Mathematical Modelling and Analysis
Volume 16 Number 2, June 2011, 315–325
Doi:10.3846/13926292.2011.580788

www.tandf.co.uk/journals/TMMA
Publisher: Taylor&Francis and VGTU
Online ISSN: 1648-3510
Print ISSN: 1392-6292

Exact Periodic Wave Solutions of a Singular Integrable Equation

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Received December 04, 2010; revised March 14, 2011; published online May 1, 2011

Abstract. In this paper, theory of dynamical systems is employed to investigate periodic waves of a singular integrable equation. These periodic waves contain smooth periodic waves, periodic cusp waves and periodic cusp loop waves. Under fixed parameter conditions, their exact parametric expressions are given.

Keywords: Singular integrable equation, closed orbit, smooth periodic wave, periodic cusp wave, periodic cusp loop wave.

AMS Subject Classification: 35Q51; 35Q58; 37K50.

1 Introduction and Main Results

Qiao [4] presented a completely integrable water wave equation:

$$u_t - u_{xxt} + 3u^2u_x - u_x^3 = (4u - 2u_{xx})u_xu_{xx} + (u^2 - u_x^2)u_{xxx}, \tag{1.1}$$

where u is the fluid velocity and subscripts denote the partial derivatives. This equation can be derived from the two-dimensional Euler equation by using the approximation procedure. He has proved that Eq. (1.1) has Lax pair and bi-Hamiltonian structures, which implies the integrability of the equation. Qiao [4, 5] obtained the new cuspons, one-peak solitons, W-shape-peaks and M-shape-peaks solutions. Apparently, if $u(x,t)$ is a solution of Eq. (1.1), then $-u(x,t)$ is a solution also. So, when $u(x,t)$ is a W-shape-peak solution of Eq. (1.1), then $-u(x,t)$ is an M-shape-peak solution. Taking special wave speed and using integral method, Qiao [4] showed a W-shape-peak explicit solution as follows:

$$u(\chi) = 2 - 3\cosh^2\chi + (\cosh\chi + 1/3)\sqrt{3(3\cosh\chi + 1)(\cosh\chi - 1)}, \tag{1.2}$$

where $\chi = |x - \frac{11}{3}t|/2 - \ln 2$.

Li and Zhang [3] called Eq. (1.1) a singular travelling wave equation. Following them, we call Eq. (1.1) a singular integrable equation. Using bifurcation method of dynamical systems, Li and Zhang [3] showed that there exist smooth solitary solutions and periodic waves of Eq. (1.1) when some parameter conditions are satisfied. They explained why the so-called W-shape-peaks and M-shape-peaks solutions can be created, gave the determined parameter conditions and got exact parametric expressions for all solitary wave solutions of Eq. (1.1). But they did not obtain exact parametric expressions of the smooth periodic waves.

In this paper, we employ the method of dynamical systems [3, 6, 7, 8] to investigate the periodic waves of the Eq. (1.1). Firstly, we derive travelling wave equation and system. Then we draw bifurcation curves and bifurcation phase portraits of the travelling wave system. By using these closed orbits, the exact periodic wave solutions of Eq. (1.1) are obtained. Corresponding to the special closed orbits the periodic waves have special loop cusp shape. We call them periodic cusp loop waves. The limit of the periodic cusp loop waves are cusp loop solitary waves. Our method is similar to the one used in [3]. Comparing our results with Qiao [4, 5] and Li *et al.* [3], we note that the periodic cusp loop waves and cusp loop solitary waves are new.

In order to state our main results in a compact form, for a given constant $c>0$ and $g\neq 0$, let the φ_1, φ_2 and φ_3 are three simple real zeros of $f(\varphi)=\varphi^3-c\varphi+g$, $\varphi_{\sqrt{c}}$ $(-\sqrt{c}<\varphi_{\sqrt{c}}<\sqrt{c})$ is a simple real root of $(c-\varphi^2)^2+4g\varphi=4g\sqrt{c}$, $\varphi_{-\sqrt{c}}$ $(-\sqrt{c}<\varphi_{-\sqrt{c}}<\sqrt{c})$ is a simple real root of $(c-\varphi^2)^2+4g\varphi=-4g\sqrt{c}$, and the φ_0 is original value.

Proposition 1. (1) *If* $\frac{8c\sqrt{c}}{27}<g<\frac{2c}{3}\sqrt{\frac{c}{3}}$ *and* $\varphi_2<\varphi_0<\varphi_3$, *then the Eq.* (1.1) *has a smooth periodic wave, and the smooth periodic wave becomes a smooth solitary wave when* φ_0 *tends to* φ_2.

(2) *If* $g=\frac{8c\sqrt{c}}{27}$ *and* $\varphi_2<\varphi_0<\varphi_3$, *then the Eq.* (1.1) *has a smooth periodic wave, and the smooth periodic wave becomes a peakon wave when* φ_0 *tends to* φ_2.

