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A Constraint Preconditioner for Solving Symmetric Positive Definite Systems and Application to the Helmholtz Equations and Poisson Equations

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Abstract. In this paper, first, by using the diagonally compensated reduction and incomplete Cholesky factorization methods, we construct a constraint preconditioner for solving symmetric positive definite linear systems and then we apply the preconditioner to solve the Helmholtz equations and Poisson equations. Second, according to theoretical analysis, we prove that the preconditioned iteration method is convergent. Third, in numerical experiments, we plot the distribution of the spectrum of the preconditioned matrix $\mathbb{M}^{-1}A$ and give the solution time and number of iterations comparing to the results of [5, 19].

Keywords: Helmholtz equations, constraint preconditioner, preconditioned iteration method, incomplete Cholesky factorization.

AMS Subject Classification: 65F10; 65N22; 65F50.

1 Introduction

Let us consider the Helmholtz equation

$$\begin{cases} -\triangle u - \lambda u = f, \quad \text{in} \quad \Omega, \\ u = g, \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω and $\partial\Omega$ are bounded domain in $\mathbb{R}^2$ and its boundary, respectively. Here, f and g are given real continuous functions on $\bar{\Omega}$ and $\partial\Omega$, respectively, and $\lambda \geq 0$ is a real constant coefficient.

From the discretization of the Helmholtz equation (1.1) by using the five-point finite difference scheme, we obtain the following linear system

$$Ax = b, \tag{1.2}$$

where $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix. For the convenience of our studies and statements, we write A in the following form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where $A_{11} \in \mathbb{R}^{p\times p}$, $A_{22} \in \mathbb{R}^{q\times q}$ with $p + q = n$, and $A_{12}^T = A_{21}$. Here A_{12}^T denotes the transpose of A_{12}. Let

$$x = \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} f \\ g \end{pmatrix}$$

be partitions of the vectors x and b corresponding to the partitions of the matrix A.

Reusken studied in [19] the Helmholtz equation (1.1) with homogeneous Dirichlet boundary condition on the boundary of the domain $\Omega = (0,1)\times(0,1)$. He used the five-point finite difference scheme to discretize equation (1.1) and obtained the linear system (1.2). By straightforward computation, Reusken obtained that the smallest eigenvalue of the matrix A of the linear system (1.2) is

$$\lambda_{\min}(A) = 19.73921 - \lambda + O(h^2),$$

where h is the mesh size. Reusken further pointed out that the coefficient matrix A is symmetric positive definite (s.p.d.), when the inequality $\lambda < 19.73921$ holds, and it is indefinite when the inequality $\lambda > 19.73921$ is satisfied.

Researchers usually solve Partial Differential Equations (PDEs) by using the finite difference approximation or the finite element method. From the discretized version, one knows that the resulting linear systems of equations are generally large and sparse. It is not considerably practicable to directly solve this type of linear systems. In order to obtain the more efficient solution, one often adopts preconditioned iteration methods to solve this kind of linear systems (see [20]).

The Helmholtz equation is a fundamental equation for time-harmonic wave propagation and it is widely applied in underwater acoustics, medicine, geophysics and so on. In fact, solving Helmholtz equations is a computationally challenging problem in computational electromagnetics. Various preconditioners and iteration solution techniques have been proposed and many researches have been contributed to numerical methods for solution of this type of equation. Mainly two kinds of methods were considered: the boundary element method [7] and the domain based method [9]. In order to avoid the indefinite systems, Heikkola *et al.* [9] have used the time-dependent equations instead of the complex-valued time-harmonic equations.

To solve the system (1.2), we apply preconditioned iteration methods based on the Krylov subspace methods, e.g., PCG [16] method, the GMRES [21] method and the BICGSTAB [22] method. By using finite difference scheme or finite element method, we discretize the three-dimensional Helmholtz problem, especially, the mid-frequency and the high-frequency problems. The resulting linear systems are extremely large, non-Hermitian, indefinite and badly conditioned. While the Krylov subspace methods are directly applied to solve this type of linear systems, the convergence rate is very slow and the computational cost is expensive. In order to improve the speed of convergence and reduce the computational cost, Erlangga et al. [10, 11] proposed the shifted-Laplacian preconditioner by adding additional damping to solve the low-frequency and the mid-frequency problems, and used the algebraic multigrid (AMG) method [14, 15] as a tool for the approximation of the inversion of the damped Helmholtz operator. The two-dimensional acoustic scattering experiments showed that the shifted-Laplacian preconditioner is particularly effective and practicable.

