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Comparison of Speeds of Convergence in Riesz-Type Families of Summability Methods. II*

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Abstract. Certain summability methods for functions and sequences are compared by their speeds of convergence. The authors are extending their results published in paper [9] for Riesz-type families $\{A_\alpha\}$ $(\alpha > \alpha_0)$ of summability methods A_α. Note that a typical Riesz-type family is the family formed by Riesz methods $A_\alpha = (R, \alpha), \alpha > 0$. In [9] the comparative estimates for speeds of convergence for two methods A_γ and A_β in a Riesz-type family $\{A_\alpha\}$ were proved on the base of an inclusion theorem. In the present paper these estimates are improved by comparing speeds of three methods A_γ, A_β and A_δ on the base of a Tauberian theorem. As a result, a Tauberian remainder theorem is proved. Numerical examples given in [9] are extended to the present paper as applications of the Tauberian remainder theorem proved here.

Keywords: speed of convergence, Tauberian remainder theorem, Riesz-type family of summability methods, Riesz methods, generalized integral Nörlund methods, Borel-type methods.

AMS Subject Classification: 40C10; 40E05; 40G05; 40G10.

1 Introduction and Basic Notions

We continue comparing speeds of convergence in Riesz-type families of summability methods started in paper [9]. In the mentioned paper any two methods in a Riesz-type family were compared by speed of convergence. In the present paper we improve our estimates comparing by speed of convergence any three methods in a Riesz-type family.

1.1. We begin our paper recalling the basic notions used in [9]. Let us consider functions $x = x(u)$ defined for $u \geq 0$, bounded and Lebesgue-measurable on every finite interval $[0, u_0]$. Let us denote the set of all such functions by X.

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Suppose that A is a transformation of functions $x = x(u)$ (or, in particular, of sequences $x = (x_n)$) into functions $Ax = y = y(u) \in X$. If the limit $\lim_{u\to\infty} y(u) = s$ exists then we say that $x = x(u)$ is convergent to s with respect to the summability method A, and write $x(u) \to s(A)$. If $y = y(u)$ is bounded then we say that x is bounded with respect to A, and write $x(u) = O(A)$. We denote by ωA the set of all these functions x, where the transformation A is applied, and by cA and mA the set of all functions x which are convergent and bounded with respect to the method A, respectively. The method A is said to be regular if $\lim_{u\to\infty} x(u) = s$ implies $\lim_{u\to\infty} y(u) = s$ whenever $x \in X$. Further we use the notation c_0 for the set of all functions $x \in X$ having $\lim_{u\to\infty} x(u) = 0$.

One of the most common summability method for functions x is an integral method A is defined with the help of transformation

$$y(u) = \int_0^\infty a(u,v)x(v)\,dv,$$

where $a(u,v)$ is a certain function of two variables $u \geq 0$ and $v \geq 0$. We say also that the integral method A is defined by the function $a(u,v)$. An example of an integral summability method is the generalized integral Nörlund method $(N, P(u), Q(u))$ defined with the help of transformation

$$y(u) = \frac{1}{R(u)} \int_0^u P(u-v)\,Q(v)x(v)\,dv \quad (u > 0),$$

where $P = P(u)$ and $Q = Q(u)$ are non-negative functions from X such that

$$R(u) = \int_0^u P(u-v)\,Q(v)\,dv \neq 0 \quad \text{for} \ \ u > 0.$$

In particular, if $Q(u) = 1$ and $P(u) = u^{\alpha-1}$ for $u > 0$ and $\alpha > 0$, we get the Riesz method (R, α).

For sequences $x = (x_n)$ we focus ourselves on certain semi-continuous summability methods A defined by transformations

$$y(u) = \sum_{n=0}^\infty a_n(u)\,x_n \quad (u \geq 0),$$

where $a_n(u)\,(n = 0, 1, \ldots)$ are some functions from X. An example of a semi-continuous method is the Borel method B defined by the transformation

$$y(u) = \frac{1}{e^u} \sum_{n=0}^\infty \frac{u^n}{n!}\,x_n. \tag{1.1}$$

1.2. One of the basic notions in this paper is the "speed of convergence". We use here definitions based on the definitions for sequences (see [4] and [5]) and extended for functions in [8] and [12]. Let $\lambda = \lambda(u)$ be a positive function from X such that $\lambda(u) \to \infty$ as $u \to \infty$. It is said that a function $x = x(u)$ is convergent to s with speed λ (shortly: λ-convergent) if the finite limit $\lim_{u\to\infty} \lambda(u)\,[x(u) - s]$ exists. If $\lambda(u)\,[x(u) - s] = O(1)$ as $u \to \infty$, then x is said to be λ-bounded.