(3) *If* $0<g<\frac{8c\sqrt{c}}{27}$ *and* $\varphi_{\sqrt{c}}<\varphi_0<\varphi_3$, *then the Eq.* (1.1) *has a smooth periodic wave.*

(4) *If* $0<g<\frac{8c\sqrt{c}}{27}$ *and* $\varphi_0=\varphi_{\sqrt{c}}$, *then the Eq.* (1.1) *has a periodic cusp wave.*

(5) *If* $0<g<\frac{8c\sqrt{c}}{27}$ *and* $\varphi_2<\varphi_0<\varphi_{\sqrt{c}}$, *then the Eq.* (1.1) *has a periodic cusp loop wave, and it becomes a cusp loop solitary wave when* φ_0 *tends to* φ_2.

Under one of the above conditions, the periodic solution $u(x,t)=\varphi(\xi)$ *of the Eq.* (1.1) *has parametric type as follows*:

$$\begin{cases} \varphi=\dfrac{1}{4g}\left(h_0-\left(\dfrac{z_2-z_1n_1^2\mathrm{sn}^2(w,k_1)}{1-n_1^2\mathrm{sn}^2(w,k_1)}\right)^2\right), \\ \xi=\dfrac{4}{\sqrt{(z_4-z_2)(z_3-z_1)}}\left(z_1w+(z_2-z_1)\Pi(\arcsin(\mathrm{sn}(w,k_1)),n_1^2,k_1)\right), \end{cases} \tag{1.3}$$

where $h_0=(c-\varphi_0^2)^2+4g\varphi_0$, *the* z_1,z_2,z_3 *and* z_4 *are four real simple real zeros of* $G(z)=\frac{1}{16g^2}(z^4-2h_0z^2+16g^2z+h_0^2-16g^2c)$, $w=\frac{\sqrt{(z_4-z_2)(z_3-z_1)}}{4}\upsilon$ *is a* *parameter variable,* $k_1 = \sqrt{\frac{(z_3-z_2)(z_4-z_1)}{(z_4-z_2)(z_3-z_1)}}$ *is the modulus of Jacobian elliptic function and* $n_1 = \sqrt{\frac{z_3-z_2}{z_3-z_1}}$.

Proposition 2. (6) *If* $-\frac{2c}{3}\sqrt{\frac{c}{3}} < g < -\frac{8c\sqrt{c}}{27}$ *and* $\varphi_1 < \varphi_0 < \varphi_2$*, then the Eq.* (1.1) *has a smooth periodic wave, and the smooth periodic wave becomes a smooth solitary wave when* φ_0 *tends to* φ_2.

(7) *If* $g = -\frac{8c\sqrt{c}}{27}$ *and* $\varphi_1 < \varphi_0 < \varphi_2$*, then the Eq.* (1.1) *has a smooth periodic wave, and the smooth periodic wave becomes a peakon wave when* φ_0 *tends to* φ_2.

(8) *If* $-\frac{8c\sqrt{c}}{27} < g < 0$ *and* $\varphi_1 < \varphi_0 < \varphi_{-\sqrt{c}}$*, then the Eq.* (1.1) *has a smooth periodic wave.*

(9) *If* $-\frac{8c\sqrt{c}}{27} < g < 0$ *and* $\varphi_0 = \varphi_{-\sqrt{c}}$*, then the Eq.* (1.1) *has a periodic cusp wave.*

(10) *If* $-\frac{8c\sqrt{c}}{27} < g < 0$ *and* $\varphi_{-\sqrt{c}} < \varphi_0 < \varphi_2$*, then the Eq.* (1.1) *has a periodic cusp loop wave, and it becomes a cusp loop solitary wave when* φ_0 *tends to* φ_2.

Under one of the above conditions, the periodic solution $u(x,t) = \varphi(\xi)$ *of the Eq.* (1.1) *has parametric type as follows*:

$$\begin{cases} \varphi = \frac{1}{4g}\left(h_0 - \left(\dfrac{z_3 - z_4 n_2^2 \mathrm{sn}^2(w,k_2)}{1 - n_2^2\mathrm{sn}^2(w,k_2)}\right)^2\right), \\ \xi = \dfrac{4}{\sqrt{(z_4-z_2)(z_3-z_1)}}\left(z_4 w + (z_3-z_4)\Pi(\arcsin(\mathrm{sn}(w,k_2)), n_2^2, k_2)\right), \end{cases} \tag{1.4}$$

where $h_0 = (c-\varphi_0^2)^2 + 4g\varphi_0$*, the* z_1, z_2, z_3 *and* z_4 *are four real simple real zeros of* $G(z) = \frac{1}{16g^2}(z^4 - 2h_0 z^2 - 16g^2 z + h_0^2 - 16g^2 c)$, $w = \frac{\sqrt{(z_4-z_2)(z_3-z_1)}}{4}\upsilon$ *is a parameter variable,* $k_2 = \sqrt{\frac{(z_3-z_2)(z_4-z_1)}{(z_4-z_2)(z_3-z_1)}}$ *is the modulus of Jacobian elliptic function and* $n_2 = \sqrt{\frac{z_3-z_2}{z_4-z_2}}$.

