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# Fractional Integro-Differential Equations with Nonlocal Conditions and $\psi$–Hilfer Fractional Derivative

**Mohammed S. Abdo[a,b], Satish K. Panchal[a] and Hussien Shafei Hussien[c]**

[a] *Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University*
Aurangabad, 431004 (M.S.), India
[b] *Department of Mathematics, Hodeidah University, Yemen*
Drehemi road, P.O.Box 3114, 250416 Al-Hodeidah, Yemen
[c] *Department of Mathematics, Faculty of science, South Valley University*
83523 Qena, Egypt

E-mail(*corresp.*): msabdo1977@gmail.com
E-mail: drpanchalsk@gmail.com
E-mail: hshafei@sci.svu.edu.eg

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**Abstract.** In this paper, we consider a fractional integro-differential equation with nonlocal condition involving a general form of Hilfer fractional derivative. We show that Cauchy-type problem is equivalent to a Volterra fractional integral equation. We also employ the Banach fixed point theorem and Krasnoselskii's fixed point theorem to obtain existence and uniqueness of solutions. Ulam-Hyers-Rassias stability results are established. Further, Mittag-Leffler least squares method is used to approximate the resulting nonlinear implicit analytic solution of the problem. An example is provided to illustrate our main results.

**Keywords:** fractional integro-differential equations, $\psi$ -Hilfer fractional derivative, $\psi$-fractional integral, existence and and Ulam-Hyers stability, fixed point theorem, Mittag-Leffler function, least squares method.

**AMS Subject Classification:** 34K37; 26A33; 34A12; 47H10; 65N35; 11T23.

# 1 Introduction

Fractional calculus has grasped the attention of many researchers through recent decades as it is a solid and growing work each in the theoretical and applied concept. Several researchers tried to suggest various kinds of fractional operators that deal with derivatives and integrals of non-integer orders and their applications, (for more details see [1,17,18,22,24]). There are some articles that presented studies about theory and analysis of $\psi$- fractional differential equations, we mention here some works on fractional differential equations including $\psi-$fractional derivative with respect to another function (see [2,3,16,19,20,25] and references therein). Recently, Kilbas et al. in [18] introduced the properties of fractional integrals and fractional derivatives of a function with respect to another function. Some of generalized fractional integral and differential operators and their properties were introduced by Agrawal in [4]. On the other hand, Furati and Kassim, [15] studied the existence, uniqueness and stability of global solutions for a Cauchy-type problem involving Hilfer fractional derivative

$$D_{a^+}^{\alpha,\beta}u(t) = f(t,u(t)),\ 0<\alpha<1, 0\leq\beta\leq 1,\ t>a,$$
$$I_{a^+}^{1-\gamma}u(a^+) = u_a,\ \ \gamma=\alpha+\beta-\alpha\beta.$$

Very recently, Sousa and Oliveira [9] proposed a $\psi-$ Hilfer fractional operator and extended few previous works dealing with the Hilfer [15, 17]. Moreover, they discussed some important qualitative properties of solutions such as existence, uniqueness, dependence continuous, and stability results, see [8, 10, 11, 12, 13, 14].

In the same context, Harikrishnan et al. [16], studied the existence and uniqueness results of fractional pantograph differential equations with $\psi$–Hilfer fractional derivative and nonlocal conditions

$$D_{a^+}^{\alpha,\beta;\psi}u(t) = f(t,u(t),\lambda u(t)),\ \ 0<\lambda<1,\ t\in(a,b],$$
$$I_{a^+}^{1-\gamma;\psi}u(t)\mid_{t=a} = \sum_{i=1}^{m} c_i u(\tau_i),\ \tau_i\in(a,b],$$

where $0<\alpha<1$, $0\leq\beta\leq 1$ and $\gamma=\alpha+\beta-\alpha\beta$.

