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Linear/Linear Rational Spline Interpolation*

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Abstract. For a strictly monotone function y on $[a, b]$ we describe the construction of an interpolating linear/linear rational spline S of smoothness class C^1. We show that for the linear/linear rational splines we obtain $\|S(x_i) - y(x_i)\|_\infty = O(h^4)$ on uniform mesh $x_i = a + ih$, $i = 0, \ldots, n$. We prove also the superconvergence of order h^3 for the first derivative and of order h^2 for the second derivative of S in certain points. Numerical examples support the obtained theoretical results.

Keywords: rational spline, interpolation, superconvergence.

AMS Subject Classification: 65D07.

1 Introduction

For a smooth function y and interpolating linear/linear rational spline S it is known that $\|S - y\|_\infty = O(h^3)$, for the proof see, e.g., [8]. The linear/linear rational splines of class C^1 have the same accuracy as the classical quadratic splines. In some cases, the error is less for the quadratic splines and in some cases, the error is less for the linear/linear rational splines. For the quadratic splines, the expansions on subintervals via the derivatives of the smooth function to interpolate could be found, e.g., in [4, 5]. They give the superconvergence of the spline values and its derivatives in certain points. We will study such a problem in the case of linear/linear rational spline interpolant.

Note that, linear/linear rational splines, being strictly monotone or constant everywhere, cannot interpolate nonmonotone data. For consistent data, the linear/linear rational spline interpolant of class C^1 always exists and is unique [9]. Let us mention that $O(h^2)$ convergence rate of quadratic spline collocation for boundary value problems is based on superconvergence property of interpolating splines. This was discovered in [3] and developed extensively in [7, 10]. Polynomial interpolants are used to establish convergence results

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of quasi-interpolants, see e.g., [6] and the references therein. For linear/linear rational spline histopolation, convergence rates could be found in [2].

Let us point out that, while the interpolation problem is a linear one, the linear/linear rational spline interpolation is, in nature, a nonlinear method because it leads to a nonlinear system with respect to the spline parameters. It was shown in [1] that any strict convexity preserving interpolation method having certain regularity properties cannot be linear. Hopefully, similar result should also hold for strict monotonicity preservation.

2 Interpolation by Linear/Linear Rational Splines

Let $a = x_0 < x_1 < \ldots < x_n = b$ be a uniform partition of the interval $[a, b]$ with knots $x_i = a + ih$, $i = 0, \ldots, n$, $h = (b-a)/n$, $n \in \mathbb{N}$. We also need the points $\xi_i = x_{i-1} + h/2$, $i = 1, \ldots, n$. Linear/linear rational spline on each particular interval $[x_{i-1}, x_i]$ is a function S of the form

$$S(x) = a_i + \frac{c_i(x-\xi_i)}{1 + d_i(x-\xi_i)}, \quad x \in [x_{i-1}, x_i], \tag{2.1}$$

where $1 + d_i(x - \xi_i) > 0$. Using the notation $S(x_i) = S_i$, $i = 0, \ldots, n$, and $S(\xi_i) = \bar{S}_i$, $i = 1, \ldots, n$, we get from (2.1)

$$\bar{S}_i = a_i, \quad S_{i-1} = a_i - \frac{hc_i}{2 - hd_i}, \quad S_i = a_i + \frac{hc_i}{2 + hd_i},$$

which allows to express uniquely a_i, c_i and d_i via the spline values and to have the representation

$$S(x) = \bar{S}_i + \frac{4(S_i - \bar{S}_i)(\bar{S}_i - S_{i-1})(x-\xi_i)}{h(S_i - S_{i-1}) + 2((\bar{S}_i - S_{i-1}) - (S_i - \bar{S}_i))(x - \xi_i)}, \ x \in [x_{i-1}, x_i]. \tag{2.2}$$

This also gives for $x \in [x_{i-1}, x_i]$

$$S'(x) = \frac{4h(S_i - \bar{S}_i)(\bar{S}_i - S_{i-1})(S_i - S_{i-1})}{(h(S_i - S_{i-1}) + 2((\bar{S}_i - S_{i-1}) - (S_i - \bar{S}_i))(x - \xi_i))^2}, \tag{2.3}$$

$$S''(x) = -\frac{16h(S_i - \bar{S}_i)(\bar{S}_i - S_{i-1})((\bar{S}_i - S_{i-1}) - (S_i - \bar{S}_i))}{(h(S_i - S_{i-1}) + 2((\bar{S}_i - S_{i-1}) - (S_i - \bar{S}_i))(x - \xi_i))^3}. \tag{2.4}$$

