac{\varepsilon^3\mu_k + T_{k1} + T_{k2} + H_k + O_4(r, \theta, x_3, \ldots, x_m)}{\omega + T_2 + H_2/r},\end{aligned} \tag{2.5}$$

for $k = 3, \ldots, m$, where for simplicity we have omitted the dependence on the variables θ, r, $x_3, \ldots, x_m$. Here, each $O_4(r, \theta, x_3, \ldots, x_m)$ denotes a term of order 4 in $r, x_3, \ldots, x_m$. We note that this system is 2π-periodic in the independent variable θ. Moreover, performing the rescaling $(r, x_3, \ldots, x_m) = \varepsilon(\rho, y_3, \ldots, y_m)$, system (2.5) has the appropriate form to apply averaging theory. Namely, setting

$$\begin{aligned}\tilde T_{k1}(\theta, \rho, y_3, \ldots, y_m) &= \frac{1}{\varepsilon^2} T_{k1}(\theta, r, x_3, \ldots, x_m),\\ \tilde T_{k2}(\theta, \rho, y_3, \ldots, y_m) &= \frac{1}{\varepsilon^3} T_{k2}(\theta, r, x_3, \ldots, x_m),\\ \tilde T_2(\theta, \rho, y_3, \ldots, y_m) &= \frac{1}{\varepsilon} T_2(\theta, r, x_3, \ldots, x_m),\end{aligned}$$

in the variables $\rho, y_3, \ldots, y_m$ the system has the form

$$\begin{aligned}\frac{d\rho}{d\theta} &= \varepsilon f_{11}(\theta, \rho, y_3, \ldots, y_m) + \varepsilon^2 f_{12}(\theta, \rho, y_3, \ldots, y_m)\\ &\quad + \varepsilon^3 g_1(\theta, \rho, y_3, \ldots, y_m, \varepsilon),\\ \frac{dy_k}{d\theta} &= \varepsilon f_{k1}(\theta, \rho, y_3, \ldots, y_m) + \varepsilon^2 f_{k2}(\theta, \rho, y_3, \ldots, y_m)\\ &\quad + \varepsilon^3 g_k(\theta, \rho, y_3, \ldots, y_m, \varepsilon),\end{aligned} \tag{2.6}$$

for some functions $g_1, g_3, \dots, g_m$, where

$$f_{11}(\theta,\rho,y_3,\dots,y_m) = \frac{\tilde{T}_{11}(\theta,\rho,y_3,\dots,y_m)}{\omega},$$
$$f_{12}(\theta,\rho,y_3,\dots,y_m) = \frac{\tilde{T}_{12}(\theta,\rho,y_3,\dots,y_m) + \tilde{H}_1(\theta,\rho,y_3,\dots,y_m)}{\omega} - \frac{\tilde{T}_2(\theta,\rho,y_3,\dots,y_m)\tilde{T}_{11}(\theta,\rho,y_3,\dots,y_m)}{\omega^2},$$

and

$$f_{k1}(\theta,\rho,y_3,\dots,y_m) = \frac{\tilde{T}_{k1}(\theta,\rho,y_3,\dots,y_m)}{\omega},$$
$$f_{k2}(\theta,\rho,y_3,\dots,y_m) = \frac{\mu_k + \tilde{T}_{k2}(\theta,\rho,y_3,\dots,y_m) + \tilde{H}_k(\theta,\rho,y_3,\dots,y_m)}{\omega} - \frac{\tilde{T}_2(\theta,\rho,y_3,\dots,y_m)\tilde{T}_{k1}(\theta,\rho,y_3,\dots,y_m)}{\omega^2},$$

for $k = 3,\dots,m$. Moreover, each $\tilde{H}_k = \tilde{H}_k(\theta,\rho,y_3,\dots,y_m)$ is a sum of terms of order 3 in $\rho, y_3,\dots,y_m$, multiplied by a function of θ among the ones in (2.4). We note that

$$\int_0^{2\pi} f_{j1}(\theta,\rho,y_3,\dots,y_m)\,d\theta = \int_0^{2\pi} \tilde{H}_j(\theta,\rho,y_3,\dots,y_m)\,d\theta = 0,$$

for $j = 1,3,4,\dots,m$ (the first integral vanishes in view of the conditions (H1)–(H4). Thus, system (2.6) can be written as system (1.5) taking $x = (\rho, y_3,\dots,y_k)$, $t = \theta$, $T = 2\pi$,

$$f = (f_{11}, f_{31},\dots,f_{m1}), \qquad g = (f_{12}, f_{32},\dots,f_{m2}), \qquad R = (g_1, g_3,\dots,g_m).$$