Motivated by above work, in this paper, we prove the existence, uniqueness, and Ulam–Hyers–Rassias stability of solutions of a nonlinear fractional integro-differential equation with nonlocal condition and $\psi-$Hilfer fractional derivatives of the form:

$$D_{a^+}^{\alpha,\beta;\psi}u(t) = f\left(t,u(t),\int_0^t \mathcal{K}(t,s,u(s))ds\right),\ \ t\in(a,b], \tag{1.1}$$
$$I_{a^+}^{1-\gamma;\psi}u(t)\mid_{t=a} = u_a+g(u),\ \ \alpha\leq\gamma=\alpha+\beta-\alpha\beta, \tag{1.2}$$

where $0<\alpha<1$, $0\leq\beta\leq 1$, $u_a$ is a constant, $D_{a^+}^{\alpha,\beta;\psi}(\cdot)$ is the generalized Hilfer fractional derivative introduced by Sousa and de Oliveira in [9], $I_{a^+}^{1-\gamma;\psi}(\cdot)$ is the $\psi-$fractional integral in the sense of Riemann-Liouville, $f:(a,b]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ and $g: C([a,b],\mathbb{R})\to\mathbb{R}$ are appropriate functions with $g(u)=\sum_{k=1}^{m}c_k u(\tau_k)$,

$\tau_k \in (a,b)$, $u_a \in \mathbb{R}$, $\tau_k$ $(k=0,1,..,m)$ are prefixed points satisfying $a<\tau_1<\tau_2<...<\tau_m<b$, $c_k$ is real numbers, and $\mathcal{K}:\mathbb{D}\times\mathbb{R}\longrightarrow\mathbb{R}$ is continuous on $\mathbb{D}=\{(t,s): a\le s\le t\le b\}$. For brevity let us take $Hu(t)=\int_0^t \mathcal{K}(t,s,u(s))ds$.

The plan of this paper is as follows. In Section 2, we present some basic definitions, preliminary facts that will be useful throughout the paper. In Section 3, we list the hypotheses and obtain an equivalent integral equation of the Cauchy type problem (1.1)–(1.2) in weighted space. Further, we prove the existence of solution of the problem (1.1)–(1.2). The uniqueness result, Ulam-Hyers and Ulam-Hyers-Rassias stability to such equations in the weighted space $C_{1-\gamma;\psi}[a,b]$ are discussed in Section 4. In Section 5, we investigate Mittag operational matrix to approximate any finite integration. An example with numerical results is provided in Section 6. Finally, the conclusions are given.

## 2 Mathematical preliminaries

In this section, we introduce some notations, definitions, and preliminary facts related by fractional calculus. The following observations are taken from [9,18]. The two-parameters Mittag-Leffler function is given by:

$$E_{\mu,\nu}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu k+\nu)}, \mu,\nu>0, z\in\mathbb{R}. \tag{2.1}$$

where $\Gamma(\cdot)$ is the Euler Gamma function. We consider the weighted spaces $C_{\gamma;\psi}[a,b]$ and $C^n_{\gamma;\psi}[a,b]$ as follows

$$C_{\gamma;\psi}[a,b]=\{h:(a,b]\to\mathbb{R}:(\psi(t)-\psi(a))^{\gamma}h(t)\in C[a,b]\},$$
$$C^n_{\gamma;\psi}[a,b]=\{h:(a,b]\to\mathbb{R}:h(t)\in C^{n-1}[a,b]; h^{(n)}(t)\in C_{\gamma;\psi}[a,b]\},$$

where $0\le\gamma<1$, $n\in\mathbb{N}$, with the norms $\|h\|_{C_{\gamma;\psi}[a,b]}=\max\{|(\psi(t)-\psi(a))^{\gamma}h(t)|, t\in[a,b]\}$, and $\|h\|_{C^n_{\gamma;\psi}[a,b]}=\sum_{k=0}^{n-1}\left\|h^{(k)}\right\|_{C[a,b]}+\left\|h^{(n)}\right\|_{C_{\gamma;\psi}[a,b]}$, respectively. In particular, if $n=0$, we have $C^0_{\gamma;\psi}[a,b]=C_{\gamma;\psi}[a,b]$.