According to the representation (2.2) we have $3n$ parameters to determine for constructing the spline. We require for C^1 continuity of S on $[a, b]$ which involves $2(n-1)$ conditions, namely that S and S' must be continuous at all interior knots $x_1, \ldots, x_{n-1}$. When the continuity of S is guaranteed by the representation (2.2), the continuity of S' at interior knots leads to the equations

$$\frac{(S_i - \bar{S}_i)(S_i - S_{i-1})}{\bar{S}_i - S_{i-1}} = \frac{(\bar{S}_{i+1} - S_i)(S_{i+1} - S_i)}{S_{i+1} - \bar{S}_{i+1}}, \quad i = 1, \ldots, n-1. \tag{2.5}$$

Given the data $\bar{y}_i$, $i = 1, \dots, n$, let us require that the interpolation conditions

$$S(\xi_i) = \bar{y}_i, \quad i = 1, \dots, n, \tag{2.6}$$

are satisfied. In addition, we impose some boundary conditions, e.g.:

$$S(a) = \alpha_1, \quad S(b) = \alpha_2, \tag{2.7}$$

or

$$S'(a) = \alpha_3, \quad S'(b) = \alpha_4, \tag{2.8}$$

and a combination with one condition from (2.7) and another from (2.8) at different endpoints is also allowed. We will specify the choice of numbers α_1, α_2, α_3 and α_4 in the following section.

Replacing the values $\bar{S}_i$, $i = 1, \dots, n$, from (2.6) in the internal equations (2.5) and considering them with two boundary conditions we obtain a nonlinear system with respect to the unknowns $S_0, \dots, S_n$.

3 Expansions of the Interpolant

We derive our superconvergence rate results basing on the expansions of the interpolant which will be established in this section. First, we analyze the nonlinear system with respect to the unknowns $S_0, \dots, S_n$.

Let us write equations (2.5) with replaced values $\bar{S}_i$ from (2.6) in the form

$$\begin{aligned}\varphi_i(S_{i-1}, S_i, S_{i+1}) &:= (S_i - \bar{y}_i)(S_i - S_{i-1})(S_{i+1} - \bar{y}_{i+1}) \\ &- (\bar{y}_{i+1} - S_i)(S_{i+1} - S_i)(\bar{y}_i - S_{i-1}) = 0,\ i = 1, \dots, n-1,\end{aligned} \tag{3.1}$$

introducing at the same time functions φ_i. Then the system consisting of the boundary conditions (2.7) and the internal equations (3.1) can be written as

$$\begin{cases} h^2(y_0')^2(S_0 - \alpha_1) = 0, \\ (S_i - \bar{y}_i)(S_i - S_{i-1})(S_{i+1} - \bar{y}_{i+1}) \\ \qquad -(\bar{y}_{i+1} - S_i)(S_{i+1} - S_i)(\bar{y}_i - S_{i-1}) = 0, \quad i = 1, \dots, n-1, \\ h^2(y_n')^2(S_n - \alpha_2) = 0. \end{cases} \tag{3.2}$$

Suppose now that we have a function $y : [a, b] \to \mathbb{R}$ to interpolate and $\bar{y}_i = y(\xi_i)$, $i = 1, \dots, n$. Denote $y_i = y(x_i)$, $i = 0, \dots, n$, similar notation will be used in the case of derivatives. At (3.1) the Taylor expansion gives

$$\begin{aligned}\varphi_i(S_{i-1}, S_i, S_{i+1}) &= \varphi_i(y_{i-1}, y_i, y_{i+1}) + \frac{\partial \varphi_i}{\partial S_{i-1}}(y_{i-1}, y_i, y_{i+1})(S_{i-1} - y_{i-1}) \\ &+ \frac{\partial \varphi_i}{\partial S_i}(y_{i-1}, y_i, y_{i+1})(S_i - y_i) + \frac{\partial \varphi_i}{\partial S_{i+1}}(y_{i-1}, y_i, y_{i+1})(S_{i+1} - y_{i+1}) + \frac{\varphi_i''}{2!}(\xi_\lambda)\bar{h}^2,\end{aligned} \tag{3.3}$$