Example 1. Letting $c = 20$ and $g = 20$, then $\varphi_1 \doteq -4.906733293$, $\varphi_2 \doteq 1.059459801$ $\varphi_3 \doteq 3.847273492$ and $\varphi_{\sqrt{c}} \doteq 3.063233706$. Taking $\varphi_0 = 3.5$, we have $h_0 = 340.0625$, $z_1 \doteq -30.10556009$, $z_2 \doteq 2.693759228$, $z_3 \doteq 7.749999997$ and $z_4 \doteq 19.66180086$. Substituting these data into (1.3), on $\xi - u$ plane we draw smooth periodic wave graph as Fig. 1 (a).

Taking $\varphi_0 = \varphi_{\sqrt{c}}$, we have $h_0 \doteq 357.7708764$, $z_1 \doteq -30.42816464$, $z_2 = 0$, $z_3 \doteq 10.61659926$ and $z_4 \doteq 19.81156537$. Substituting these data into (1.3), on $\xi - u$ plane we draw periodic cusp wave graph as Fig. 1 (b).

Taking $\varphi_0 = 1.5$, we have $h_0 = 435.0625$, $z_1 \doteq -31.79311168$, $z_2 \doteq -5.543893946$, $z_3 = 17.75$ and $z_4 \doteq 19.58700563$. Substituting these data into (1.3), on $\xi - u$ plane we draw periodic cusp loop wave graph as Fig. 1 (c).

Taking $\varphi_0 = \varphi_2$, we have $h_0 \doteq 441.1184867$, $z_1 \doteq -31.89728257$, $z_2 \doteq -5.857807291$ and $z_3 = z_4 \doteq 18.87749737$. Substituting these data into (1.3), on $\xi - u$ plane we draw cusp loop solitary wave graph as Fig. 1 (d).

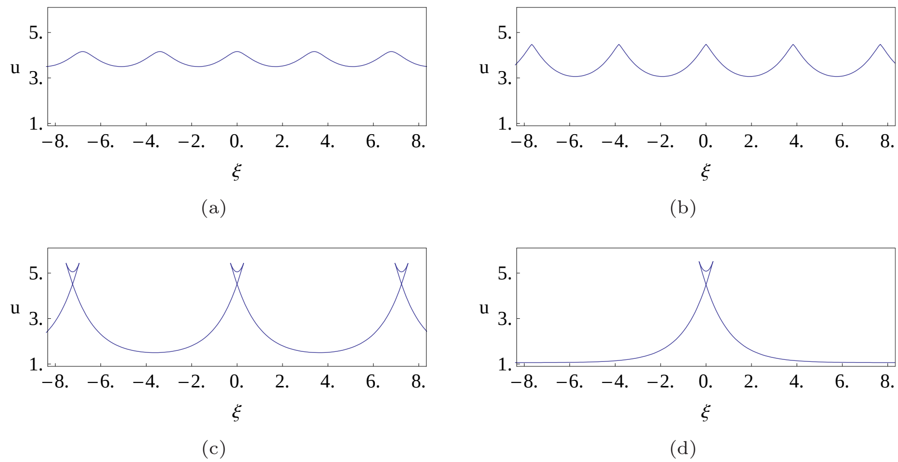

Figure 1. The smooth periodic wave, periodic cusp wave, periodic cusp loop wave and cusp loop solitary wave of Eq. (1.1) with $c = 20$ and $g = 20$: (a) $\varphi_0 = 3.5$, (b) $\varphi_0 = \varphi_{\sqrt{c}}$, (c) $\varphi_0 = 1.5$, (d) $\varphi_0 = \varphi_2$.

Example 2. Letting $c = 20$ and $g = -20$, then $\varphi_1 \doteq -3.847273492$, $\varphi_2 \doteq -1.059459801$ $\varphi_3 \doteq 4.906733293$ and $\varphi_{-\sqrt{c}} \doteq -3.063233706$. Taking $\varphi_0 = -3.5$, we have $h_0 = 340.0625$, $z_1 \doteq -19.66180086$, $z_2 \doteq -7.749999997$, $z_3 \doteq -2.693759228$ and $z_4 \doteq 30.10556009$. Substituting these data into (1.4), on $\xi - u$ plane we draw smooth periodic wave graph as Fig. 2 (a).

Taking $\varphi_0 = \varphi_{-\sqrt{c}}$, we have $h_0 = 357.7708764$, $z_1 \doteq -19.81156537$, $z_2 \doteq -10.61659926$, $z_3 = 0$ and $z_4 \doteq 30.42816464$. Substituting these data into (1.4), on $\xi - u$ plane we draw periodic cusp wave graph as Fig. 2 (b).

Taking $\varphi_0 = -1.5$, we have $h_0 = 435.0625$, $z_1 \doteq -19.58700563$, $z_2 = -17.75$, $z_3 \doteq 5.543893946$ and $z_4 \doteq 31.79311168$. Substituting these data into (1.4), on $\xi - u$ plane we draw periodic cusp loop wave graph as Fig. 2 (c).