Definition 1. ( [18]) Let $\alpha>0$ and $\psi$ be an increasing function, having a continuous derivative $\psi'$ on $(a,b)$. The left-sided fractional integral of a function $h$ with respect to $\psi$ on $[a,b]$ is defined by

$$I_{a^+}^{\alpha,\psi}h(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\psi'(s)\left[\psi(t)-\psi(s)\right]^{\alpha-1}h(s)ds,\ t>a$$

provided that $I_{a^+}^{\alpha,\psi}$ is exists. Note that when $\psi(t)=t$, we obtain the known classical Riemann-Liouville fractional integral.

Definition 2. ( [6]) Let $\alpha>0$, $n$ be the smallest integer greater than or equal to $\alpha$ and $h\in L^p[a,b]$, $p\ge1$, let $\psi\in C^n[a,b]$ an increasing function such that $\psi'(t)\neq0$, for all $t\in[a,b]$. The left-sided $\psi-$Riemann–Liouville fractional derivative of $h$ of order $\alpha$ is given by

$$D_{a^+}^{\alpha;\psi}h(t)=\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a^+}^{n-\alpha,\psi}h(t).$$

Definition 3. ( [5]) The left-sided $\psi$–Caputo fractional derivative of function $h \in C^n[a,b]$ $(n-1<\alpha<n=[\alpha]+1)$ with respect to another function $\psi$ is defined by

$$^{c}D_{a^+}^{\alpha;\psi}h(t) = I_{a^+}^{n-\alpha;\psi}h_{\psi}^{[n]}(t),$$

where $[\alpha]$ denotes the integer part of the real number $\alpha$, $h_{\psi}^{[n]}(t)=\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n h(t)$ and $\psi$ as in Definition 2.

The fractional derivative that we will deal in our work is a Hilfer type operator and it is defined by the following definition.

Definition 4. ( [9,20]) Let $0<\alpha<1$, $0\le\beta\le1$ and $h,\psi\in C^1[a,b]$ be two functions such that $\psi'(t)\neq0$, for all $t\in[a,b]$. Then the left-sided $\psi$–Hilfer fractional derivative $D_{a^+}^{\alpha,\beta;\psi}(\cdot)$ of a function $h$ of order $\alpha$ and type $\beta$ is defined by

$$D_{a^+}^{\alpha,\beta;\psi}h(t) = I_{a^+}^{\beta(1-\alpha);\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)I_{a^+}^{(1-\beta)(1-\alpha);\psi}h(t).$$

On the other hand, we have

$$D_{a^+}^{\alpha,\beta;\psi}h(t) = I_{a^+}^{\beta(1-\alpha);\psi}D_{a^+}^{\gamma;\psi}h(t),\ t>a, \tag{2.2}$$

where $D_{a^+}^{\gamma;\psi}h(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)I_{a^+}^{(1-\beta)(1-\alpha);\psi}h(t)$ and $\gamma=\alpha+\beta-\alpha\beta$.

**Lemma 1.** *( [18]) Let $\alpha,\beta>0$. Then we have the following semigroup property*

$$I_{a^+}^{\alpha;\psi}I_{a^+}^{\beta;\psi}h(t) = I_{a^+}^{\alpha+\beta;\psi}h(t),\ t>a.$$

**Lemma 2.** *( [9]) Let $\alpha>0$, $0\le\beta\le1$ and $0\le\gamma=\alpha+\beta-\alpha\beta<1$. If $h\in L^1(a,b)$ and $D_{a^+}^{\beta(1-\alpha);\psi}h(\cdot)$ exists on $L^1(a,b)$, then*

$$D_{a^+}^{\alpha,\beta;\psi}I_{a^+}^{\alpha;\psi}h(t) = I_{a^+}^{\beta(1-\alpha);\psi}D_{a^+}^{\beta(1-\alpha);\psi}h(t),\ t\in(a,b].$$