with the difference vector $\bar{h} = (S_{i-1} - y_{i-1}, S_i - y_i, S_{i+1} - y_{i+1})$, some $\lambda \in (0,1)$ and $\xi_\lambda = (y_{i-1}, y_i, y_{i+1}) + \lambda \bar{h}$. From (3.1) we find for $i = 1, \dots, n-1$

$$\begin{aligned}
\frac{\partial \varphi_i}{\partial S_{i-1}} &= -(S_i - \bar{y}_i)(S_{i+1} - \bar{y}_{i+1}) + (\bar{y}_{i+1} - S_i)(S_{i+1} - S_i), \\
\frac{\partial \varphi_i}{\partial S_i} &= (S_i - S_{i-1})(S_{i+1} - \bar{y}_{i+1}) + (S_i - \bar{y}_i)(S_{i+1} - \bar{y}_{i+1}) \\
&\quad + (S_{i+1} - S_i)(\bar{y}_i - S_{i-1}) + (\bar{y}_{i+1} - S_i)(\bar{y}_i - S_{i-1}), \\
\frac{\partial \varphi_i}{\partial S_{i+1}} &= (S_i - \bar{y}_i)(S_i - S_{i-1}) - (\bar{y}_{i+1} - S_i)(\bar{y}_i - S_{i-1}).
\end{aligned}$$

Suppose in the following that $y \in C^4[a,b]$. Let us expand y_{i-1}, $\bar{y}_i$, $\bar{y}_{i+1}$ and y_{i+1} at the point x_i by Taylor formula up to the forth derivative as

$$\begin{aligned}
y_{i-1} &= y_i - h y_i' + \frac{h^2}{2} y_i'' - \frac{h^3}{6} y_i''' + \frac{h^4}{24} y_i^{IV} + o(h^4), \\
\bar{y}_i &= y_i - \frac{h}{2} y_i' + \frac{h^2}{8} y_i'' - \frac{h^3}{48} y_i''' + \frac{h^4}{384} y_i^{IV} + o(h^4), \\
\bar{y}_{i+1} &= y_i + \frac{h}{2} y_i' + \frac{h^2}{8} y_i'' + \frac{h^3}{48} y_i''' + \frac{h^4}{384} y_i^{IV} + o(h^4), \\
y_{i+1} &= y_i + h y_i' + \frac{h^2}{2} y_i'' + \frac{h^3}{6} y_i''' + \frac{h^4}{24} y_i^{IV} + o(h^4)
\end{aligned}$$

with the error terms $O(h^{4+\alpha})$ in the case $y^{IV} \in \mathrm{Lip}\,\alpha$, $0 < \alpha \le 1$. Then direct calculations yield

$$\begin{aligned}
\frac{\partial \varphi_i}{\partial S_{i-1}}(y_{i-1}, y_i, y_{i+1}) &= \frac{h^2}{4} y_i'^2 + \frac{h^3}{4} y_i' y_i'' + O(h^4), \\
\frac{\partial \varphi_i}{\partial S_i}(y_{i-1}, y_i, y_{i+1}) &= \frac{3}{2} h^2 y_i'^2 + O(h^4), \qquad (3.4) \\
\frac{\partial \varphi_i}{\partial S_{i+1}}(y_{i-1}, y_i, y_{i+1}) &= \frac{h^2}{4} y_i'^2 - \frac{h^3}{4} y_i' y_i'' + O(h^4), \quad \text{and} \\
\varphi_i(y_{i-1}, y_i, y_{i+1}) &= \frac{h^6}{64}\Big((y_i')^2 y_i^{IV} - 4 y_i' y_i'' y_i''' + 3(y_i'')^3\Big) + o(h^6). \qquad (3.5)
\end{aligned}$$

The entries in the matrix φ_i'' consisting of the second order partial derivatives of φ_i are of order $O(h)$. This with the help of $\|\bar{h}\|_\infty = O(h^3)$ (recall that $\|S - y\|_\infty = O(h^3)$) gives $\varphi_i''(\xi_\lambda)\bar{h}^2 = O(h^7)$.