We use the notations c^λ and m^λ for the sets of all λ-convergent and λ-bounded functions x, respectively. It is said that x is convergent or bounded with speed λ with respect to the summability method A if $Ax \in c^\lambda$ or $Ax \in m^\lambda$, respectively.

1.3. The main subject of the paper is a Riesz-type family of summability methods ([8, 13]). Let $\{A_\alpha\}$ be a family of summability methods A_α where [1] $\alpha \underset{(-)}{>} \alpha_1$ and which are defined by transformations of functions $x = x(u) \in \omega A_\alpha \subset X$ into $A_\alpha x = y_\alpha = y_\alpha(u) \in X$. Suppose that for any $\beta > \gamma \underset{(-)}{>} \alpha_1$ we have

$$\omega A_\gamma \subset \omega A_\beta. \tag{1.2}$$

Definition 1. ([8], Definition 1; [13], Definition 2) A family $\{A_\alpha\}$ $(\alpha \underset{(-)}{>} \alpha_1)$ is said to be a Riesz-type family if for every $\beta > \gamma \underset{(-)}{>} \alpha_1$ the relation (1.2) holds and the methods A_γ and A_β are connected through

$$y_\beta(u) = \frac{M_{\gamma,\beta}}{r_\beta(u)} \int_0^u (u-v)^{\beta-\gamma-1} r_\gamma(v) y_\gamma(v)\,dv \qquad (u > 0), \tag{1.3}$$

$$r_\beta(u) = M_{\gamma,\beta} \int_0^u (u-v)^{\beta-\gamma-1} r_\gamma(v)\,dv \qquad (u > 0), \tag{1.4}$$

where $r_\gamma = r_\gamma(u)$ and $r_\beta = r_\beta(u)$ are some positive functions from X and $M_{\gamma,\beta}$ is a constant depending on γ and β.

Example 1. Let $\{A_\alpha\}$ be the family of generalized Nörlund methods $A_\alpha = (N, p_\alpha(u), q(u))$ $(\alpha > \alpha_0)$ defined by positive functions $p = p(u) \in X$ and $q = q(u) \in X$ and a number α_0 such that

$$r_\alpha(u) = \int_0^u p_\alpha(u-v) q(v)\,dv > 0 \qquad (u > 0,\ \alpha > \alpha_0),$$

where $p_\alpha(u) = \int_0^u (u-v)^{\alpha-1} p(v)\,dv$. It is known that relations (1.3) together with (1.4) and (1.6) hold here for any $\beta > \gamma > \alpha_0$ (see [14]), and thus this family is a Riesz-type family.

Example 2. Consider the Borel-type methods $A_\alpha = (B, \alpha, q_n)$ (see [13]). Let (q_n) be a non-negative sequence such that the power series $\sum q_n u^n$ has the radius of convergence $R = \infty$ and $q_n > 0$ at least for one $n \in \mathbf{N}$. Denote

$$r_\alpha(u) = \sum_{n=1}^{\infty} \frac{n! q_n u^{n+\alpha-1}}{\Gamma(n+\alpha)} \tag{1.5}$$

and define the methods (B, α, q_n) $(\alpha > -1/2)$ for converging sequences $x = (x_n)$ with the help of transformation

$$y_\alpha(u) = \frac{1}{r_\alpha(u)} \sum_{n=1}^{\infty} \frac{n! q_n u^{n+\alpha-1}}{\Gamma(n+\alpha)} x_n \qquad (u > 0).$$

[1] The notation $\alpha \underset{(-)}{>} \alpha_1$ means that we consider parameter values $\alpha > \alpha_1$ or $\alpha \geq \alpha_1$ with some fixed number α_1.