*Moreovere, if $h\in C_{1-\gamma;\psi}[a,b]$, $I_{a^+}^{\beta(1-\alpha);\psi}h\in C^1_{1-\gamma;\psi}[a,b]$, then $D_{a^+}^{\alpha,\beta;\psi}I_{a^+}^{\alpha;\psi}h(\cdot)$ exists on $(a,b]$ and*

$$D_{a^+}^{\alpha,\beta;\psi}I_{a^+}^{\alpha;\psi}h(t) = h(t),\ t\in(a,b].$$

**Lemma 3.** *( [3]) Let $0<\alpha<1$, $0\le\beta\le1$ and $\gamma=\alpha+\beta-\alpha\beta$. If $h\in C^{\gamma}_{1-\gamma;\psi}[a,b]$ then*

$$I_{a^+}^{\gamma;\psi}D_{a^+}^{\gamma;\psi}h(t) = I_{a^+}^{\alpha;\psi}\ D_{a^+}^{\alpha,\beta;\psi}h(t),\quad D_{a^+}^{\gamma;\psi}I_{a^+}^{\alpha;\psi}h(t) = D_{a^+}^{\beta(1-\alpha);\psi}h(t).$$

**Proposition 1.** *( [18]) Let $\alpha,\delta>0$ and $t>a$. Then $\psi-$fractional integral and derivative of a power function are given by*

$$I_{a^+}^{\alpha;\psi}\left[\psi(t)-\psi(a)\right]^{\delta-1} = \frac{\Gamma(\delta)}{\Gamma(\delta+\alpha)}\left[\psi(t)-\psi(a)\right]^{\alpha+\delta-1}, \tag{2.3}$$

*and* $D_{a^+}^{\alpha;\psi}\left[\psi(t)-\psi(a)\right]^{\alpha-1}=0,\ 0<\alpha<1.$

**Lemma 4.** *( [2, 8]) Let $0<\alpha<1$, $0\leq\beta\leq 1$, and $\gamma=\alpha+\beta-\alpha\beta$ and let $\psi\in C^{1}[a,b]$ an increasing function such that $\psi'(t)\neq 0$, for all $t\in[a,b]$. Then $I_{a^+}^{\alpha;\psi}(\cdot)$ is bounded from $C_{1-\gamma;\psi}[a,b]$ into $C_{1-\gamma;\psi}[a,b]$.*

**Lemma 5.** *( [2]) Let $\alpha>0$, $0\leq\gamma<1,\psi$ as in Definition 4, and $h\in C_{1-\gamma;\psi}[a,b]$. If $\alpha>\gamma$, then $I_{a^+}^{\alpha;\psi}h\in C[a,b]$ and $I_{a^+}^{\alpha;\psi}h(a)=\lim_{t\to a^+}I_{a^+}^{\alpha;\psi}h(t)=0$.*

***Theorem 1.*** *( [9, 21]) Let $h\in C_{1-\gamma;\psi}[a,b]$, and $I_{a^+}^{1-\alpha;\psi}h\in C^{1}_{1-\gamma;\psi}[a,b]$, $0<\alpha<1$, $0\leq\gamma<1$. Then*

$$I_{a^+}^{\alpha;\psi}D_{a^+}^{\alpha;\psi}h(t)=h(t)-\frac{I_{a^+}^{1-\alpha;\psi}h(a)}{\Gamma(\alpha)}\left[\psi(t)-\psi(a)\right]^{\alpha-1},\ \text{for all } t\in(a,b].$$

**Theorem 2.** *( [18]) (Banach fixed point theorem) Let $(X,d)$ be a nonempty complete metric space with $T:X\to X$ is a contraction mapping. Then map $T$ has a fixed point.*

**Theorem 3.** *( [7]) (Krasnoselskii's fixed point theorem) Let $X$ be a Banach space, let $\Omega$ be a bounded closed convex subset of $X$ and let $T_1$, $T_2$ be mapping from $\Omega$ into $X$ such that $T_1x+T_2y\in\Omega$ for every pair $x,y\in\Omega$. If $T_1$ is contraction and $T_2$ is completely continuous, then the equation $T_1x+T_2x=x$ has a solution on $\Omega$.*