It is easy to verify that system (2.6) satisfies the assumptions of the averaging theory of the second order described in Theorem 1, taking $\Omega = U \cap (\mathbb{R}^+ \times \mathbb{R}^{m-2})$ for some open disc U centered at the origin in $\mathbb{R}^{m-1}$, and taking $\varepsilon_0 > 0$ sufficiently small.

Functions in Theorem 1

To apply Theorem 1 (see (1.7)) we need to compute the functions g and f^1 (see (1.3)), which we write in the form

$$g = \begin{pmatrix} f_{12}(\rho,y_3,\dots,y_m) \\ f_{32}(\rho,y_3,\dots,y_m) \\ \vdots \\ f_{m2}(\rho,y_3,\dots,y_m) \end{pmatrix} \quad \text{and} \quad f^1 = \begin{pmatrix} F_1(\rho,y_3,\dots,y_m) \\ F_3(\rho,y_3,\dots,y_m) \\ \vdots \\ F_m(\rho,y_3,\dots,y_m) \end{pmatrix},$$

for some functions $F_1, F_3,\dots,F_m$. We also let

$$G_i(\rho,y_3,\dots,y_m) = \frac{1}{2\pi}\int_0^{2\pi} \left[f_{i2}(\theta,\rho,y_3,\dots,y_m) + F_i(\theta,\rho,y_3,\dots,y_m)\right] d\theta. \quad (2.7)$$

Now we start computing the functions G_i, which are the components of the sum $g^0 + (f^1)^0$ (see (1.6)). We first observe that

$$\frac{1}{2\pi}\int_0^{2\pi} f_{12}(\theta,\rho,y_3,\ldots,y_m)\,d\theta = \frac{1}{2\pi\omega}\int_0^{2\pi} \tilde{T}_{12}(\theta,\rho,y_3,\ldots,y_m)\,d\theta - \frac{1}{2\pi\omega^2}\int_0^{2\pi} \tilde{T}_{11}(\theta,\rho,y_3,\ldots,y_m)\tilde{T}_2(\theta,\rho,y_3,\ldots,y_m)\,d\theta, \tag{2.8}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} f_{k2}(\theta,\rho,y_3,\ldots,y_m)\,d\theta = \mu_k + \frac{1}{2\pi\omega}\int_0^{2\pi} \tilde{T}_{k2}(\theta,\rho,y_3,\ldots,y_m)\,d\theta - \frac{1}{2\pi\omega^2}\int_0^{2\pi} \tilde{T}_{k1}(\theta,\rho,y_3,\ldots,y_m)\tilde{T}_2(\theta,\rho,y_3,\ldots,y_m)\,d\theta,$$

for $k = 3,\ldots,m$. Furthermore, following (1.3) we let

$$Y_j = Y_j(\theta,\rho,y_3,\ldots,y_m) = \int_0^{\theta} \tilde{T}_{j1}(\psi,\rho,y_3,\ldots,y_m)\,d\psi, \tag{2.9}$$

for $j = 1,3,\ldots,m$. Then the function y^1 in (1.3) is given by

$$y^1 = \frac{1}{\omega}(Y_1, Y_3,\ldots,Y_m).$$

Moreover

$$F_i(\theta,\rho,y_3,\ldots,y_m) = \frac{1}{\omega}\left(\frac{\partial f_{i1}}{\partial\rho}Y_1 + \sum_{j=3}^{m}\frac{\partial f_{i1}}{\partial y_j}Y_j\right). \tag{2.10}$$