For solving a 3D exterior Helmholtz problem, Chen et al., [6] presented a new and high-order collocation method. They used domain integrals to remove the hypersingularity and then used singularity subtraction to avoid domain integrals. Based on a marching finite-difference scheme, Yuriy et al., [24] developed two inverse algorithms: H-method and p-method, they presented an efficient preconditioner and used the preconditioned conjugate gradient method for rapid solving of the 2D Helmholtz equation.

Axelsson and Kolotilina [2] proposed diagonally compensated reduction method which modifies a symmetric, positive definite matrix leading to an M-matrix by reducing positive off-diagonal entries and subsequently adding them to the corresponding diagonal entry. Based on this method, the s.p.d. matrix A was split into the sum $A = B + R$, where R is a symmetric, positive semi-definite matrix and its nonzero entries are the corresponding off-diagonal positive entries of A. Let us denote by $\tilde{A} = A + D$ the following matrix

$$\tilde{A} = \begin{pmatrix} A_{11} + D_1 & A_{12} \\ A_{21} & A_{22} + D_2 \end{pmatrix} \triangleq \begin{pmatrix} \tilde{A}_{11} & A_{12} \\ A_{21} & \tilde{A}_{22} \end{pmatrix}, \tag{1.3}$$

where $D = \mathrm{diag}(D_1, D_2)$ is a diagonally compensated reduction matrix (see [2]), such that the following equality

$$De = Re$$

holds, where all entries of the vector e are equal to one. Let $\mathbb{M} = B + D$. Then matrix A is split as

$$A = \mathbb{M} - (D - R),$$

and matrix $\tilde{A}$ is split as

$$\tilde{A} = \mathbb{M} - (-R).$$

The way of perturbing the diagonal is very similar to the real-shifted Laplacian method [3, 10, 11]. The only difference is only how the preconditioner presented in this paper is inverted.

Keller, Gould and Wathen [13] introduced the following constraint preconditioner

$$\mathbb{P} = \begin{pmatrix} E & A_{12} \\ A_{21} & 0 \end{pmatrix},$$

where E is symmetric and it approximates A. The preconditioner was designed for solving the linear system (1.2) with the (2, 2) block A_{22} being zero. Dollar

[8] extended this idea to the case where the (2, 2) block is symmetric and positive semidefinite.

In this paper, for solving symmetric positive definite linear systems in general and the Helmholtz equations and Poisson equations as particular examples, we construct a new constraint preconditioner. By using diagonally compensated reduction and selecting proper diagonal matrix Λ corresponding to A, we prove that $\Lambda+A$ is a s.p.d. H-matrix, where A is the coefficient matrix of the linear system (1.2). Further, by employing incomplete Cholesky factorization for the $(1,1)$ block principal submatrix of $D+A$, we construct the constraint preconditioner $\mathbb{M}$ and theoretically prove the convergence of the preconditioned iterative method with the preconditioned matrix $\mathbb{M}^{-1}\tilde{A}$. According to Yun and Kim [23], when we apply the preconditioner $\mathbb{M}$ to the original linear system (1.2), we know that the preconditioned iteration method with the preconditioned matrix $\mathbb{M}^{-1}A$ is also convergent. In the numerical experiments, we plot the distribution of the spectrum of the preconditioned matrix $\mathbb{M}^{-1}A$ and give the solution time and number of iterations required to solve the test systems. The obtained results are compared with the results of [5, 19].

2 Convergence Analysis of the Constraint Preconditioned Method

In this section, we use the diagonally compensated reduction and incomplete Cholesky factorization to construct a new type of preconditioners for solving s.p.d. systems. In the following sections, we study the convergence properties of this type of preconditioned iterative methods.

For the convenience of our statements and proofs, we give some notations and lemmas. Let $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ and $B = (b_{ij}) \in \mathbb{R}^{m\times n}$, where $\mathbb{R}^{m\times n}$ denotes the set of $m \times n$ real matrices. We write $A \geq B$, if $a_{ij} \geq b_{ij}$ for any i and j and denote by $|A| = (|a_{ij}|)$ the absolute matrix of A.

$A \in Z_n = \{A \in \mathbb{R}^{n\times n} | a_{ij} \leq 0, i \neq j; i, j = 1, \cdots, n\}$ is an M-matrix if $A = sI - B$, where $s \geq 0$, $B \geq 0$, $s > \rho(B)$ and $\rho(B)$ is the spectral radius of B. A complex $n \times n$ matrix A is called an H-matrix if $\mu(A) \in M_n$, where $\mu(A) = (\mu_{ij})$ is the comparison matrix of A defined by

$$\mu_{ij} = \begin{cases} -|a_{ij}|, & i \neq j, \\ \ \ |a_{ij}|, & i = j. \end{cases}$$

Denote by M_n, H_n the set of $n \times n$ M-, H-matrices, respectively.