Taking now into account (3.4), (3.5) and the order of the error term in (3.3), system (3.2) reduces to

$$h^2 (y_0')^2 (S_0 - \alpha_1) = 0, \qquad (3.6)$$

$$\frac{h^6}{64}\Big((y_i')^2 y_i^{IV} - 4 y_i' y_i'' y_i''' + 3(y_i'')^3\Big) + \Big(\frac{h^2}{4}(y_i')^2 + \frac{h^3}{4} y_i' y_i'' + O(h^4)\Big)$$

$$\times (S_{i-1}-y_{i-1}) + \Big(\frac{3}{2}h^2(y_i')^2 + O(h^4)\Big)(S_i - y_i) + \Big(\frac{h^2}{4}(y_i')^2 - \frac{h^3}{4}y_i'y_i'' + O(h^4)\Big)$$
$$\times (S_{i+1}-y_{i+1}) + o(h^6) = 0, \quad i = 1, \dots, n-1,$$
$$h^2(y_n')^2(S_n - \alpha_2) = 0.$$

It is known, see, e.g., [8, 9] that a linear/linear rational spline interpolant exist only if y is strictly monotone or constant everywhere. Thus, we assume that $y'(x) > 0$ for all $x \in [a, b]$ or $y'(x) < 0$ for all $x \in [a, b]$ which means that y is strictly monotone. Consider (3.6) as a linear system with respect to the unknowns $S_i - y_i$, $i = 0, \dots, n$. Then its matrix has the diagonal dominance in rows for sufficiently small h. We look for the solution such that

$$S_i = y_i + h^4\big[\psi(y)\big]_i + \beta_i, \quad i = 0, \dots, n, \tag{3.7}$$

where the continuous function $\psi(y)$ and numbers β_i will be specified later. The continuity of $\psi(y)$ gives $[\psi(y)]_{i-1} = [\psi(y)]_i + o(1)$, $[\psi(y)]_{i+1} = [\psi(y)]_i + o(1)$. Replacing now (3.7) in the internal equations of (3.6) we get

$$\begin{aligned}
&\frac{h^6}{64}\Big((y_i')^2y_i^{IV} - 4y_i'y_i''y_i''' + 3(y_i'')^3\Big) + \Big(\frac{h^2}{4}(y_i')^2 + \frac{h^3}{4}y_i'y_i'' + O(h^4)\Big) \\
&\quad \times \Big(h^4[\psi(y)]_i + \beta_{i-1}\Big) + \Big(\frac{3}{2}h^2(y_i')^2 + O(h^4)\Big)\Big(h^4[\psi(y)]_i + \beta_i\Big) \\
&\quad + \Big(\frac{h^2}{4}(y_i')^2 - \frac{h^3}{4}y_i'y_i'' + O(h^4)\Big)\Big(h^4[\psi(y)]_i + \beta_{i+1}\Big) + o(h^6) = 0.
\end{aligned}$$

Determine the function $\psi(y)$ so that the coefficient at h^6 is equal to 0, i.e.,

$$\frac{1}{64}\Big((y_i')^2y_i^{IV} - 4y_i'y_i''y_i''' + 3(y_i'')^3\Big) + 2(y_i')^2[\psi(y)]_i = 0$$

which means that

$$[\psi(y)]_i = -\frac{1}{128}\Big(y_i^{IV} - 4\frac{y_i''y_i'''}{y_i'} + 3\frac{(y_i'')^3}{(y_i')^2}\Big), \quad i = 1, \dots, n-1. \tag{3.8}$$

Extend (3.8) for $i = 0$ and $i = n$ as well, then choose $\beta_0 = o(h^4)$ and $\beta_n = o(h^4)$ (e.g., it may be $\beta_0 = \beta_n = 0$). This determines the values of α_1 and α_2. Thus, we pose the boundary conditions (2.7) in the form

$$\begin{aligned}
S(a) &= y(a) - \frac{h^4}{128}\Big(y^{IV}(a) - 4\frac{y''(a)y'''(a)}{y'(a)} + 3\frac{(y''(a))^3}{(y'(a))^2}\Big) + o(h^4), \\
S(b) &= y(b) - \frac{h^4}{128}\Big(y^{IV}(b) - 4\frac{y''(b)y'''(b)}{y'(b)} + 3\frac{(y''(b))^3}{(y'(b))^2}\Big) + o(h^4).
\end{aligned} \tag{3.9}$$