Computation of the functions G_i

Now we proceed with the explicit computation of the integrals giving the functions G_i. We first observe that

$$\frac{1}{2\pi\omega}\int_0^{2\pi} \tilde{T}_{12}(\theta,\rho,y_3,\ldots,y_m)\,d\theta = \frac{1}{16\omega}\Big[\rho^3\big(f_{111}^1 + f_{122}^1 + f_{112}^2 + f_{222}^2\big) + 8\rho\sum_{i=3}^{m}\sum_{j=i}^{m}\big(\delta_{1ij}f_{1ij}^1 + \delta_{2ij}f_{2ij}^2\big)y_iy_j\Big]. \tag{2.11}$$

Moreover, with the help of an algebraic manipulator such as Mathematica we obtain

$$-\frac{1}{2\pi\omega^2}\int_0^{2\pi} \tilde{T}_{11}(\theta,\rho,y_3,\ldots,y_m)\tilde{T}_2(\theta,\rho,y_3,\ldots,y_m)\,d\theta$$
$$= \frac{\rho^3}{8}\left(\frac{1}{2}f_{11}^1f_{12}^1 + \frac{1}{2}f_{12}^1f_{22}^1 - \frac{1}{2}f_{11}^1f_{11}^2 - \frac{1}{2}f_{11}^2f_{12}^2 + \frac{1}{2}f_{22}^1f_{22}^2 - \frac{1}{2}f_{12}^2f_{22}^2\right)$$

$$+\frac{\rho}{4}\sum_{i=3}^{m}\sum_{j=i}^{m}\delta_{ij}\Big(f_{ij}^1f_{12}^1-\frac{1}{2}f_{ij}^1f_{11}^2+\frac{1}{2}f_{ij}^1f_{22}^2$$
$$-\frac{1}{2}f_{ij}^2f_{11}^1+\frac{1}{2}f_{ij}^2f_{22}^1-f_{ij}^2f_{12}^2\Big)y_iy_j. \tag{2.12}$$

Adding (2.11) and (2.12) we obtain the first component of g^0. Now we consider the remaining components. For $k=3,\dots,m$ we have

$$\frac{1}{2\pi\omega}\int_0^{2\pi}\tilde{T}_{k2}(\theta,\rho,y_3,\dots,y_m)\,d\theta$$
$$=\frac{1}{\omega}\left[\frac{\rho^2}{4}\sum_{j=3}^{m}(f_{11j}^k+f_{22j}^k)y_j+\sum_{i=3}^{m}\sum_{j=i}^{m}\sum_{l=j}^{m}\delta_{ijl}f_{ijl}^ky_iy_jy_l\right],$$

and after some tedious computations we obtain

$$-\frac{1}{2\pi\omega^2}\int_0^{2\pi}\tilde{T}_{k1}(\theta,\rho,y_3,\dots,y_m)\tilde{T}_2(\theta,\rho,y_3,\dots,y_m)\,d\theta$$
$$=-\frac{1}{2\omega^2}\sum_{i=3}^{m}\sum_{j=i}^{m}\sum_{l=3}^{m}\delta_{i,j}\big(f_{ij}^2f_{1l}^k-f_{ij}^1f_{2l}^k\big)y_iy_jy_l$$
$$+\frac{\rho^2}{8\omega^2}\sum_{j=3}^{m}\Big(f_{2j}^1f_{22}^k+f_{1j}^2f_{22}^k-2f_{2j}^2f_{12}^k+f_{12}^1f_{1j}^k$$
$$-\frac{3}{2}f_{11}^2f_{1j}^k-\frac{1}{2}f_{22}^2f_{1j}^k+\frac{1}{2}f_{22}^kf_{2j}^k+\frac{3}{2}f_{22}^1f_{2j}^k-f_{12}^2f_{2j}^k\Big).$$

These formulas conclude the computation of the integral in (2.8).

Now we compute the integrals $\int_0^{2\pi}F_i(\theta,\rho,y_3,\dots,y_m)\,d\theta$. Setting $j=1$ in (2.9) we obtain

$$Y_1=Y_1(\theta,\rho,y_3,\dots,y_m)=\int_0^{\theta}\tilde{T}_{11}(\psi,\rho,y_3,\dots,y_m)\,d\psi,$$