Taking $\varphi_0 = \varphi_2$, we have $h_0 \doteq 441.1184867$, $z_1 = z_2 \doteq -18.87759249$, $z_3 \doteq 5.857807291$ and $z_4 \doteq 31.89728257$. Substituting these data into (1.4), on $\xi - u$ plane we draw cusp loop solitary wave graph as Fig. 2 (d).

2 Preliminary

In order to derive the expressions of the above solutions , we establish a planar system corresponding to Eq. (1.1). For a given constant c, substituting $\xi = x - ct$ and $u(x,t) = \varphi(\xi)$ in Eq. (1.1), it follows that

$$(\varphi^2 - (\varphi')^2 - c)(\varphi - \varphi'')' + 2\varphi'(\varphi - \varphi'')^2 = 0. \tag{2.1}$$

Integrating (2.1) once with respect to ξ, we have the following travelling wave equation.

$$(\varphi - \varphi'')(\varphi^2 - (\varphi')^2 - c) + g = 0, \tag{2.2}$$

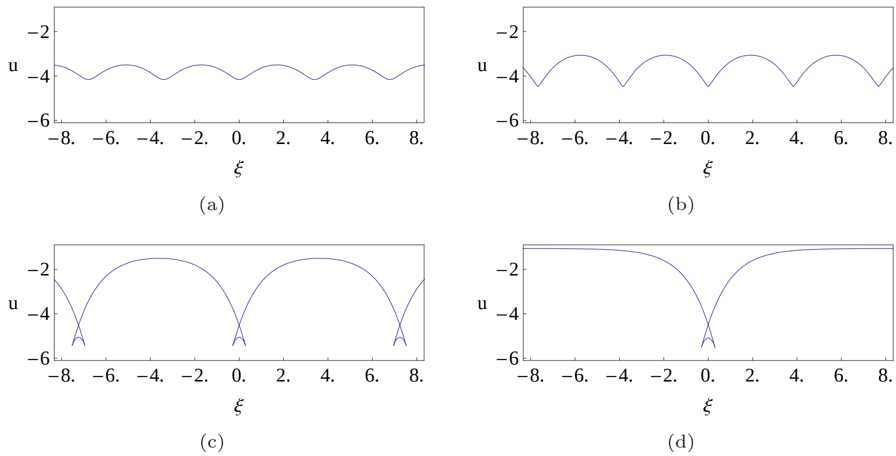

Figure 2. The smooth periodic wave, periodic cusp wave, periodic cusp loop wave and cusp loop solitary wave of Eq. (1.1) with $c = 20$ and $g = -20$: (a) $\varphi_0 = -3.5$, (b) $\varphi_0 = \varphi_{-\sqrt{c}}$, (c) $\varphi_0 = -1.5$, (d) $\varphi_0 = \varphi_2$.

where g is integral constant. We suppose that $g \neq 0$, and letting $\varphi' = y$, the Eq. (2.2) becomes a planar system

$$\begin{cases} \dfrac{d\varphi}{d\xi} = y, \\ \dfrac{dy}{d\xi} = \dfrac{\varphi(c - \varphi^2 + y^2) - g}{c - \varphi^2 + y^2}. \end{cases} \tag{2.3}$$

Clearly, on the hyperbola $\varphi^2 - y^2 = c$, the system (2.3) is discontinuous. Such system is called singular travelling wave system [3].

In what follows, without loss of generality we can assume $c > 0$ and make the transformation

$$d\xi = (c - \varphi^2 + y^2)d\tau,$$

where τ is parametric variable. Thus system (2.3) becomes

$$\begin{cases} \dfrac{d\varphi}{d\tau} = (c - \varphi^2 + y^2)y, \\ \dfrac{dy}{d\tau} = \varphi(c - \varphi^2 + y^2) - g. \end{cases} \tag{2.4}$$

Obviously, systems (2.3) and (2.4) have the same first integral

$$H(\varphi, y) = (c - \varphi^2 + y^2)^2 + 4g\varphi = h. \tag{2.5}$$

Therefore, both systems (2.3) and (2.4) have the same topological phase portraits except the hyperbola $\varphi^2 - y^2 = c$.

Let $f_0(\varphi) = \varphi^3 - c\varphi$, then for a fixed $c > 0$, the $\varphi = \pm\sqrt{\frac{c}{3}}$ are extreme points of $f_0(\varphi)$, $f_0(-\sqrt{\frac{c}{3}}) = \frac{2c}{3}\sqrt{\frac{c}{3}}$ is maximum, and $f_0(\sqrt{\frac{c}{3}}) = -\frac{2c}{3}\sqrt{\frac{c}{3}}$ is minimum.

Let $f(\varphi) = \varphi^3 - c\varphi + g$, then the following facts hold.

(1) $f(\varphi)$ only has a zero $\varphi_1 < -\sqrt{c}$ when $g > \frac{2c}{3}\sqrt{\frac{c}{3}}$. $(\varphi_1, 0)$ is a saddle point of system (2.4).