Now we may write (3.6) as follows

$$\begin{cases}
h^2(y_0')^2\beta_0 = o(h^6), \\
\Big(\frac{h^2}{4}(y_i')^2 + \frac{h^3}{4}y_i'y_i'' + O(h^4)\Big)\beta_{i-1} + \Big(\frac{3}{2}h^2(y_i')^2 + O(h^4)\Big)\beta_i \\
\quad + \Big(\frac{h^2}{4}(y_i')^2 - \frac{h^3}{4}y_i'y_i'' + O(h^4)\Big)\beta_{i+1} + o(h^6) = 0, \quad i = 1, \dots, n-1, \\
h^2(y_n')^2\beta_n = o(h^6).
\end{cases}$$

This system has the matrix form $A\bar{\beta} = g$, where $\bar{\beta} = (\beta_0, \dots, \beta_n)$ and $\|g\|_\infty = o(h^6)$. Since $\|A^{-1}\|_\infty = O(h^{-2})$, we obtain

$$\|\bar{\beta}\|_\infty \le \|A^{-1}\|_\infty \|g\|_\infty = o(h^4),$$

where $\|A^{-1}\|$ is the matrix norm corresponding to the uniform vector norm. In total, we have

$$S_i = y_i - \frac{h^4}{128}\Big(y_i^{IV} - 4\frac{y_i'' y_i'''}{y_i'} + 3\frac{(y_i'')^3}{(y_i')^2}\Big) + o(h^4), \quad i = 0, \dots, n, \tag{3.10}$$

which could be also transformed into the form

$$S_i = y_i - \frac{h^4}{128}\Big(y_i'\Big(\frac{y'''}{y'}\Big)_i' - 3y_i''\Big(\frac{y''}{y'}\Big)_i'\Big) + o(h^4), \quad i = 0, \dots, n.$$

Our next aim is to establish the expansions of interpolant S and its first and second derivatives on the whole particular interval. First, we write the representation (2.2) with obvious notations A and B in the form

$$S(x) = \bar{S}_i + \frac{A}{B} = y(x) + \frac{A - (y(x) - \bar{S}_i)B}{B}.$$

Then using $\bar{S}_i = \bar{y}_i$ and Taylor expansions at ξ_i

$$\begin{aligned}
S_i &= y_i - \frac{h^4}{128}\psi_i + o(h^4) \\
&= \bar{y}_i + \bar{y}_i'\frac{h}{2} + \frac{\bar{y}_i''}{2}\Big(\frac{h}{2}\Big)^2 + \frac{\bar{y}_i'''}{6}\Big(\frac{h}{2}\Big)^3 + \frac{\bar{y}_i^{IV}}{24}\Big(\frac{h}{2}\Big)^4 - \frac{h^4}{128}\bar{\psi}_i + o(h^4), \\
S_{i-1} &= y_{i-1} - \frac{h^4}{128}\psi_{i-1} + o(h^4) \\
&= \bar{y}_i - \bar{y}_i'\frac{h}{2} + \frac{\bar{y}_i''}{2}\Big(\frac{h}{2}\Big)^2 - \frac{\bar{y}_i'''}{6}\Big(\frac{h}{2}\Big)^3 + \frac{\bar{y}_i^{IV}}{24}\Big(\frac{h}{2}\Big)^4 - \frac{h^4}{128}\bar{\psi}_i + o(h^4), \\
y(x) &= \bar{y}_i + \bar{y}_i' th + \frac{\bar{y}_i''}{2}(th)^2 + \frac{\bar{y}_i'''}{6}(th)^3 + \frac{\bar{y}_i^{IV}}{24}(th)^4 + o(h^4)
\end{aligned}$$

in the fractional term, we arrive at the expansion for $x \in [x_{i-1}, x_i]$

$$\begin{aligned}
S(x) = y(x) &+ \frac{t(1-4t^2)}{48}h^3\Big(2\bar{y}_i''' - 3\frac{(\bar{y}_i'')^2}{\bar{y}_i'}\Big) \\
&+ \frac{t^2}{48}h^4\Big(-(1+2t^2)\bar{y}_i^{IV} + 6\frac{\bar{y}_i''\bar{y}_i'''}{\bar{y}_i'} - 6(1-t^2)\frac{(\bar{y}_i'')^3}{(\bar{y}_i')^2}\Big) + o(h^4),
\end{aligned} \tag{3.11}$$