and it follows from (2.2) that

$$Y_1=\frac{\rho^2}{12}\bigg[-3\Big(f_{12}^1+\frac{1}{2}f_{11}^2+\frac{3}{2}f_{22}^2\Big)\cos\theta-\Big(f_{12}^1+\frac{1}{2}f_{11}^2-\frac{1}{2}f_{22}^2\Big)\cos(3\theta)$$
$$+3\Big(\frac{3}{2}f_{11}^1+\frac{1}{2}f_{22}^1+f_{12}^2\Big)\sin\theta+\Big(\frac{1}{2}f_{11}^1-\frac{1}{2}f_{22}^1-f_{12}^2\Big)\sin(3\theta)\bigg]$$
$$-\frac{\rho}{4}\sum_{j=3}^{m}\big[\big(f_{2j}^1+f_{1j}^2\big)y_j\cos(2\theta)+2f_{2j}^2y_j\sin(2\theta)\big]$$
$$+\sum_{i=3}^{m}\sum_{j=i}^{m}\delta_{ij}\big(f_{ij}^1\sin\theta-f_{ij}^2\cos\theta\big)y_iy_j. \tag{2.13}$$

Moreover, for $j=3,\dots,m$ it follows from (2.9) and (2.3) that

$$Y_j=-\frac{\rho^2}{4}\big[f_{12}^k\cos(2\theta)+f_{22}^k\sin(2\theta)\big]-\rho\sum_{j=3}^{m}\big(f_{2j}^k\cos\theta-f_{1j}^k\sin\theta\big)y_j. \tag{2.14}$$

Now we observe that by (2.2) and (2.3),

$$\frac{\partial \tilde{T}_{11}}{\partial \rho} = 2\rho\Big[\frac{1}{2}f_{11}^1\cos^3\theta + \Big(f_{12}^1 + \frac{1}{2}f_{11}^2\Big)\cos^2\theta\sin\theta + \Big(\frac{1}{2}f_{22}^1 + f_{12}^2\Big)\cos\theta\sin^2\theta$$
$$+ \frac{1}{2}f_{22}^2\sin^3\theta\Big] + \sum_{j=3}^{m}\big[f_{2j}^2(\sin^2\theta - \cos^2\theta) + \big(f_{2j}^1 + f_{1j}^2\big)\cos\theta\sin\theta\big]y_j,$$

$$\frac{\partial \tilde{T}_{11}}{\partial y_j} = \rho\big[f_{2j}^2(\sin^2\theta - \cos^2\theta) + \big(f_{2j}^1 + f_{1j}^2\big)\cos\theta\sin\theta\big]$$
$$+ 2\sum_{i=j}^{m}\delta_{ji}\big(f_{ji}^1\cos\theta + f_{ji}^2\sin\theta\big)y_i,$$

and

$$\frac{\partial \tilde{T}_{k1}}{\partial \rho} = 2\rho\Big[-\frac{1}{2}f_{22}^k(\cos^2\theta - \sin^2\theta) + f_{12}^k\cos\theta\sin\theta\Big]$$
$$+ \sum_{j=3}^{m}\big(f_{1j}^k\cos\theta + f_{2j}^k\sin\theta\big)y_j,$$
$$\frac{\partial \tilde{T}_{k1}}{\partial y_j} = \rho\big(f_{1j}^k\cos\theta + f_{2j}^k\sin\theta\big),$$

for $k = 3, \ldots, m$.

After straightforward computations it follows from (2.13) and (2.14) that

$$\frac{1}{2\pi\omega^2}\int_0^{2\pi}\frac{\partial \tilde{T}_{11}}{\partial \rho}Y_1\,d\theta = \frac{\rho}{4\omega^2}\sum_{i=3}\sum_{j=i}^{m}\delta_{ij}\Big(f_{ij}^1f_{12}^1 + \frac{1}{2}f_{ij}^1f_{11}^2 + \frac{3}{2}f_{ij}^1f_{22}^2$$
$$- \frac{3}{2}f_{ij}^2f_{11}^1 - \frac{1}{2}f_{ij}^2f_{22}^1 - f_{12}^2f_{ij}^2\Big)y_iy_j,$$

$$\frac{1}{2\pi\omega^2}\sum_{j=3}^{m}\int_0^{2\pi}\frac{\partial \tilde{T}_{11}}{\partial y_j}Y_j\,d\theta = \frac{\rho^3}{8\omega^2}\sum_{j=3}^{m}\Big(-\frac{1}{2}f_{2j}^1f_{22}^j - \frac{1}{2}f_{1j}^2f_{22}^j + f_{2j}^2f_{12}^j\Big)$$
$$+ \frac{\rho}{\omega^2}\sum_{j=3}^{m}\sum_{i=j}^{m}\sum_{l=3}^{m}\delta_{ij}\big(f_{ij}^2f_{1l}^j - f_{ij}^1f_{21}^j\big)y_iy_l.$$