The methods $A_\alpha = (B, \alpha, q_n)$ satisfy relations (1.3) and (1.4) with $r_\alpha(u)$ defined by (1.5) and $M_{\gamma,\beta} = 1/\Gamma(\beta - \gamma)$ (see [13]) and form therefore a Riesz-type family. In particular, if $q_n = \frac{1}{n!}$ we get the Borel-type methods $(B, \alpha) = (B, \alpha, 1/n!)$ (see [1, 2]). If, in addition, $\alpha = 1$, we have the Borel method $B = (B, 1)$.

Example 3. Consider the family of generalized Nörlund methods $A_\alpha{=}(N, u^{\alpha-1}, q(u))$ where $\alpha > 0$ and $q = q(u)$ is a positive function from X. These methods are defined by transformation of x into $A_\alpha x = y_\alpha(u)$ with

$$y_\alpha(u) = \frac{1}{r_\alpha(u)} \int_0^u (u - v)^{\alpha-1} q(v)x(v)\, dv \qquad (u > 0),$$

where $r_\alpha = r_\alpha(u) = \int_0^u (u - v)^{\alpha-1} q(v)\, dv$. This family satisfies relations (1.3) and (1.4) with

$$M_{\gamma,\beta} = \frac{\Gamma(\beta)}{\Gamma(\gamma)\Gamma(\beta - \gamma)} \tag{1.6}$$

(see [9], Example 1) and therefore it is a Riesz-type family. In particular, if $q(u) = 1\,(u \geq 0)$ we have Riesz methods $(N, u^{\alpha-1}, 1) = (R, \alpha)$.

2 Preliminary Results

We need some results proved in [9].

2.1. Speeds of convergence of any two methods in a Riesz-type family were compared in [9] on the base of an inclusion theorem which will be formulated as the following proposition.

Proposition 1. *Let* $\{A_\alpha\}$ $(\alpha \underset{(-)}{>} \alpha_1)$ *be a Riesz-type family. Then we have for functions* $x = x(u)$ *and numbers* s *and* $\beta > \gamma \underset{(-)}{>} \alpha_1$ *that*

i) $x(u) = O\,(A_\gamma) \Longrightarrow x(u) = O\,(A_\beta)$, *ii)* $x(u) \to s(A_\gamma) \Longrightarrow x(u) \to s(A_\beta)$, *provided in case ii) that* $\lim_{u\to\infty} \int_0^u r_{\alpha_1}(v)\, dv = \infty$ *is satisfied if* $\gamma = \alpha_1$ *is included.*

The next theorem (see [9], Theorem 1) describes how the speed of convergence changes if we go from one summability method in the family to a stronger one.

Theorem A. *Let* $\{A_\alpha\}(\alpha > \alpha_0)$ *be a Riesz-type family. Let some positive function* $\lambda = \lambda(u) \to \infty$ (as $u \to \infty$) *from* X *and some number* $\gamma > \alpha_0$ *such that* $\dfrac{r_\gamma(u)}{\lambda(u)} \in X$ *be given.*

i) Then we have for functions $x = x(u)$ *and numbers* s *and* $\beta > \gamma$ *that*

$$\lambda(u)\,[y_\gamma(u) - s] = O(1) \implies \lambda_\beta(u)\,[y_\beta(u) - s] = O(1), \tag{2.1}$$

where the speeds are related through the formulas

$$\lambda_\beta(u) = \frac{r_\beta(u)}{b_\beta(u)},\ b_\beta(u) = M_{\gamma,\beta} \int_0^u (u{-}v)^{\beta-\gamma-1} b_\gamma(v)\, dv,\ b_\gamma(u) = \frac{r_\gamma(u)}{\lambda(u)}. \tag{2.2}$$

ii) Moreover, we have that

$$\lambda(u)\,[y_\gamma(u) - s] \to t \implies \lambda_\beta(u)\,[y_\beta(u) - s] \to t, \tag{2.3}$$

provided that

$$\lim_{u\to\infty} \int_0^u b_\gamma(v)\,dv = \infty. \tag{2.4}$$

Under restriction (2.4) the condition $\lambda(u) \to \infty$ implies $\lambda_\beta(u) \to \infty$ in Theorem A (see [9], Remark 2). We note also that Theorem A can be considered as a generalization of case A) of Theorem 1 from [12], which was proved for matrix case. Certain evaluations for speed of convergence for Riesz and Nörlund matrix methods in Banach spaces were proved in recent papers [6] and [7].