**Lemma 6.** *( [8]) (Gronwall's lemma) Let $x,y$, be two integrable functions and $h$ continuous, with domain $[a,b]$. Let $\psi\in C^{1}[a,b]$ an increasing function such that $\psi'(t)\neq 0,\forall t\in[a,b]$. Assume that $x$ and $y$ are nonnegative and $h$ is nonnegative and nondecreasing. If*

$$x(t)\leq y(t)+h(t)\int_a^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}x(s)ds,$$

*then, for all $t\in[a,b]$,we have*

$$x(t)\leq y(t)+\int_a^t\sum_{k=1}^{\infty}\frac{\left[h(t)\Gamma(\alpha)\right]^{k}}{\Gamma(\alpha k)}\psi'(s)(\psi(t)-\psi(s))^{\alpha k-1}y(s)ds. \tag{2.4}$$

## 3 Existence result

In this section, we obtain an equivalent integral and existence result of the problem (1.1)–(1.2) in weighted space of continuous functions by means of fixed point theorem of Krasnoselskii. Firstly, we introduce the weighted spaces

$$C^{\alpha,\beta}_{1-\gamma;\psi}[a,b]=\{h\in C_{1-\gamma;\psi}[a,b],D_{a^+}^{\alpha,\beta;\psi}h\in C_{1-\gamma;\psi}[a,b]\},$$
$$C^{\gamma}_{1-\gamma;\psi}[a,b]=\{h\in C_{1-\gamma;\psi}[a,b],D_{a^+}^{\gamma;\psi}h\in C_{1-\gamma;\psi}[a,b]\},$$

where $\gamma=\alpha+\beta-\alpha\beta$. Since $D_{a^+}^{\alpha,\beta;\psi}h=I_{a^+}^{\beta(1-\alpha);\psi}D_{a^+}^{\gamma;\psi}h$ it is obvious that, $C^{\gamma}_{1-\gamma;\psi}[a,b]\subset C^{\alpha,\beta}_{1-\gamma;\psi}[a,b]$.

**Theorem 4.** *Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$ and $\gamma = \alpha + \beta - \alpha\beta$. Assume that $f(\cdot, u(\cdot), Hu(\cdot)) \in C_{1-\gamma;\psi}[a,b]$ for any $u \in C_{1-\gamma;\psi}[a,b]$. If $u \in C^{\gamma}_{1-\gamma;\psi}[a,b]$ then $u$ satisfies the problem (1.1)–(1.2) if and only if $u$ satisfies the integral equation*

$$\begin{aligned} u(t) &= \frac{[\psi(t)-\psi(a)]^{\gamma-1}}{B}\left[\sum_{k=1}^{m}\frac{c_k}{\Gamma(\alpha)}\int_a^{\tau_k}\ominus_{\psi}^{\alpha}(\tau_k,s)h(s)ds + u_a\right] \\ &\quad +\frac{1}{\Gamma(\alpha)}\int_a^t \ominus_{\psi}^{\alpha}(t,s)h(s)ds, \end{aligned} \tag{3.1}$$

*where* $\ominus_{\psi}^{\alpha}(\tau_k,s) := \psi'(s)\left[\psi(\tau_k)-\psi(s)\right]^{\alpha-1}$, $h(s) := f(s,u(s),Hu(s))$ *and* $0 \neq B := \Gamma(\gamma) - \sum_{k=1}^{m} c_k\left[\psi(\tau_k)-\psi(a)\right]^{\gamma-1}$.

*Proof.* ($\Longrightarrow$) Let $u \in C^{\gamma}_{1-\gamma;\psi}[a,b]$ be a solution of the problem (1.1)–(1.2). We prove that $u$ is also a solution of Equation (3.1). From the definition of $C^{\gamma}_{1-\gamma;\psi}[a,b]$, Lemma 4, and using Definition 4, we have

$$I_{a^+}^{1-\gamma;\psi}u \in C_{1-\gamma;\psi}[a,b] \text{ and } D_{a^+}^{\gamma;\psi}u = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)I_{a^+}^{1-\gamma;\psi}u \in C_{1-\gamma;\psi}[a,b].$$