Lemma 1. [4, 12]. *Let A and $B \in \mathbb{R}^{n\times n}$. If $A \in M_n$, $B \in Z_n$ and $A \leq B$, then $B \in M_n$.*

Lemma 2. *Let $A \in \mathbb{R}^{n\times n}$ and $B \in M_n$. If $\mu(A) \geq B$, then $A \in H_n$ and*

$$B^{-1} \geq |A^{-1}| \geq 0.$$

For the proof, see, e.g., [12] p. 117 and p. 131.

Lemma 3. [1]. *Let $A \in H_n$. The splitting $A = M - N$ is called an H-compatible splitting if $\mu(A) = \mu(M) - |N| \in M_n$ and $\mu(M) \in M_n$. Then $A = M - N$ is a convergent splitting, i.e.,*

$$\rho(M^{-1}N) \leq \rho\{[\mu(M)]^{-1}|N|\} < 1.$$

Let the incomplete LU factorizations of A_{11} and $\mu(A_{11})$ be

$$A_{11} = LU - R, \tag{2.1}$$

and

$$\mu(A_{11}) = \tilde{L}\tilde{U} - \tilde{R}, \tag{2.2}$$

where U and $\tilde{U}$ are upper triangular matrices and L, $\tilde{L}$ are lower triangular matrices.

Lemma 4. [17]. *Let the incomplete LU factorization of A_{11} and $\mu(A_{11})$ be defined as in (2.1) and (2.2). Then we have*

$$|L^{-1}| \leq \tilde{L}^{-1}, \quad |U^{-1}| \leq \tilde{U}^{-1}, \quad |R| \leq \tilde{R}, \tag{2.3}$$

$$|(LU)^{-1}R| \leq (\tilde{L}\tilde{U})^{-1}\tilde{R}. \tag{2.4}$$

Proof. According to [17] Theorem 3.3, we review the following important results

$$U = A_{n-1}, \quad \tilde{U} = B_{n-1}, \quad L = \left(\prod_{k=1}^{n-1} L_{n-k}\right)^{-1},$$

$$\tilde{L} = \left(\prod_{k=1}^{n-1} \tilde{L}_{n-k}\right)^{-1}, \quad R = \sum_{k=1}^{n-1} R_k, \quad \tilde{R} = \sum_{k=1}^{n-1} \tilde{R}_k,$$

and for $k = 1, 2, \ldots, n-1$, the following inequalities hold

$$|A_k^{-1}| \leq |B_k^{-1}|, \quad |L_k| \leq \tilde{L}_k, \quad |R_k| \leq \tilde{R}_k,$$

where $A_k, B_k, R_k, \tilde{R}_k, R_k, \tilde{L}_k$ and L_k are defined as in [17] Theorem 3.3.

So, we obtain the following inequalities:

$$|U^{-1}| \leq \tilde{U}^{-1}, \quad |L^{-1}| = \left|\prod_{k=1}^{n-1} L_{n-k}\right| \leq \prod_{k=1}^{n-1} \tilde{L}_{n-k} \leq \tilde{L}^{-1}, \quad |R| \leq \tilde{R}.$$

According to the above analysis, we further have

$$|(LU)^{-1}R| \leq |(LU)^{-1}||R| \leq |U^{-1}||L^{-1}||R| \leq (\tilde{L}\tilde{U})^{-1}\tilde{R}.$$

For the sake of convenience, we let

$$M_{11} = LU, \quad \tilde{M}_{11} = \tilde{L}\tilde{U}. \tag{2.5}$$

Then the matrix M may be presented as:

$$\mathbb{M} \triangleq \begin{pmatrix} M_{11} & A_{12} \\ A_{21} & \tilde{A}_{22} \end{pmatrix}. \tag{2.6}$$

□

Lemma 5. *Let A be a symmetric positive definite matrix, $\tilde{A} = A + D$ be a diagonally dominated matrix of A and D be a diagonally compensated reduction matrix. Let $\tilde{A} = \mathbb{M} - \tilde{N}$ be a splitting, $\mathbb{M}$ be a symmetric positive definite matrix and $N = \mathbb{M} - A$. If the splitting $\tilde{A} = \mathbb{M} - \tilde{N}$ is convergent, then the splitting $A = \mathbb{M} - N$ is also convergent.* (cf. [23] Lemma 3.10).