2.2. The speeds $\lambda = \lambda(u)$ and $\lambda_\beta = \lambda_\beta(u)$ defined in Theorem A can be compared by the inequalities.

Let $a = a(u)$ and $b = b(u)$ be two positive functions from X. If there exist positive numbers c_1, c_2 and u_0 such that the condition

$$c_1 b(u) \le a(u) \le c_2 b(u) \tag{2.5}$$

holds for every $u > u_0$, we write $a(u) \approx b(u)$. If $b = b(u)$ is nondecreasing and condition (2.5) is satisfied with some positive c_1 and c_2 for any $u > 0$, then we say that $a = a(u)$ is almost nondecreasing.

The following proposition is proved in [9] (see [9], Propositions 2 and 3).

Proposition 2. *Let a Riesz-type family* $\{A_\alpha\}$ $(\alpha > \alpha_0)$ *and a positive function* $\lambda = \lambda(u) \in X$ *be given. Fix some* $\gamma > \alpha_0$ *and suppose that* $\lambda_\beta = \lambda_\beta(u)$ $(\beta > \gamma > \alpha_0)$ *is defined through* (2.2). *Then for* $\beta > \gamma > \alpha_0$ *we have:*

i) $\lambda_\beta(u) \le L\,\lambda(u)$ $(u > 0)$ *provided that* $\lambda = \lambda(u)$ *is almost nondecreasing,*

ii) $\lambda_\beta(u) \ge \dfrac{K r_\beta(u)}{r_\gamma(u) u^{\beta-\gamma}} \lambda(u)$ $\quad (u > 0)$ *provided that* $b_\gamma(u) = r_\gamma(u)/\lambda(u)$ *is almost nondecreasing, where* L *and* K *are some positive constants independent from* u.

Previous result state that switching to a stronger method, the speed of convergence can not be improved but also it cannot become too much worse. This is consistent with results known for matrix methods (see e.g. [4, 6, 12]).

3 Main Results. A Tauberian Remainder Theorem

First we prove a convexity theorem.

Theorem 1. *Let* $\{A_\alpha\}$ $(\alpha \underset{(-)}{>} \alpha_1)$ *be a Riesz-type family satisfying the condition*

$$r_\beta(u)/r_\alpha(u) \approx u^{\beta-\alpha} \quad (u > 0) \tag{3.1}$$

for all $\beta > \alpha > \alpha_1$. *Then we have for functions* $x = x(u)$ *and numbers* s *and* $\beta > \delta > \gamma \underset{(-)}{>} \alpha_1$ *that*

$$x(u) = O\,(A_\gamma), \quad x(u) \to s\,(A_\beta) \Longrightarrow x(u) \to s\,(A_\delta). \tag{3.2}$$

Proof. Suppose first that $\gamma > \alpha_1$. Without a loss of generality we may take $\beta = \gamma + 1$ and $s = 0$. Suppose that

$$y_{\gamma+1}(u) \to 0 \text{ as } u \to \infty, \quad y_\gamma(u) = O(1) \tag{3.3}$$

for a function $x = x(u)$ and some value γ of the parameter, and show that

$$y_\delta(u) \to 0 \text{ as } u \to \infty \tag{3.4}$$

for any δ such that $\gamma < \delta < \gamma + 1$. By relation (1.3) we have that

$$y_\delta(u) = \frac{M_{\gamma,\delta}}{r_\delta(u)} \int_0^u (u-v)^{\delta-\gamma-1} r_\gamma(v) y_\gamma(v)\, dv \qquad (u > 0).$$

Choose some $\theta \in (1/2; 1)$ and divide $y_\delta(u)$ into two parts:

$$\begin{aligned} y_\delta(u) &= \frac{M_{\gamma,\delta}}{r_\delta(u)} \int_0^{\theta u} (u-v)^{\delta-\gamma-1} r_\gamma(v) y_\gamma(v)\, dv \\ &+ \frac{M_{\gamma,\delta}}{r_\delta(u)} \int_{\theta u}^u (u-v)^{\delta-\gamma-1} r_\gamma(v) y_\gamma(v)\, dv = I_1(u,\theta) + I_2(u,\theta). \end{aligned} \tag{3.5}$$