Summarizing, by (2.10) we conclude that

$$\frac{\omega^2}{2\pi}\int_0^{2\pi}F_1(\rho, y_3, \ldots, y_m)\,d\theta$$
$$= \frac{\rho^3}{8}\Big(\frac{1}{2}f_{11}^1f_{12}^1 + \frac{1}{2}f_{12}^1f_{22}^1 - \frac{1}{2}f_{11}^1f_{11}^2 - \frac{1}{2}f_{11}^2f_{12}^2 + \frac{1}{2}f_{22}^1f_{22}^2$$
$$- \frac{1}{2}f_{12}^2f_{22}^2 - \frac{1}{2}\sum_{j=3}^{m}f_{2j}^1f_{22}^j - \frac{1}{2}\sum_{j=3}^{m}f_{1j}^2f_{22}^j + \sum_{j=3}^{m}f_{2j}^2f_{12}^j\Big)$$

$$+\frac{\rho}{2}\sum_{i=3}\sum_{j=i}^{m}\delta_{ij}\big(f_{ij}^1 f_{12}^1 + f_{ij}^1 f_{22}^2 - f_{ij}^2 f_{11}^1 - f_{ij}^2 f_{12}^2\big) y_i y_j$$
$$+\rho\sum_{j=3}^{m}\sum_{i=j}^{m}\sum_{l=3}^{m}\delta_{ij}\big(f_{ij}^2 f_{1l}^j - f_{ij}^1 f_{2l}^j\big) y_i y_l$$
$$+\frac{\omega}{16}\Big[\rho^3\big(f_{111}^1 + f_{122}^1 + f_{112}^2 + f_{222}^2\big) + 8\rho\sum_{i=3}^{m}\sum_{j=i}^{m}\big(\delta_{1ij} f_{1ij}^1 + \delta_{2ij} f_{2ij}^2\big) y_i y_j\Big].$$

Moreover, for $k = 3, \dots, m$ we have

$$\frac{1}{2\pi\omega^2}\int_0^{2\pi}\frac{\partial\tilde{T}_{k1}}{\partial y_j} Y_j \, d\theta = 0,$$

and

$$\frac{1}{2\pi\omega^2}\int_0^{2\pi}\frac{\partial\tilde{T}_{k1}}{\partial \rho} Y_1 \, d\theta = -\frac{1}{2\omega^2}\sum_{i=3}^{m}\sum_{j=i}^{m}\sum_{l=3}^{m}\delta_{ij}\big(f_{ij}^2 f_{1l}^k - f_{ij}^1 f_{2l}^k\big) y_i y_j y_l$$
$$+\frac{\rho^2}{8\omega^2}\sum_{j=3}^{m}\Big(f_{2j}^1 f_{22}^k + f_{1j}^2 f_{22}^k - 2 f_{2j}^2 f_{12}^k - f_{12}^1 f_{1j}^k - \frac{1}{2} f_{11}^2 f_{1j}^k$$
$$-\frac{3}{2} f_{22}^2 f_{1j}^k + \frac{3}{2} f_{22}^k f_{2j}^k + \frac{1}{2} f_{22}^1 f_{2j}^k + f_{12}^2 f_{2j}^k\Big) y_j.$$