Theorem 1. *Let $\tilde{A}$ be an H-matrix and be partitioned into* (1.3). *Assume that M_{11} and $\mathbb{M}$ are defined as in* (2.5) *and* (2.6), *respectively. Then $\tilde{A} = \mathbb{M} - (\mathbb{M} - \tilde{A}) \equiv \mathbb{M} - \tilde{N}$ is an H–compatible splitting, and*

$$\rho(\mathbb{M}^{-1}\tilde{N}) < 1.$$

Proof. Notice that we get $\tilde{R} \geq 0$ from (2.3), thus $\tilde{L}\tilde{U} \geq \mu(A_{11})$. Further, since $\tilde{L}^{-1} \geq 0$ and $\tilde{U}^{-1} \geq 0$, then $\tilde{M}_{11}^{-1} \leq [\mu(\tilde{A}_{11})]^{-1}$. Therefore, by (2.3), (2.5) and the definition of the comparison matrix, we have

$$\begin{aligned}\mu(\tilde{A}_{22} - A_{21}M_{11}^{-1}A_{12}) &\geq \mu(\tilde{A}_{22}) - |A_{21}||M_{11}^{-1}||A_{12}| \\ &= \mu(\tilde{A}_{22}) - |A_{21}||U^{-1}L^{-1}||A_{12}| \geq \mu(\tilde{A}_{22}) - |A_{21}|\tilde{U}^{-1}\tilde{L}^{-1}|A_{12}| \\ &= \mu(\tilde{A}_{22}) - |A_{21}|\tilde{M}_{11}^{-1}|A_{12}| \geq \mu(\tilde{A}_{22}) - |A_{21}|[\mu(\tilde{A}_{11})]^{-1}|A_{12}|.\end{aligned}$$

Though $(LU)^{-1}$ and $(\tilde{L}\tilde{U})^{-1}$ are not an H-matrix and an M-matrix respectively, however, due to $\mu(\tilde{A}_{22}) - |A_{21}|[\mu(\tilde{A}_{11})]^{-1}|A_{12}|$ is an M-matrix (see [4, (5.1)]), then, according to Lemma 1, we know that $\mu(\tilde{A}_{22} - A_{21}M_{11}^{-1}A_{12})$ is still an M-matrix. Similarly, we can get $\mu(\tilde{A}_{22}) - |A_{21}|\tilde{M}_{11}^{-1}|A_{12}|$ is also an M-matrix. Further, by the definition of H-matrix, $\tilde{A}_{22} - A_{21}M_{11}^{-1}A_{12}$ is an H-matrix. Therefore, from [12], the following inequality holds

$$\left|(\tilde{A}_{22} - A_{21}M_{11}^{-1}A_{12})^{-1}\right| \leq \left(\mu(\tilde{A}_{22}) - |A_{21}|\tilde{M}_{11}^{-1}|A_{12}|\right)^{-1}.$$

By simple manipulation, we get

$$|\mathbb{M}^{-1}\tilde{N}| = \left|\begin{pmatrix} M_{11} & A_{12} \\ A_{21} & \tilde{A}_{22}\end{pmatrix}^{-1}\begin{pmatrix} R - D_1 & 0 \\ 0 & 0\end{pmatrix}\right| = \begin{pmatrix} \Theta & 0 \\ \Phi & 0\end{pmatrix},$$

where

$$\begin{aligned}\Theta &= |M_{11}^{-1}(R - D_1) + M_{11}^{-1}A_{12}(\tilde{A}_{22} - A_{21}M_{11}^{-1}A_{12})^{-1}A_{21}M_{11}^{-1}(R - D_1)|, \\ \Phi &= | - (\tilde{A}_{22} - A_{21}M_{11}^{-1}A_{12})^{-1}A_{21}M_{11}^{-1}(R - D_1)|.\end{aligned}$$

According to (2.4) and the above analysis, the following inequalities are valid

$$\begin{aligned}\Theta &\leq |M_{11}^{-1}(R - D_1)| + |M_{11}^{-1}||A_{12}||(\tilde{A}_{22} - A_{21}M_{11}^{-1}A_{12})^{-1}||A_{21}||M_{11}^{-1}(R - D_1)| \\ &\leq \tilde{M}_{11}^{-1}(\tilde{R} - D_1) + \tilde{M}_{11}^{-1}|A_{12}|[\mu(\tilde{A}_{22}) - |A_{21}|\tilde{M}_{11}^{-1}|A_{12}|]^{-1}|A_{21}|\tilde{M}_{11}^{-1}(\tilde{R} - D_1),\end{aligned}$$

and

$$\begin{aligned}\Phi &= | - (\tilde{A}_{22} - A_{21}M_{11}^{-1}A_{12})^{-1}A_{21}M_{11}^{-1}(R - D_1)| \\ &\leq |(\tilde{A}_{22} - A_{21}M_{11}^{-1}A_{12})^{-1}||A_{21}||M_{11}^{-1}(R - D_1)| \\ &\leq [\mu(\tilde{A}_{22}) - |A_{21}|\tilde{M}_{11}^{-1}|A_{12}|]^{-1}|A_{21}|\tilde{M}_{11}^{-1}(\tilde{R} - D_1).\end{aligned}$$