with $x = \xi_i + th$, $t \in [-1/2, 1/2]$. Clearly, the expansion (3.11) at $x = x_i$ or $t = 1/2$ coincides with (3.10). From (2.3), proceeding similarly, we get

$$\begin{aligned}
S'(x) = y'(x) &+ \frac{1-12t^2}{48}h^2\Big(2\bar{y}_i''' - 3\frac{(\bar{y}_i'')^2}{\bar{y}_i'}\Big) \\
&- \frac{t}{24}h^3\Big((1+4t^2)\bar{y}_i^{IV} - 6\frac{\bar{y}_i''\bar{y}_i'''}{\bar{y}_i'} + 6(1-2t^2)\frac{(\bar{y}_i'')^3}{(\bar{y}_i')^2}\Big) + o(h^3),
\end{aligned} \tag{3.12}$$

and, finally, from (2.4)

$$S''(x) = y''(x) + th\Big(-\bar{y}_i''' + \frac{3}{2}\frac{\bar{y}_i''^2}{\bar{y}_i'}\Big) + \frac{h^2}{24}\Big(-(1+12t^2)\bar{y}_i^{IV} + 6\frac{\bar{y}_i''\bar{y}_i'''}{\bar{y}_i'} - 6(1-6t^2)\frac{(\bar{y}_i'')^3}{(\bar{y}_i')^2}\Big) + o(h^2), \quad (3.13)$$

where, as before, $x = \xi_i + th$, $t \in [-1/2, 1/2]$.

Similar reasoning allows us to establish the expansions (3.10)–(3.13) in the case of boundary conditions (2.8) provided we pose them in the form

$$S'(a) = y'(a) - \frac{h^2}{12}\Big(y'''(a) - \frac{3}{2}\frac{(y''(a))^2}{y'(a)}\Big) + o(h^3), \quad (3.14)$$
$$S'(b) = y'(b) - \frac{h^2}{12}\Big(y'''(b) - \frac{3}{2}\frac{(y''(b))^2}{y'(b)}\Big) + o(h^3).$$

We have proved the following theorem.

Theorem 1. *Let y be a strictly monotone function and $y \in C^4[a,b]$. Then the linear/linear rational spline S of smoothness class C^1 satisfying interpolation conditions* (2.6) *and boundary conditions* (3.9) *or* (3.14) *expands like* (3.10)–(3.13).

Remark 1. If $y^{IV} \in \mathrm{Lip}\,\alpha$, $0 < \alpha \le 1$, then in previous formulae all the error terms written as $o(h^k)$ for some k could be replaced by $O(h^{k+\alpha})$.

Basing on expansions (3.11)–(3.13) it is now immediate consequence to obtain superconvergence assertions. From (3.11) we get $S(x_i) = y(x_i) + O(h^4)$, $i = 0, \dots, n$, (3.12) yields $S'(x) = y'(x) + O(h^3)$ in points $x = \xi_i + th$, corresponding to $t = \pm\sqrt{3}/6$, and (3.13) gives $S''(\xi_i) = y''(\xi_i) + O(h^2)$, $i = 1, \dots, n$.

Similar expansions for quadratic spline interpolants were known earlier. They are given, e.g., in [4] in a slightly different form

$$S(x) = y(x) - \frac{t(1-t)(1-2t)}{12}h^3y'''(x) - \frac{(1-2t)^2(1+4t-4t^2)}{128}h^4y^{IV}(x) + o(h^4),$$
$$S'(x) = y'(x) - \frac{1-6t-6t^2}{12}h^2y'''(x) - \frac{t(1-t)(1-2t)}{6}h^3y^{IV}(x) + o(h^3),$$
$$S''(x) = y''(x) + \frac{1-2t}{2}hy'''(x) - \frac{1-6t-6t^2}{6}h^2y^{IV}(x) + o(h^2),$$
$$x \in [x_{i-1}, x_i], \quad x = x_{i-1} + th, \quad t \in [0,1].$$

We can check directly that here the superconvergence takes place at the same points as for linear/linear rational spline interpolants.