(2) $f(\varphi)$ has one simple zero φ_1, a double zero $\varphi_2 = \sqrt{\frac{c}{3}}$, and $\varphi_1 < -\sqrt{c} < \varphi_2$ when $g = \frac{2c}{3}\sqrt{\frac{c}{3}}$. $(\varphi_1, 0)$ is a saddle point, and $(\varphi_2, 0)$ is a cusp point of system (2.4).

(3) $f(\varphi)$ has three simple zeros φ_1, φ_2 and φ_3, and $\varphi_1 < -\sqrt{c} < 0 < \varphi_2 < \sqrt{\frac{c}{3}} < \varphi_3 < \sqrt{c}$ when $0 < g < \frac{2c}{3}\sqrt{\frac{c}{3}}$. $(\varphi_1, 0)$ and $(\varphi_2, 0)$ are two saddle points, and $(\varphi_3, 0)$ is a center point of system (2.4).

(4) $f(\varphi)$ has three simple zeros φ_1, φ_2 and φ_3, and $-\sqrt{c} < \varphi_1 < -\sqrt{\frac{c}{3}} < \varphi_2 < 0 < \sqrt{c} < \varphi_3$ when $-\frac{2c}{3}\sqrt{\frac{c}{3}} < g < 0$. $(\varphi_2, 0)$ and $(\varphi_3, 0)$ are two saddle points, and $(\varphi_1, 0)$ is a center point of system (2.4).

(5) $f(\varphi)$ has one simple zero φ_2, a double zero $\varphi_1 = -\sqrt{\frac{c}{3}}$, and $\varphi_1 < \sqrt{c} < \varphi_2$ when $g = -\frac{2c}{3}\sqrt{\frac{c}{3}}$. $(\varphi_1, 0)$ is a cusp point, and $(\varphi_2, 0)$ is a saddle point of system (2.4).

(6) $f(\varphi)$ only has a zero $\varphi_1 > \sqrt{c}$ when $g < -\frac{2c}{3}\sqrt{\frac{c}{3}}$. $(\varphi_1, 0)$ is a saddle point of system (2.4).

When $|g| < \frac{2c}{3}\sqrt{\frac{c}{3}}$ and $g \neq 0$, the homoclinic orbit is defined by $H(\varphi, y) = H(\varphi_2, 0) = h_2$. The homoclinic orbit that passes the point $(\sqrt{c}, 0)$ (or $(-\sqrt{c}, 0)$) is defined by $g = \frac{8c\sqrt{c}}{27}$ (or $g = -\frac{8c\sqrt{c}}{27}$).

According to the above analysis, we obtain the bifurcation phase portraits of systems (2.3) and (2.4) as given in Fig. 3.

3 The Proof of Main Results

It is well known that the closed orbit of the travelling system gives a periodic wave solution of the corresponding nonlinear wave equation. To find the exact parametric expressions of periodic wave solutions, we assume that $(\varphi_0, 0)$ is the initial point of a closed orbit, and it has expression from (2.5) that

$$y^2 = \varphi^2 + \delta\sqrt{h_0 - 4g\varphi} - c, \tag{3.1}$$

where $\delta = \pm 1$ and $h_0 = H(\varphi_0, 0)$. Let $z^2 = h_0 - 4g\varphi$, we have

$$\varphi = \frac{h_0 - z^2}{4g}, \quad y^2 = G(z), \tag{3.2}$$

where $G(z) = \frac{1}{16g^2}(z^4 - 2h_0 z^2 + \delta 16 g^2 z + h_0^2 - 16g^2 c)$. From first equation of (2.3), we obtain $\dfrac{dz}{d\xi} = -\dfrac{2g}{z}y$. Let