Then, we further get

$$|\mathbb{M}^{-1}\tilde{N}| = \left| \begin{pmatrix} M_{11} & A_{12} \\ A_{21} & \tilde{A}_{22} \end{pmatrix}^{-1} \begin{pmatrix} R-D_1 & 0 \\ 0 & 0 \end{pmatrix} \right|$$
$$\leq \begin{pmatrix} \tilde{M}_{11} & -|A_{12}| \\ -|A_{21}| & \mu(\tilde{A}_{22}) \end{pmatrix}^{-1} \begin{pmatrix} \tilde{R}-D_1 & 0 \\ 0 & 0 \end{pmatrix} \equiv \tilde{M}^{-1}\hat{N}.$$

Therefore, according to [18], we have $\rho(|\mathbb{M}^{-1}\tilde{N}|) \leq \rho(\tilde{M}^{-1}\hat{N})$. From [23] Lemma 3.12, we see that $\mu(A) = \tilde{M} - \hat{N}$ is an regular splitting, and $\mu(A)$ is an M-matrix, thus $\rho(\tilde{M}^{-1}\hat{N}) < 1$. So, we further have

$$\rho(|\mathbb{M}^{-1}\tilde{N}|) \leq \rho(\tilde{M}^{-1}\hat{N}) < 1.$$

Then, we complete this proof. □

Theorem 2. *Let A be a s.p.d. matrix and $\tilde{A}$ satisfy the condition of Theorem 2.4. Assume $\mathbb{M}$ is defined as in* (2.6) *and $N = \mathbb{M} - A$, then the splitting $A = \mathbb{M} - N$ is a convergent splitting, so, the following inequality holds:*

$$\rho(\mathbb{M}^{-1}N) < 1.$$

Proof. By Lemma 3 and Theorem 1, we get the above inequality. □

3 Application to the Helmholtz and Poisson equations

In this section, in order to validate the performance of the constraint preconditioner $\mathbb{M}$, we present some numerical experiments for solving the Helmholtz equation (1.1) with $\lambda \neq 0$ and Poisson equation with $\lambda = 0$. Using the standard finite difference discretization, we obtain the linear system (1.2). In order to obtain faster convergence rate and reduce solution time, we use the constraint preconditioner to accelerate the Krylov subspace methods, e.g., PCG, GMRES and BiCGSTAB.

In our numerical experiments, we choose the zero vector as the initial guess and take the right-hand-side vector b so that the exact solutions x and y are the unity vectors with all entries equal to one. We use the estimate

$$\|b - Ax^k\|_2 < 10^{-6}\|b\|_2$$

as a stopping criterion, where x^k is the solution at the kth iterate. We denote by $\|\cdot\|_2$ the 2-norm and let the maximum number of iterations of PCG , GMRES and BiCGSTAB be 100. We denote by IT the number of iterations and by CPU the solution time of PCG, BiCGSTAB and GMRES iteration methods with the preconditioners $\mathbb{M}$, $\mathbb{M}_1$, $\mathbb{M}_2$ and $\mathbb{M}_3$, respectively.

According to the partition in (1.2), in the following examples without loss of generalize, we denote by p the integer part of $n/3$ and $q = n - p$. We perform all numerical experiments with MATLAB 7.01 and use the computer which is a PC-Intel(R), Core(TM)2 CPU T7200 2.0 GHz, 1024 M of RAM.