Therefore, for $j = 3, \dots, k$ we have

$$\frac{\omega^2}{2\pi}\int_0^{2\pi} F_j(\rho, y_3, \dots, y_m) \, d\theta = -\sum_{i=3}^{m}\sum_{j=i}^{m}\sum_{l=3}^{m}\delta_{ij}\big(f_{ij}^2 f_{1l}^k - f_{ij}^1 f_{2l}^k\big) y_i y_j y_l$$
$$+\frac{\rho^2}{4}\sum_{j=3}^{m}\big(f_{2j}^1 f_{22}^k + f_{1j}^2 f_{22}^k - 2 f_{2j}^2 f_{12}^k$$
$$- f_{11}^2 f_{1j}^k - f_{22}^2 f_{1j}^k + f_{22}^k f_{2j}^k + f_{22}^1 f_{2j}^k\big) y_j$$
$$+\omega\Big[\frac{\rho^2}{4}\sum_{j=3}^{m}\big(f_{11j}^k + f_{22j}^k\big) y_j + \sum_{i=3}^{m}\sum_{j=i}^{m}\sum_{l=j}^{m}\delta_{ijl} f_{ijl}^k y_i y_j y_l\Big].$$

This concludes the computation of the functions G_i, and the proof of Theorem 2 is complete.

3 Proof of Theorem 3

When $m = 3$ the averaged system (1.9) becomes

$$\frac{d\rho}{d\theta} = \varDelta_1 \rho^3 + \varDelta_2 \rho y_3^2,$$
$$\frac{dy_3}{d\theta} = \varDelta_3 \rho^2 y_3 + \varDelta_4 y_3^3.$$

We can easily verify that it has as a single equilibrium point with $\rho > 0$, given by

$$p = \left(\frac{|2\omega\mu_3|^{1/3}\sqrt{|\Delta_2|}}{|\Delta_1|^{1/6}|\Delta_3\Delta_2 - \Delta_1\Delta_4|^{1/3}}, \frac{(2\omega\mu_3\Delta_1)^{1/3}}{(\Delta_3\Delta_2 - \Delta_1\Delta_4)^{1/3}} \right).$$

A simple computation shows that the Jacobian matrix at the point p has determinant $12|\omega\mu_3|\sqrt{|\Delta_1\Delta_2|} \neq 0$, and thus we can apply Theorem 1, and we get Theorem 3.

4 Details of Example 1

The purpose of this section is to provide the details of Example 1, showing that the averaging theory of the second order can be applied to system (1.10).

In coordinates r, θ, x_3, x_4 system (1.10) becomes

$$\begin{aligned}
\dot{r} &= x_4^2\cos\theta + \frac{rx_3(x_4+x_3)\cos^2\theta}{\omega} - \frac{4r^3\cos^4\theta}{3\omega} + r^2\cos^2\theta\sin\theta, \\
\dot{\theta} &= -\frac{x_4^2\sin\theta}{r} + \frac{4r^2\cos^3\theta\sin\theta}{3\omega} - r\cos\theta\sin^2\theta \\
&\quad + \frac{\omega^2\cos^2\theta - x_3x_4\cos\theta\sin\theta - x_3^2\cos\theta\sin\theta + \omega^2\sin^2\theta}{\omega}, \\
\dot{x}_3 &= (\varepsilon^3 + x_4x_3^2)\mu_3 - 2r^2(x_4+x_3)\mu_3\cos^2\theta, \\
\dot{x}_4 &= \frac{-x_4x_3^2 + \varepsilon^3\omega\mu_4 + x_4x_3^2\omega\mu_4}{\omega} - 2r^2(x_4+x_3)\mu_4\cos^2\theta + rx_3\sin\theta.
\end{aligned}$$

Performing the rescaling $\rho = r\varepsilon$, $x_3 = y_3\varepsilon$ and $x_4 = y_4\varepsilon$, in the region $\dot{\theta} \neq 0$ we obtain the reduced system