$$d\xi = z d\upsilon, \tag{3.3}$$

then

$$\frac{dz}{d\upsilon} = -2gy. \tag{3.4}$$

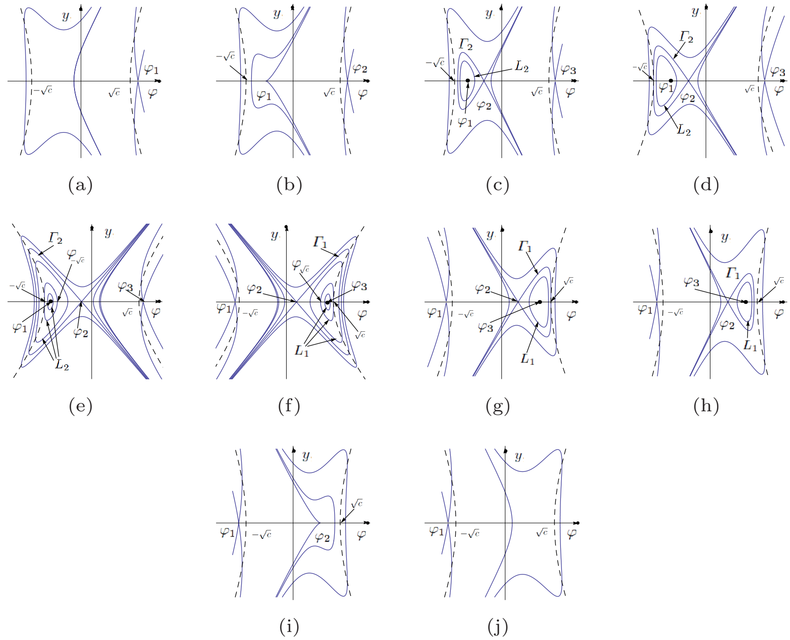

Figure 3. The bifurcation phase portraits of systems (2.3) and (2.4) with $c > 0$ and $g \neq 0$: (a) $g < -\frac{2c}{3}\sqrt{\frac{c}{3}}$; (b) $g = -\frac{2c}{3}\sqrt{\frac{c}{3}}$; (c) $-\frac{2c}{3}\sqrt{\frac{c}{3}} < g < -\frac{8c\sqrt{c}}{27}$; (d) $g = -\frac{8c\sqrt{c}}{27}$; (e) $-\frac{8c\sqrt{c}}{27} < g < 0$; (f) $0 < g < \frac{8c\sqrt{c}}{27}$; (g) $g = \frac{8c\sqrt{c}}{27}$; (h) $\frac{8c\sqrt{c}}{27} < g < \frac{2c}{3}\sqrt{\frac{c}{3}}$; (i) $g = \frac{2c}{3}\sqrt{\frac{c}{3}}$; (j) $g > \frac{2c}{3}\sqrt{\frac{c}{3}}$.

3.1 The proof of Proposition 1

(1) $\frac{8c\sqrt{c}}{27} < g < \frac{2c}{3}\sqrt{\frac{c}{3}}$ and $\varphi_2 < \varphi_0 < \varphi_3$. In this case, the closed orbit L_1 which passes through the point $(\varphi_0, 0)$ and the homoclinic orbit Γ_1 which passes through the point $(\varphi_2, 0)$ have no intersection point with the hyperbola $\varphi^2 - y^2 = c$ (see Fig. 3 (h)). Thus, corresponding to L_1, the Eq. (1.1) has a smooth periodic wave, and corresponding to Γ_1, the Eq. (1.1) has a smooth solitary wave. The smooth periodic wave becomes a smooth solitary wave when φ_0 tends to φ_2.

(2) $g = \frac{8c\sqrt{c}}{27}$ and $\varphi_2 < \varphi_0 < \varphi_3$. In this case, the closed orbit L_1 which passes through the point $(\varphi_0, 0)$ has no intersection point with the hyperbola $\varphi^2 - y^2 = c$, and The homoclinic orbit Γ_1 which passes through the point $(\varphi_2, 0)$ has only one intersection point $(\sqrt{c}, 0)$ with the hyperbola $\varphi^2 - y^2 = c$ (see Fig. 3 (g)). Thus, corresponding to L_1, the Eq. (1.1) has a smooth periodic wave, and corresponding to Γ_1, the Eq. (1.1) has a peakon. The smooth periodic wave becomes a peakon when φ_0 tends to φ_2.

(3) $0 < g < \frac{8c\sqrt{c}}{27}$ and $\varphi_{\sqrt{c}} < \varphi_0 < \varphi_3$, where $H(\varphi_{\sqrt{c}}, 0) = H(\sqrt{c}, 0)$. In this case, the closed orbit L_1 which passes through the point $(\varphi_0, 0)$ has no intersection point with the hyperbola $\varphi^2 - y^2 = c$ (see Fig. 3 (f)). Thus,

corresponding to L_1, the Eq. (1.1) has a smooth periodic wave.

(4) $0 < g < \frac{8c\sqrt{c}}{27}$ and $\varphi_0 = \varphi_{\sqrt{c}}$. In this case, the closed orbit L_1 which passes through the point $(\varphi_0, 0)$ has only a intersection point $(\sqrt{c}, 0)$ with the hyperbola $\varphi^2 - y^2 = c$ (see Fig. 3 (f)). Thus, corresponding to L_1, the Eq. (1.1) has a periodic cusp wave.

(5) $0 < g < \frac{8c\sqrt{c}}{27}$ and $\varphi_2 < \varphi_0 < \varphi_{\sqrt{c}}$.

In this case, the closed orbit L_1 which passes through the point $(\varphi_0, 0)$ and the homoclinic orbit Γ_1 which passes through the point $(\varphi_2, 0)$ have two intersection points with the hyperbola $\varphi^2 - y^2 = c$ (see Fig. 3 (f)). Thus, corresponding to L_1, the Eq. (1.1) has a periodic cusp loop wave, and corresponding to Γ_1, the Eq. (1.1) has a cusp loop solitary wave. The periodic cusp loop wave becomes cusp loop solitary wave when φ_0 tends to φ_2.

Under the one of conditions (1), (2), (3), (4) and (5), the closed orbit L_1 has expression as (3.1), where $\delta = 1$, and $G(z) = \frac{1}{16g^2}(z^4 - 2hz^2 + 16g^2 z + h^2 - 16g^2 c) = \frac{1}{16g^2}(z - z_1)(z - z_2)(z - z_3)(z - z_4)$, z_1, z_2, z_3 and z_4 are four real roots of the $G(z) = 0$. Thus, the second equation of (3.2) becomes

$$y = \pm\frac{1}{4g}\sqrt{(z_4 - z)(z_3 - z)(z - z_2)(z - z_1)}, \tag{3.5}$$

where $z_1 < z_2 \le z \le z_3 < z_4$.