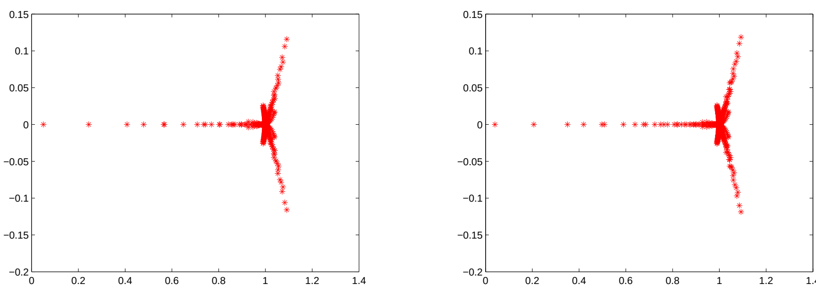
Example 1. Consider the Helmholtz equation (1.1) with $\lambda \neq 0$ and use five-point finite difference scheme, we obtain the coefficient matrix of linear system (1.2) as

$$A = \begin{pmatrix} B & C & & & \\ C & B & C & & \\ & \ddots & \ddots & \ddots & \\ & & C & B & C \\ & & & C & B \end{pmatrix},$$

where

$$B = \begin{pmatrix} \frac{4}{h^2} - \lambda & -\frac{1}{h^2} & & & \\ -\frac{1}{h^2} & \frac{4}{h^2} - \lambda & -\frac{1}{h^2} & & \\ & \ddots & \ddots & & \\ & -\frac{1}{h^2} & \frac{4}{h^2} - \lambda & -\frac{1}{h^2} & \\ & & -\frac{1}{h^2} & \frac{4}{h^2} - \lambda & \end{pmatrix}, \quad C = \begin{pmatrix} -\frac{1}{h^2} & & & \\ & -\frac{1}{h^2} & & \\ & & \ddots & \\ & & & -\frac{1}{h^2} \end{pmatrix},$$

with $B, C \in \mathbb{R}^{m \times m}$, $A \in \mathbb{R}^{n \times n}$ and $n = m^2$. In this example, we compare the performance of the new preconditioner $\mathbb{M}$ and preconditioners $\mathbb{M}_1$ and $\mathbb{M}_2$ presented in [19]. For the convenience of comparison, we only consider the Galerkin approximation $\hat{S}_{bb}^1$ and $\tilde{S}_{bb}^2$ corresponding to the preconditioner $\mathbb{M}_1$ and preconditioner $\mathbb{M}_2$, respectively, where $\hat{S}_{bb}^1$ and $\tilde{S}_{bb}^2$ are defined as in [19, (10)].

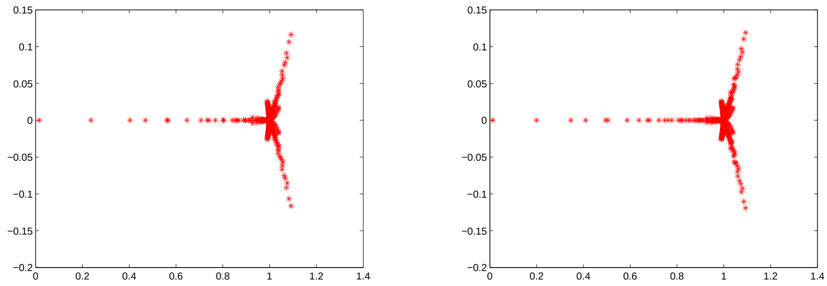


$n = 45 \times 45$ $\qquad$ $n = 50 \times 50$

Figure 1. The spectrum of the preconditioned matrix $\mathbb{M}^{-1}A$ with $\lambda = 17$.

In Figure 1 and Figure 2, we study clustering properties of the spectrum of the preconditioned matrices $\mathbb{M}^{-1}A$ for $\lambda = 17$ and $\lambda = 19$, respectively. From these pictures, we see that the eigenvalues of the preconditioned matrix $\mathbb{M}^{-1}A$ have only one clustered point (1,0) on the complex plane and are strongly clustered.

The solution time and number of iterations of BiCGSTAB and GMRES iteration methods with preconditioners $\mathbb{M}$, $\mathbb{M}_1$ and $\mathbb{M}_2$ are presented in Table 1 for $\lambda = 17$ and Table 2 for $\lambda = 19$. From results presented in tables, we see that the CPU time and number of iterations of BiCGSTAB and GMRES iteration



$n = 45 \times 45$ $\quad$ $n = 50 \times 50$

Figure 2. The spectrum of the preconditioned matrix $\mathbb{M}^{-1}A$ with $\lambda = 19$.