$$\begin{aligned}
\frac{d\rho}{d\theta} &= \frac{\varepsilon\cos\theta(y_4^2 + \rho^2\cos\theta\sin\theta)}{\omega} \\
&\quad - \frac{\varepsilon^2\cos\theta}{3\rho\omega^2}\big(-3\rho^2y_3y_4\cos^3\theta - 3\rho^2y_3^2\cos^3\theta + 4\rho^4\cos^5\theta \\
&\quad - 3y_4^4\sin\theta - 6\rho^2y_4^2\cos\theta\sin^2\theta - 3\rho^2y_3y_4\cos\theta\sin^2\theta \\
&\quad - 3\rho^2y_3^2\cos\theta\sin^2\theta + 4\rho^4\cos^3\theta\sin^2\theta - 3\rho^4\cos^2\theta\sin^3\theta\big), \\
\frac{dy_3}{d\theta} &= \frac{\varepsilon^2\mu_3(1 + y_3^2y_4 - 2\rho^2y_4\cos^2\theta - 2\rho^2y_3\cos^2\theta)}{\omega}, \\
\frac{dy_4}{d\theta} &= \frac{\varepsilon\rho y_3\sin\theta}{\omega} + \frac{\varepsilon^2}{\omega^2}\big(-y_4y_3^2\cos^2\theta + \omega\mu_4\cos^2\theta + y_4y_3^2\omega\mu_4\cos^2\theta \\
&\quad - 2\rho^2y_4\omega\mu_4\cos^4\theta - 2\rho^2y_3\omega\mu_4\cos^4\theta + y_4^2y_3\sin^2\theta - y_4y_3^2\sin^2\theta \\
&\quad + \omega\mu_4\sin^2\theta + y_4y_3^2\omega\mu_4\sin^2\theta - 2\rho^2y_4\omega\mu_4\cos^2\theta\sin^2\theta \\
&\quad - 2\rho^2y_3\omega\mu_4\cos^2\theta\sin^2\theta + \rho^2y_3\cos\theta\sin^3\theta\big).
\end{aligned}$$

We can write this system in the form (2.6) with

$$f_{11} = \frac{\cos\theta(y_4^2 + \rho^2\cos\theta\sin\theta)}{\omega}, \qquad f_{31} = 0, \qquad f_{41} = \frac{\rho y_3\sin\theta}{\omega},$$

and

$$f_{12} = -\frac{\cos\theta}{3\rho\omega^2}\big(-3\rho^2 y_3 y_4 \cos^3\theta - 3\rho^2 y_3^2 \cos^3\theta + 4\rho^4\cos^5\theta \\ - 3y_4^4\sin\theta - 6\rho^2 y_4^2\cos\theta\sin^2\theta - 3\rho^2 y_4 y_3\cos\theta\sin^2\theta \\ - 3\rho^2 y_3^2\cos\theta\sin^2\theta + 4\rho^4\cos^3\theta\sin^2\theta - 3\rho^4\cos^2\theta\sin^3\theta\big),$$

$$f_{32} = \frac{\mu_3(1 + y_4 y_3^2 - 2\rho^2 y_4\cos^2\theta - 2\rho^2 y_3\cos^2\theta)}{\omega},$$

$$f_{42} = \frac{1}{\omega^2}\big(-y_4 y_3^2\cos^2\theta + \omega\mu_4\cos^2\theta + y_4 y_3^2\omega\mu_4\cos^2\theta \\ - 2\rho^2 y_4\omega\mu_4\cos^4\theta - 2\rho^2 y_3\omega\mu_4\cos^4\theta + y_4^2 y_3\sin^2\theta - y_4 y_3^2\sin^2\theta \\ + \omega\mu_4\sin^2\theta + y_4 y_3^2\omega\mu_4\sin^2\theta - 2\rho^2 y_4\omega\mu_4\cos^2\theta\sin^2\theta \\ - 2\rho^2 y_3\omega\mu_4\cos^2\theta\sin^2\theta + \rho^2 y_3\cos\theta\sin^3\theta\big).$$

Note that

$$\int_0^{2\pi} f_{11}(\theta,\rho,y_3,y_4)\,d\theta = \int_0^{2\pi} f_{41}(\theta,\rho,y_3,y_4)\,d\theta = 0.$$

Now we study the averages of the second order. Setting

$$Y_1(\theta,\rho,y_3,y_4) = \int_0^{\theta} f_{11}(\psi,\rho,y_3,y_4)\,d\theta, \quad Y_3 = 0,$$

$$Y_4(\theta,\rho,y_3,y_4) = \int_0^{\theta} f_{41}(\psi,\rho,y_3,y_4)\,d\theta,$$

with the notation of (2.7) we obtain

$$G_1 = \frac{1}{2\pi}\int_0^{2\pi} f_{12}(\theta,\rho,y_3,y_4)\,d\theta \\ + \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial f_{11}(\theta,\rho,y_3,y_4)}{\partial\rho} Y_1(\theta,\rho,y_3,y_4)\,d\theta \\ + \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial f_{11}(\theta,\rho,y_3,y_4)}{\partial y_4} Y_4(\theta,\rho,y_3,y_4)\,d\theta \\ = \frac{\rho}{2\omega^2}\big(-\rho^2 + y_4^2 - y_3 y_4 + y_3^2\big),$$