Substituting (3.5) into (3.4) and integrating it along L_1, we have

$$\int_{z_2}^{z} \frac{1}{\sqrt{(z_4 - s)(z_3 - s)(s - z_2)(s - z_1)}}\, ds = \mp\frac{1}{2}\int_0^{\upsilon} ds. \tag{3.6}$$

By using formula 254.00 in [1], from (3.6), we obtain

$$z = \frac{z_2 - z_1 n_1^2 \mathrm{sn}^2(w, k_1)}{1 - n_1^2 \mathrm{sn}^2(w, k_1)}, \tag{3.7}$$

where $w = \frac{\sqrt{(z_4 - z_2)(z_3 - z_1)}}{4}\upsilon$ is a parameter variable, $k_1 = \sqrt{\frac{(z_3 - z_2)(z_4 - z_1)}{(z_4 - z_2)(z_3 - z_1)}}$ is the modulus of Jacobian elliptic function and $n_1 = \sqrt{\frac{z_3 - z_2}{z_3 - z_1}}$.

Substituting (3.7) into (3.3) and integrating it, we have

$$\int_0^{\xi} ds = \frac{4}{\sqrt{(z_4 - z_2)(z_3 - z_1)}} \int_0^{w} \frac{z_2 - z_1 n_1^2 \mathrm{sn}^2(s, k_1)}{1 - n_1^2 \mathrm{sn}^2(s, k_1)}\, ds. \tag{3.8}$$

By using formula 400.01 in [1], from (3.8), we obtain

$$\xi = \frac{4}{\sqrt{(z_4 - z_2)(z_3 - z_1)}}\left(z_1 w + (z_2 - z_1)\Pi(\arcsin(\mathrm{sn}(w, k_1)), n_1^2, k_1)\right). \tag{3.9}$$

Thus, from (3.2) we obtain the periodic wave solution $u(x,t) = \varphi(\xi)$ of parametric type as follows:

$$\begin{cases} \varphi = \dfrac{1}{4g}(h_0 - z^2), \\ \xi = \dfrac{4}{\sqrt{(z_4 - z_2)(z_3 - z_1)}}\left(z_1 w + (z_2 - z_1)\Pi(\arcsin(\mathrm{sn}(w, k_1)), n_1^2, k_1)\right), \end{cases} \tag{3.10}$$

Here we complete the proof of Proposition 1.

3.2 The proof of Proposition 2

(6) $-\frac{2c}{3}\sqrt{\frac{c}{3}} < g < -\frac{8c\sqrt{c}}{27}$ and $\varphi_1 < \varphi_0 < \varphi_2$. In this case, the closed orbit L_2 which passes through the point $(\varphi_0, 0)$ and the homoclinic orbit Γ_2 which passes through the point $(\varphi_2, 0)$ have no intersection point with the hyperbola $\varphi^2 - y^2 = c$ (see Fig. 3 (c)). Thus, corresponding to L_2, the Eq. (1.1) has a smooth periodic wave, and corresponding to Γ_2, the Eq. (1.1) has a smooth solitary wave. The smooth periodic wave becomes a smooth solitary wave when φ_0 tends to φ_2.

(7) $g = -\frac{8c\sqrt{c}}{27}$ and $\varphi_1 < \varphi_0 < \varphi_2$. In this case, the closed orbit L_2 which passes through the point $(\varphi_0, 0)$ has no intersection point with the hyperbola $\varphi^2 - y^2 = c$, and the homoclinic orbit Γ_2 which passes through the point $(\varphi_2, 0)$ has only one intersection point $(-\sqrt{c}, 0)$ with the hyperbola $\varphi^2 - y^2 = c$ (see Fig. 3 (d)). Thus, corresponding to L_2, the Eq. (1.1) has a smooth periodic wave, and corresponding to Γ_2, the Eq. (1.1) has a peakon. The smooth periodic wave becomes a peakon when φ_0 tends to φ_2.

(8) $-\frac{8c\sqrt{c}}{27} < g < 0$ and $\varphi_1 < \varphi_0 < \varphi_{-\sqrt{c}}$, where $H(\varphi_{-\sqrt{c}}, 0) = H(-\sqrt{c}, 0)$.

In this case, the closed orbit L_2 which passes through the point $(\varphi_0, 0)$ has no intersection point with the hyperbola $\varphi^2 - y^2 = c$ (see Fig. 3 (e)). Thus, corresponding to L_2, the Eq. (1.1) has a smooth periodic wave.

(9) $-\frac{8c\sqrt{c}}{27} < g < 0$ and $\varphi_0 = \varphi_{-\sqrt{c}}$.

In this case, the closed orbit L_2 which passes through the point $(\varphi_0, 0)$ has only a intersection point $(-\sqrt{c}, 0)$ with the hyperbola $\varphi^2 - y^2 = c$ (see Fig. 3 (e)). Thus, corresponding to L_2, the Eq. (1.1) has a periodic cusp wave.