Table 1. Solution time in seconds and number of iterations of GMRES and BiCGSTAB iteration methods with preconditioner $\mathbb{M}$ presented in this paper and preconditioners $\mathbb{M}_1$, $\mathbb{M}_2$ proposed in [19] for $\mu = 10$, $\alpha = 0.5$ and $\beta = 2.2$.

	m	25	30	40	45	50
$\mathbb{M}$-bicgstab	CPU	**1.328**	**3.86**	**27.297**	**60.985**	**140.578**
	IT	**5.5**	**6.5**	**8.5**	**9.5**	**10.50**
$\mathbb{M}_1$-bicgstab	CPU	3.031	8.672	61.313	112.984	282.937
	IT	14.5	14.5	19	17.5	22.50
$\mathbb{M}_2$-bicgstab	CPU	3.438	10.328	69.312	135.391	280.391
	IT	16.5	17.5	21.5	21	23
$\mathbb{M}$-GMRES(10)	CPU	**0.968**	**2.922**	**19.125**	**44.703**	**107.703**
	IT	**8**	**9**	**11**	**3**	**4**
$\mathbb{M}_1$-GMRES(10)	CPU	3.141	4.718	54.547	60.64	202.453
	IT	9	5	13	8	13
$\mathbb{M}_2$-GMRES(10)	CPU	3.547	6.516	64.063	120.094	360.75
	IT	13	11	9	13	13

methods with constraint preconditioner $\mathbb{M}$ are less than that of BiCGSTAB and GMRES iteration methods with $\mathbb{M}_1$ and $\mathbb{M}_2$, i.e., the constraint preconditioner $\mathbb{M}$ is more efficient and accurate than the preconditioners $\mathbb{M}_1$ and $\mathbb{M}_2$.

Example 2. Consider the Helmholtz equation (1.2) and use central difference scheme, we obtain the coefficient matrix of linear system (1.2) as follows

$$A = \begin{pmatrix} A_1 & D & & & \\ D^T & A_1 & D & & \\ & \ddots & \ddots & & \\ & D^T & A_1 & D^T & \\ & & D^T & A_1 \end{pmatrix},$$

Table 2. Solution time in seconds and number of iterations of GMRES and BiCGSTAB iteration methods with preconditioner $\mathbb{M}$ presented in this paper and preconditioners $\mathbb{M}_1$, $\mathbb{M}_2$ proposed in [19] for $\mu = 10$, $\alpha = 0.5$ and $\beta = 2.2$.

	m	25	30	40	45	50
$\mathbb{M}$-bicgstab	CPU	**1.234**	**4.2030**	**30.50**	**68.2180**	**136.5470**
	IT	**6**	**7**	**9.50**	**10.50**	**11**
$\mathbb{M}_1$-bicgstab	CPU	3.234	9.0470	59.75	116.079	273.578
	IT	15.5	15	18.50	18	22.50
$\mathbb{M}_2$-bicgstab	CPU	3.532	10.594	71.141	134.171	320.063
	IT	17	17.5	22	20.50	26.50
$\mathbb{M}$-GMRES(10)	CPU	**1.4060**	**3.0150**	**22.4060**	**56.3590**	**119.046**
	IT	**8**	**8**	**3**	**4**	**9**
$\mathbb{M}_1$-GMRES(10)	CPU	1.4690	5.672	93.5780	86.953	393.672
	IT	8	8	7	11	10
$\mathbb{M}_2$-GMRES(10)	CPU	1.4690	10.157	307.578	177.156	931.078
	IT	10	10	34	14	18

where $A \in \mathbb{R}^{n\times n}$, $n = 16m$ and

$$A_1 = \begin{pmatrix} B & C & & \\ C^T & B & C & \\ & C^T & B & C \\ & & C^T & B \end{pmatrix}, \qquad D = \begin{pmatrix} E & & & \\ & E & & \\ & & E & \\ & & & E \end{pmatrix}$$

$$B = \begin{pmatrix} d & -e & -f & 0 \\ -e & d & 0 & -f \\ -f & 0 & d & -e \\ 0 & -f & -e & d \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -e & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -e & 0 \end{pmatrix}$$

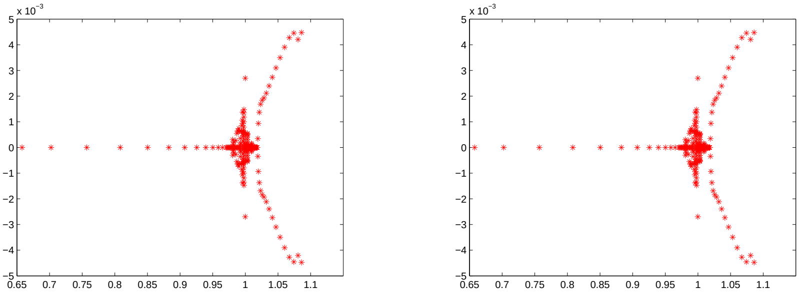
$$E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -f & 0 & 0 & 0 \\ 0 & -f & 0 & 0 \end{pmatrix}.$$

From the equation (1.3) in [5], the elements of the matrix $d = 139.5$, $e = 20.25$, $f = 49$ are obtained.