Thus we have the equality $y_\delta(u) = I_1(u,\theta) + I_2(u,\theta)$. Note that $I_1(u,\theta)$ and $I_2(u,\theta)$ depend also on γ, δ. Integrating by parts, we get for $I_1(u,\theta)$ the following form:

$$\begin{aligned} I_1(u,\theta) &= \frac{M_{\gamma,\delta}}{r_\delta(u)} \Big((u-v)^{\delta-\gamma-1} \int_0^v r_\gamma(t) y_\gamma(t) dt \Big)\Big|_0^{\theta u} \\ &+ \frac{M_{\gamma,\delta}}{r_\delta(u)} \int_0^{\theta u} \Big[(\delta-\gamma-1)(u-v)^{\delta-\gamma-2} \int_0^v r_\gamma(t) y_\gamma(t) dt \Big] dv = I_1'(u,\theta) + I_1''(u,\theta), \end{aligned}$$

where

$$\begin{aligned} I_1'(u,\theta) &= \frac{M_{\gamma,\delta}}{r_\delta(u)} ((u-v)^{\delta-\gamma-1} \int_0^v r_\gamma(t) y_\gamma(t) dt)\Big|_0^{\theta u} = \frac{M_{\gamma,\delta}}{r_\delta(u)} (u-\theta u)^{\delta-\gamma-1} \\ &\times \int_0^{\theta u} r_\gamma(t) y_\gamma(t) dt = \frac{M_{\gamma,\delta}}{M_{\gamma,\gamma+1}} \frac{(u-\theta u)^{\delta-\gamma-1}}{r_\delta(u)} r_{\gamma+1}(\theta u) y_{\gamma+1}(\theta u) \end{aligned}$$

and

$$\begin{aligned} I_1''(u,\theta) &= \frac{M_{\gamma,\delta}}{r_\delta(u)} \int_0^{\theta u} \Big[(\delta-\gamma-1)(u-v)^{\delta-\gamma-2} \int_0^v r_\gamma(t) y_\gamma(t) dt \Big] dv \\ &= \frac{M_{\gamma,\delta}}{M_{\gamma,\gamma+1}} \frac{1}{r_\delta(u)} \int_0^{\theta u} (\delta-\gamma-1)(u-v)^{\delta-\gamma-2} r_{\gamma+1}(v) y_{\gamma+1}(v)\, dv. \end{aligned}$$

Using conditions (3.1) and (3.3) we get

$$\begin{aligned} I_1'(u,\theta) &= O(1) \frac{(u-\theta u)^{\delta-\gamma-1}}{r_\delta(u)} r_{\gamma+1}(u) y_{\gamma+1}(u) = O(1) u^{\gamma+1-\delta} u^{\delta-\gamma-1} \\ &\times (1-\theta)^{\delta-\gamma-1} y_{\gamma+1}(u) = o(1)(1-\theta)^{\delta-\gamma-1} = o_\theta(1) \text{ as } u \to \infty. \end{aligned}$$

Thus we have $I_1'(u,\theta) = o_\theta(1)$ as $u \to \infty$. Let us show that also $I_1''(u,\theta) = o_\theta(1)$ as $u \to \infty$. Denoting

$$c'_{\gamma,\delta}(u,v) = \begin{cases} \dfrac{1}{r_\delta(u)}(u-v)^{\delta-\gamma-2} r_{\gamma+1}(v), & \text{if } 0 \le v \le \theta u, \\ 0, & \text{if } v > \theta u, \end{cases}$$

we will show that the integral transformation defined by $c'_{\gamma,\delta}(u,v)$ is a $c_0 \to c_0$ type transformation. We use Theorem 6 from [3] which gives the sufficient conditions for the regularity of integral methods. Let us prove first that

$$\int_0^{v_0} c'_{\gamma,\delta}(u,v)\,dv = o_\theta(1) \quad \text{as} \quad u \to \infty,$$