(10) $-\frac{8c\sqrt{c}}{27} < g < 0$ and $\varphi_{-\sqrt{c}} < \varphi_0 < \varphi_2$.

In this case, the closed orbit L_2 which passes through the point $(\varphi_0, 0)$ and the homoclinic orbit Γ_2 which passes through the point $(\varphi_2, 0)$ have two intersection points with the hyperbola $\varphi^2 - y^2 = c$ (see Fig. 3 (e)). Thus, corresponding to L_2, the Eq. (1.1) has a periodic cusp loop wave, and corresponding to Γ_2, the Eq. (1.1) has a cusp loop solitary wave. The periodic cusp loop wave becomes cusp loop solitary wave when φ_0 tends to φ_2.

Under the one of conditions (6), (7), (8), (9) and (10), the closed orbit L_2 has expression as (3.1), where $\delta = -1$, and $G(z) = \frac{1}{16g^2}(z^4 - 2hz^2 - 16g^2 z + h^2 - 16g^2 c) = \frac{1}{16g^2}(z - z_1)(z - z_2)(z - z_3)(z - z_4)$, z_1, z_2, z_3 and z_4 are four real roots of the $G(z) = 0$. Thus, the second equation of (3.2) becomes

$$y = \pm \frac{1}{4|g|}\sqrt{(z_4 - z)(z_3 - z)(z - z_2)(z - z_1)}, \qquad (3.11)$$

where $z_1 < z_2 \le z \le z_3 < z_4$.

Substituting (3.11) into (3.4) and integrating it along L_2, we have

$$\int_z^{z_3} \frac{1}{\sqrt{(z_4 - s)(z_3 - s)(s - z_2)(s - z_1)}}\, ds = \pm \frac{1}{2} \int_v^0 ds. \qquad (3.12)$$

By using formula 255.00 in [1], from (3.12), we obtain

$$z = \frac{z_3 - z_4 n_2^2 \mathrm{sn}^2(w, k_2)}{1 - n_2^2 \mathrm{sn}^2(w, k_2)}, \qquad (3.13)$$

where $w = \frac{\sqrt{(z_4-z_2)(z_3-z_1)}}{4}\upsilon$ is a parameter variable, $k_2 = \sqrt{\frac{(z_3-z_2)(z_4-z_1)}{(z_4-z_2)(z_3-z_1)}}$ is the modulus of Jacobian elliptic function and $n_2 = \sqrt{\frac{z_3-z_2}{z_4-z_2}}$.

Substituting (3.13) into (3.3) and integrating it, we have

$$\int_0^{\xi} ds = \frac{4}{\sqrt{(z_4-z_2)(z_3-z_1)}} \int_0^{w} \frac{z_3 - z_4 n_2^2 \mathrm{sn}^2(s,k_2)}{1 - n_2^2 \mathrm{sn}^2(s,k_2)} ds. \tag{3.14}$$

By using formula 400.01 in [1], from (3.14), we obtain

$$\xi = \frac{4}{\sqrt{(z_4-z_2)(z_3-z_1)}} \left(z_4 w + (z_3 - z_4) \Pi(\arcsin(\mathrm{sn}(w,k_2)), n_2^2, k_2) \right). \tag{3.15}$$

Thus, from (3.2) we obtain the periodic wave solution $u(x,t) = \varphi(\xi)$ of parametric type as follows:

$$\begin{cases} \varphi = \dfrac{1}{4g}(h_0 - z^2), \\ \xi = \dfrac{4}{\sqrt{(z_4-z_2)(z_3-z_1)}} \left(z_4 w + (z_3 - z_4) \Pi(\arcsin(\mathrm{sn}(w,k_2)), n_2^2, k_2) \right), \end{cases} \tag{3.16}$$

Here we complete the proof of Proposition 2.

4 Conclusions

In this paper, we obtained exact periodic wave solutions of Eq. (1.1) by using the theory of dynamical systems. We draw the bifurcation phase portraits of the singular travelling wave system (2.3). Through studying shape of periodic waves, we have shown that the periodic waves of Eq. (1.1) contain smooth periodic waves, periodic cusp waves and periodic cusp loop waves. The limit of periodic cusp loop waves are cusp loop solitary waves (see Figs. 1 and 2). Compare the results with Qiao [4, 5] and Li et al. [3], the periodic cusp loop waves and cusp loop solitary waves are new. The Eq. (1.1) naturally has a physical meaning since it is derived from the two dimensional Euler equation (see [4]). In this paper, we successfully solve the Eq. (1.1) with smooth periodic waves, periodic cusp waves, periodic cusp loop waves and cusp loop solitary waves. The solutions of Eq. (1.1) may be applied to neuroscience for providing a mathematical model and explaining electrophysiological responses of visceral nociceptive neurons and sensitization of dorsal root reflexes [2].

Acknowledgements

The authors express their sincere thanks to the reviewer for suggestions and comments. This work was supported by the Natural Science Foundation of China (No. 10871073) and Yunnan Province (2008ZC153M).

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