In this example, we compare the performance of the new preconditioner $\mathbb{M}$ with the preconditioner $\mathbb{M}_3$ described in [5]. The preconditioner $\mathbb{M}_3$ is given as $\mathbb{M}_3 = \frac{1}{2r}(rI + F)(rI + G)$, with $r = 8.336666$ and $G = A - F$, where

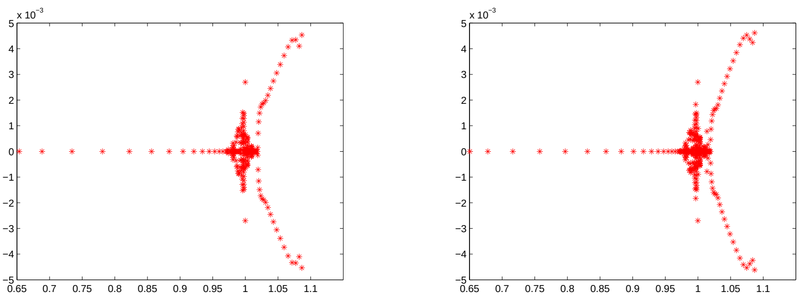
$$F = \begin{pmatrix} F_1 & & & \\ & F_1 & & \\ & & \ddots & \\ & & & F_1 \end{pmatrix}, \quad F_1 = \begin{pmatrix} \frac{d}{2} & -e & -f & 0 \\ -e & \frac{d}{2} & 0 & -f \\ -f & 0 & \frac{d}{2} & -e \\ 0 & -f & -e & \frac{d}{2} \end{pmatrix}.$$

From results given in Figure 3 and Figure 4, we see that the spectrum of the preconditioned matrix $\mathbb{M}^{-1}A$ is considerably clustered. From these pictures,



$m = 50$ $m = 60$

Figure 3. The spectrum of the preconditioned matrix $\mathbb{M}^{-1}A$ for $m = 50$ and $m = 60$



$m = 70$ $m = 80$

Figure 4. The spectrum of the preconditioned matrix $\mathbb{M}^{-1}A$ for $m = 70$ and $m = 80$

we see that almost all eigenvalues of the preconditioned matrix $\mathbb{M}^{-1}A$ are close to the point (1,0) on the complex plane. From the clustering properties of the spectrum, the preconditioner developed in this paper is very efficient.

The solution time and number of iterations of PCG, BiCGSTAB and GMRES iteration methods with preconditioners $\mathbb{M}$ and $\mathbb{M}_3$ are presented in Table 3. Here we use '-' to present the number of iterations being less than that of [5]. From the presented results, we see that the solution time and number of iterations of PCG, BiCGSTAB and GMRES iteration methods with constraint preconditioner $\mathbb{M}$ are less than that of PCG, BiCGSTAB and GMRES iteration methods with $\mathbb{M}_3$, i.e., the constraint preconditioner $\mathbb{M}$ is more efficient and accurate than the preconditioner $\mathbb{M}_3$.

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Table 3. Solution time in seconds and number of iterations of PCG, GMRES and BiCGSTAB iteration methods with preconditioners $\mathbb{M}$ presented in this paper and $\mathbb{M}_3$ proposed in [5].

	m	30	40	50	60	70	80
$\mathbb{M}$-pcg	CPU	**0.281**	**0.797**	**1.547**	**2.578**	**4.125**	**6**
	IT	**-**	**-**	**-**	**7**	**7**	**7**
$\mathbb{M}_3$-pcg	CPU	5.281	11.984	23.063	38.359	59.641	88.938
	IT	-	-	-	8	8	8
$\mathbb{M}$-GMRES(10)	CPU	**0.469**	**0.812**	**1.516**	**2.500**	**4.594**	**6.781**
	IT	**6**	**6**	**6**	**6**	**-**	**-**
$\mathbb{M}_3$-GMRES(10)	CPU	1.469	3.375	6.391	10.469	89.641	213.891
	IT	7	7	6	6	-	-
$\mathbb{M}$-bicgstab	CPU	**0.343**	**0.797**	**2.7660**	**2.594**	**4.016**	**7**
	IT	**3.5**	**3.5**	**3.5**	**3.5**	**3.5**	**4**
$\mathbb{M}_3$-bicgstab	CPU	2.704	7.719	18.578	18.765	29.61	45.078
	IT	25	31.5	24	24	25	25

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