assuming that v_0 is a fixed positive number and $v < v_0 < \theta u$. We get:

$$\int_0^{v_0} c'_{\gamma,\delta}(u,v)\,dv = \frac{1}{r_\delta(u)} \int_0^{v_0} (u-v)^{\delta-\gamma-2} r_{\gamma+1}(v)\,dv \le \frac{r_{\gamma+1}(u)}{r_\delta(u)}$$
$$\times \int_0^{v_0} (u-v)^{\delta-\gamma-2}\,dv = O(1) u^{\gamma+1-\delta}(u-v)^{\delta-\gamma-1}\Big|_0^{v_0}$$
$$= O(1)\Big[u^{\gamma+1-\delta}(u-v_0)^{\delta-\gamma-1} - 1\Big]$$
$$= O(1)\Big[(1-\frac{v_0}{u})^{\delta-\gamma-1} - 1\Big] = o_\theta(1) \text{ as } u \to \infty.$$

Following Theorem 6 from [3] it remains to show that the condition

$$\int_0^{\theta u} c'_{\gamma,\delta}(u,v)\,dv = O_\theta(1) \qquad (u > 0)$$

is also fulfilled. With the help of (3.1) we get:

$$\int_0^{\theta u} \frac{r_{\gamma+1}(v)}{r_\delta(u)}(u-v)^{\delta-\gamma-2}\,dv \le \frac{r_{\gamma+1}(u)}{r_\delta(u)} \int_0^{\theta u} (u-v)^{\delta-\gamma-2}\,dv$$
$$= O(1)u^{\gamma+1-\delta}(u-\theta u)^{\delta-\gamma-1} = O_\theta(1).$$

Thus we have shown that the integral transformation defined by $c'_{\gamma,\delta}(u,v)$ is of type $c_0 \to c_0$ for every $\theta \in (1/2;1)$, and therefore condition $I_1''(u,\theta) = o_\theta(1)$ is satisfied. By the obtained relations we have that

$$I_1(u,\theta) = I_1'(u,\theta) + I_1''(u,\theta) = o_\theta(1) \quad \text{as } u \to \infty. \tag{3.6}$$

Next we evaluate the quantity $I_2(u,\theta)$ using relations (3.1) and (3.3):

$$I_2(u,\theta) = O(1) \int_{\theta u}^{u} (u-v)^{\delta-\gamma-1} \frac{r_{\gamma+1}(v)}{v r_\delta(u)}\,dv = O(1)\frac{r_{\gamma+1}(u)}{r_\delta(u)\theta u}$$
$$\times \int_{\theta u}^{u} (u-v)^{\delta-\gamma-1}\,dv = O(1)\,u^{\gamma-\delta}(u-v)^{\delta-\gamma}\Big|_u^{\theta u} = O(1)\,(1-\theta)^{\delta-\gamma}.$$

So we have the estimate

$$I_2(u,\theta) = O(1)(1-\theta)^{\delta-\gamma}. \tag{3.7}$$

Now we are able to complete our proof showing that (3.4) is true for every $\gamma < \delta < \gamma + 1$. We choose $\varepsilon > 0$ and afterwards $\theta_\varepsilon \in (1/2, 1)$ so, that

$$I_2(u, \theta_\varepsilon) = O(1)(1 - \theta_\varepsilon)^{\delta-\gamma} < \frac{\varepsilon}{2} \text{ for any } u > 0$$

(see (3.7)). Next we choose $U = U_{\theta_\varepsilon}$ so, that $|I_1(u, \theta_\varepsilon)| < \varepsilon/2$ for all $u > U$ (see (3.6)). It follows from (3.5) that $|y_\delta(u)| < \varepsilon$ when $u > U$, i.e., (3.4) holds. Thus we have shown that implication (3.2) is true for all $\beta > \delta > \gamma > \alpha_1$.

If $\gamma = \alpha_1$, then we choose some $\gamma < \gamma_1 < \delta$ and get that $x(u) = O(A_\gamma)$ implies $x(u) = O(A_{\gamma_1})$. To finish the proof, it remains to apply implication (3.2), already proved, with γ_1 instead of γ. □

Note that Theorem 1 was formulated (but not proved) in [13] as Proposition 4 with a hint on analogy with matrix case (see [10, 11]). The following Tauberian remainder theorem extends Theorem A.