4 Numerical Examples

We interpolated the function $y(x) = x^{-2}$ on the interval $[-2, -0.2]$ and the function $y(x) = \sin x$ on the interval $[-1.5, 1.5]$ by linear/linear rational spline

S as described in Section 2. The boundary conditions (2.7) with

$$\alpha_1 = y_0 + \frac{3}{64}h^4\frac{1}{x_0^6}, \quad \alpha_2 = y_n + \frac{3}{64}h^4\frac{1}{x_n^6}$$

for the function $y(x) = x^{-2}$ and

$$\alpha_1 = y_0 + \frac{3}{128}h^4\frac{\sin x_0}{\cos^2 x_0}, \quad \alpha_2 = y_n + \frac{3}{128}h^4\frac{\sin x_n}{\cos^2 x_n}$$

for the function $y(x) = \sin x$ were used.

Table 1. Numerical results for $y(x) = x^{-2}$, $\varepsilon_n = S(z_i) - y(z_i)$, $i = 1, 2, 3$.

	$z_1 = -1.55$		$z_2 = -1.1$		$z_3 = -0.65$	
n	ε_n	$\varepsilon_{2n}/\varepsilon_n$	ε_n	$\varepsilon_{2n}/\varepsilon_n$	ε_n	$\varepsilon_{2n}/\varepsilon_n$
16	$5.383 \cdot 10^{-7}$		$4.189 \cdot 10^{-6}$		$9.697 \cdot 10^{-5}$	
32	$3.379 \cdot 10^{-8}$	15.931	$2.641 \cdot 10^{-7}$	15.861	$6.170 \cdot 10^{-6}$	15.716
64	$2.110 \cdot 10^{-9}$	15.984	$1.654 \cdot 10^{-8}$	15.967	$3.880 \cdot 10^{-7}$	15.984
128	$1.322 \cdot 10^{-10}$	15.991	$1.035 \cdot 10^{-9}$	15.981	$2.429 \cdot 10^{-8}$	15.978
256	$8.262 \cdot 10^{-12}$	16.001	$6.467 \cdot 10^{-11}$	16.004	$1.519 \cdot 10^{-9}$	15.991

Table 2. Numerical results for $y(x) = \sin x$, $\varepsilon_n = S(z_i) - y(z_i)$, $i = 1, 2$.

	$z_1 = -0.75$		$z_2 = 0.75$	
n	ε_n	$\varepsilon_{2n}/\varepsilon_n$	ε_n	$\varepsilon_{2n}/\varepsilon_n$
16	$-5.496 \cdot 10^{-5}$		$5.496 \cdot 10^{-5}$	
32	$-2.272 \cdot 10^{-6}$	24.19	$2.272 \cdot 10^{-6}$	24.19
64	$-1.435 \cdot 10^{-7}$	15.833	$1.435 \cdot 10^{-7}$	15.833
128	$-8.996 \cdot 10^{-9}$	15.952	$8.996 \cdot 10^{-9}$	15.952
256	$-5.626 \cdot 10^{-10}$	15.990	$5.626 \cdot 10^{-10}$	15.990

Table 3. Numerical results for $y(x) = x^{-2}$, $\varepsilon''_n = S''(z_i) - y''(z_i)$, $i = 1, 2$.

	$z_1 = \frac{a+b}{2} - \frac{h}{2}$		$z_2 = \frac{a+b}{2} + \frac{h}{2}$	
n	ε''_n	$\varepsilon''_{2n}/\varepsilon''_n$	ε''_n	$\varepsilon''_{2n}/\varepsilon''_n$
16	$-2.602 \cdot 10^{-3}$		$-4.789 \cdot 10^{-3}$	
32	$-7.639 \cdot 10^{-4}$	3.406	$-4.037 \cdot 10^{-3}$	4.618
64	$-1.066 \cdot 10^{-4}$	3.697	$-2.408 \cdot 10^{-4}$	4.306
128	$-5.370 \cdot 10^{-5}$	3.847	$-5.798 \cdot 10^{-5}$	4.153
256	$-1.369 \cdot 10^{-5}$	3.923	$-1.422 \cdot 10^{-5}$	4.077

The "tridiagonal" nonlinear system to determine the values of S_i was solved by Newton's method and the iterations were stopped at $\|S^k - S^{k-1}\|_\infty \le 10^{-10}$, S^k being the sequence of approximations to the vector $S = (S_0, \ldots, S_n)$. The errors $\varepsilon_n = S(z_i) - y(z_i)$ and $\varepsilon''_n = S''(z_i) - y''(z_i)$ were calculated in certain superconvergence points z_i. Results of numerical tests are presented in Tables 1–3.

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