$$G_3 = \frac{1}{2\pi}\int_0^{2\pi} f_{32}(\theta,\rho,y_3,y_4)\,d\theta = \frac{\mu_3}{\omega}\big(1 + y_4 y_3^2 - \rho^2(y_3 + y_4)\big),$$

$$G_4 = \frac{1}{2\pi}\int_0^{2\pi} f_{42}(\theta,\rho,y_3,y_4)\,d\theta \\ + \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial f_{41}(\theta,\rho,y_3,y_4)}{\partial\rho} Y_1(\theta,\rho,y_3,y_4)\,d\theta \\ + \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial f_{41}(\theta,\rho,y_3,y_4)}{\partial y_4} Y_4(\theta,\rho,y_3,y_4)\,d\theta \\ = \frac{1}{\omega^2}\big[y_4(y_4 - y_3)y_3 + \big(1 + y_4 y_3^2 - \rho^2(y_3 + y_4)\big)\omega\mu_4\big].$$

Computing the zeros of (G_1, G_3, G_4) yields

$$p_1 = (1,1,0), \qquad p_2 = (1,0,1), \qquad p_3 = (1,1,1),$$

in coordinates ρ, y_3, y_4. For the function (G_1, G_3, G_4), the Jacobians at the points p_1 and p_2 have determinant $-3\mu_3/\omega^5 \neq 0$, and the Jacobian at the point p_3 has determinant $3\mu_3/\omega^5 \neq 0$. Thus, we can apply Theorem 1. This concludes the details of the example.

Acknowledgments

The first and third authors were supported by FCT through CAMGSD, Lisbon. The second author has been supported by the grants MEC/FEDER MTM 2008–03437, AGAUR 2009SGR 410, ICREA Academia and by FP7-PEOPLE-2012-IRSES-316338.

References

[1] L. Barreira, J. Llibre and C. Valls. Periodic orbits near equilibria. *Comm. Pure Appl. Math.*, **63**:1225–1236, 2010. http://dx.doi.org/10.1002/cpa.20324.

[2] A. Buica and J. Llibre. Averaging methods for finding periodic orbits via Brower degree. *Bull. Sci. Math.*, **128**:7–22, 2004. http://dx.doi.org/10.1016/j.bulsci.2003.09.002.

[3] Y. Li and K. Zhao. 2^n positive periodic solutions to n species non-autonomous Lotka–Volterra unidirectional food chains with harvesting terms. *Math. Model. Anal.*, **15**:313–326, 2010. http://dx.doi.org/10.3846/1392-6292.2010.15.313-326.

[4] A. Liapunov. Problème général de la stabilité du mouvement. *Ann. Fac. Sci. Toulouse*, **2**:203–474, 1907. http://dx.doi.org/10.5802/afst.246.

[5] J. Llibre. Averaging theory and limit cycles for quadratic systems. *Rad. Mat.*, **11**:215–228, 2002.

[6] K. Meyer, G. Hall and D. Offin. *Introduction to Hamiltonian Dynamical Systems and the N-Body Problem*, volume 90 of *Appl. Math. Sci.* Springer, 2009.

[7] J. Moser. Periodic orbits near an equilibrium and a theorem by Alan Weinstein. *Comm. Pure Appl. Math.*, **29**:724–747, 1976. http://dx.doi.org/10.1002/cpa.3160290613.

[8] J. Sanders and F. Verhulst. *Averaging Methods in Nonlinear Dynamical Systems*, volume 59 of *Appl. Math. Sci.* Springer, 1985.

[9] I.R. Shafarevich. *Basic Algebraic Geometry.* Springer, 1974.

[10] A. Weinstein. Lagrangian submanifolds and Hamiltonian systems. *Ann. Math.*, **98**:377–410, 1973. http://dx.doi.org/10.2307/1970911.

[11] A. Weinstein. Normal modes for nonlinear hamiltonian systems. *Invent. Math.*, **20**:47–57, 1973. http://dx.doi.org/10.1007/BF01405263.