Theorem 2. *Let $\{A_\alpha\}\,(\alpha > \alpha_0)$ be a Riesz-type family. Let some positive function $\lambda = \lambda(u) \to \infty$ (as $u \to \infty$) from X and some number $\gamma > \alpha_0$ such that $r_\gamma(u)/\lambda(u) \in X$ be given. Suppose that $b_\beta(u)$ and $\lambda_\beta(u)$ are defined through* (2.2). *Suppose also that the following condition*

$$b_\beta(u)/b_\alpha(u) \approx u^{\beta-\alpha} \quad (u > 0) \tag{3.8}$$

is satisfied for any $\beta > \alpha > \gamma$. Then we have for functions $x = x(u)$ and numbers s and $\beta > \delta > \gamma$ that

$$\lambda(u)[y_\gamma(u) - s] = O(1),\ \lambda_\beta(u)[y_\beta(u) - s] \to t \implies \lambda_\delta(u)[y_\delta(u) - s] \to t. \tag{3.9}$$

Proof. We set $\alpha_1 = \gamma$ and construct another family $\{B_\alpha\}(\alpha \geq \gamma)$ on the base of relations (2.2). Namely, we define the methods B_α by the transformations of a function $y = y(u) \in X$ into $\eta_\alpha = \eta_\alpha(u)$ with

$$\eta_\alpha(u) = \frac{M_{\gamma,\alpha}}{b_\alpha(u)} \int_0^u (u - v)^{\beta-\gamma-1} b_\gamma(v) y(v)\, dv \quad (\alpha > \gamma)$$

and $\eta_\gamma(u) = y(u)$, i.e., $B_\gamma = I$. The family $\{B_\alpha\}(\alpha \geq \gamma)$ is a Riesz-type family (see Example 3) satisfying the presumptions of Theorem 1. Let us apply methods B_α to $y = \lambda(u)\,[y_\gamma(u) - s]$ and realize that $B_\alpha y = \eta_\alpha(u) = \lambda_\alpha(u)\,[y_\alpha(u) - s]$ for any $\alpha > \gamma$. Thus, implication (3.9) holds by Theorem 1 for any $\beta > \delta > \gamma$ as (3.2) in the form

$$y(u) = O\,(B_\gamma), \quad y(u) \to t\,(B_\beta) \implies y(u) \to t\,(B_\delta).$$

□

An analogous Tauberian remainder theorem for "matrix case" was proved in [12] as Theorem 2. Some Tauberian remainder theorems for Nörlund and Riesz matrix methods in Banach spaces were proved recently in [6] and [7]. Some estimates for speeds in a Riesz-type family (weaker than here) can be found also in [8].

4 Examples on Comparison of Speeds of Convergence

Here we give some numerical examples on application of Theorem 2 for comparison of speeds of convergence in special Riesz-type families. More precisely, we extend Examples 5, 7 and 9 from [9], where Theorem A was applied. In mentioned examples comparative evaluations (2.1) and (2.3) for speeds of any two methods A_γ and A_β in Riesz-type families $\{A_\alpha\}$ are presented. Here we improve these results, comparing any three methods A_γ, A_β and A_δ with the help of implication (3.9).

Example 4. We consider the Riesz methods $A_\alpha = (R, \alpha)$ $(\alpha > 0)$. Choose the speed of convergence $\lambda(u) = (u+1)^\rho$ $(\rho > 0)$ and fix some number $\gamma > 0$. Suppose that $x = x(u)$ is a function having a given speed of convergence $\lambda(u)$ with respect to the method $A_\gamma = (R, \gamma)$ and define with the help of formulas (2.2) the function $b_\beta(u)$ and afterwards the speed of convergence $\lambda_\beta(u)$ of $x = x(u)$ with respect to the methods $A_\beta = (R, \beta)$ for $\beta > \gamma$. In Example 5 in [9] the following estimates for $b_\beta(u)$ and $\lambda_\beta(u)$ were proved for any $\beta > \gamma$ if $u \to \infty$:

$$b_\beta(u) \sim M_{\gamma,\beta} B(\beta-\gamma, \gamma-\rho+1) u^{\beta-\rho}/\gamma, \quad \text{if } \rho < \gamma+1, \tag{4.1}$$

$$b_\beta(u) \approx \begin{cases} u^{\beta-\gamma-1}\log u, & \text{if } \rho = \gamma+1, \\ u^{\beta-\gamma-1}, & \text{if } \rho > \gamma+1, \end{cases} \tag{4.2}$$

$$\lambda_\beta(u) \sim \frac{\Gamma(\gamma+1)\Gamma(\beta-\rho+1)}{\Gamma(\beta+1)\Gamma(\gamma-\rho+1)}\, u^\rho \sim \frac{\Gamma(\gamma+1)\Gamma(\beta-\rho+1)}{\Gamma(\beta+1)\Gamma(\gamma-\rho+1)}\, \lambda(u), \quad \text{if } \rho < \gamma+1, \tag{4.3}$$

$$\lambda_\beta(u) \approx \begin{cases} u^\rho/\log u \sim \lambda(u)/\log u, & \text{if } \rho = \gamma+1, \\ u^\rho\, u^{\gamma-\rho+1} \sim \lambda(u)\, u^{\gamma-\rho+1}, & \text{if } \rho > \gamma+1. \end{cases} \tag{4.4}$$

Estimates (4.1)–(4.2) show that condition (3.8) is satisfied for all $\beta > \alpha > \gamma$. Thus Theorem 2 applies, and implication (3.9) is true for any $\beta > \delta > \gamma$ where speeds λ_β and λ_δ obey evaluates (4.3) and (4.4).

Example 5. Let us consider the Borel-type methods $A_\alpha = (B, \alpha, 1/n!) = (B, \alpha)$ $(\alpha > -1/2)$. Suppose that $\lambda(u) = (u+1)^\rho e^u$, fix some $\gamma > -1/2$ and find $\lambda_\beta(u)$ for $\beta > \gamma$ through (2.2) again. In Example 7 in [9] for $\beta > \gamma$ the following estimates were proved:

$$b_\beta(u) \approx \begin{cases} u^{\beta-\gamma-1}, & \text{if } \rho > 1, \\ u^{\beta-\gamma-1}\log u, & \text{if } \rho = 1, \\ u^{\beta-\gamma-\rho}, & \text{if } \rho < 1, \end{cases} \tag{4.5}$$

$$\lambda_\beta(u) \approx \begin{cases} \dfrac{e^u}{u^{\beta-\gamma-1}} \sim \dfrac{\lambda(u)}{u^{\beta-\gamma+\rho-1}}, & \text{if } \rho > 1, \\ \dfrac{e^u}{u^{\beta-\gamma-1}\log u} \sim \dfrac{\lambda(u)}{u^{\beta-\gamma}\log u}, & \text{if } \rho = 1, \\ \dfrac{e^u}{u^{\beta-\gamma-\rho}} \sim \dfrac{\lambda(u)}{u^{\beta-\gamma}}, & \text{if } \rho < 1. \end{cases} \tag{4.6}$$

Condition (3.8) is satisfied for all $\beta > \alpha > \gamma$ by relations (4.5). Therefore, Theorem 2 applies again, and implication (3.9) is true for any $\beta > \delta > \gamma$ where speeds λ_β and λ_δ obey evaluates (4.6).

Example 6. Suppose that $A_\alpha = (N, u^{\alpha-1}, e^{u^\varphi})$ $(\alpha > 0)$ where $0 < \varphi < 1$ is some fixed number. Suppose that $\lambda(u) = e^{u^\varphi}$. It was shown in Example 9 in [9] that $b_\beta(u) \approx u^{\beta-\gamma+(1-\varphi)\gamma}$ and $\lambda_\beta(u) \approx e^{u^\varphi} u^{\varphi(\gamma-\beta)} = u^{\varphi(\gamma-\beta)}\lambda(u)$ for $\beta > \gamma$. We see that (3.8) is satisfied for any $\beta > \alpha > \gamma$. Therefore, implication (3.9) is true for any $\beta > \delta > \gamma$ by Theorem 2.

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