Since $\psi \in C^1[a,b]$ and using definition of $C^n_{\gamma;\psi}[a,b]$, it follows that $I_{a^+}^{1-\gamma;\psi}u \in C^1_{1-\gamma;\psi}[a,b]$. Hence by utilize Theorem 1, then for each $t \in (a,b]$, we can write

$$I_{a^+}^{\gamma;\psi}D_{a^+}^{\gamma;\psi}u(t) = u(t) - \frac{I_{a^+}^{1-\gamma;\psi}u(a)}{\Gamma(\gamma)}\left[\psi(t)-\psi(a)\right]^{\gamma-1}. \tag{3.2}$$

By hypothesis, $D_{a^+}^{\gamma;\psi}u \in C_{1-\gamma;\psi}[a,b]$, using Lemma 3 and Equation (1.1), we have

$$I_{a^+}^{\gamma;\psi}D_{a^+}^{\gamma;\psi}u(t) = I_{a^+}^{\alpha;\psi}D_{a^+}^{\alpha,\beta;\psi}u(t) = I_{a^+}^{\alpha;\psi}h(t). \tag{3.3}$$

Comparing Equations (3.2) and (3.3), we see that

$$u(t) = \frac{I_{a^+}^{1-\gamma;\psi}u(a)}{\Gamma(\gamma)}\left[\psi(t)-\psi(a)\right]^{\gamma-1} + I_{a^+}^{\alpha;\psi}h(t). \tag{3.4}$$

Now, we substitute $t=\tau_k$ in (3.4) and multiply by $c_k$ we can write

$$c_k u(\tau_k) = \frac{c_k I_{a^+}^{1-\gamma;\psi}u(a)}{\Gamma(\gamma)}\left[\psi(\tau_k)-\psi(a)\right]^{\gamma-1} + c_k I_{a^+}^{\alpha;\psi}h(\tau_k).$$

The last equality with the nonlocal condition (1.2), gives us

$$I_{a^+}^{1-\gamma;\psi}u(a) = \sum_{k=1}^{m}c_k u(\tau_k) + u_a = \frac{\Gamma(\gamma)}{B}\left[\sum_{k=1}^{m}c_k I_{a^+}^{\alpha;\psi}h(\tau_k) + u_a\right]. \tag{3.5}$$

Substituting (3.5) into (3.4), we conclude that $u(t)$ satisfies (3.1).

($\Longleftarrow$) Assume that $u \in C^{\gamma}_{1-\gamma;\psi}[a,b]$ satisfying the integral equation (3.1). First, we prove that $u$ also satisfies the problem (1.1)–(1.2). To this end, apply the fractional derivative operator $D_{a^+}^{\gamma;\psi}$ on both sides of (3.1). Then from Lemma 3, Proposition 1 and Definition 4, we get

$$D_{a^+}^{\gamma;\psi}u(t) = D_{a^+}^{\beta(1-\alpha);\psi}h(t). \tag{3.6}$$

Since $u \in C_{1-\gamma;\psi}^{\gamma}[a,b]$, and by definition of $C_{1-\gamma;\psi}^{\gamma}[a,b]$, we have $D_{a^+}^{\gamma;\psi}u \in C_{1-\gamma;\psi}[a,b]$. The last inclusion with (3.6) gives

$$D_{a^+}^{1;\psi}I_{a^+}^{1-\beta(1-\alpha);\psi}h = D_{a^+}^{1;\psi}I_{a^+}^{1;\psi}I_{a^+}^{-\beta(1-\alpha);\psi}h = D_{a^+}^{\beta(1-\alpha);\psi}h \in C_{1-\gamma;\psi}[a,b], \tag{3.7}$$

where $D_{a^+}^{1;\psi} = \frac{1}{\psi'(t)}\frac{d}{dt}$. Also, since $h \in C_{1-\gamma;\psi}[a,b]$, by Lemma 4,

$$I_{a^+}^{1-\beta(1-\alpha);\psi}h \in C_{1-\gamma;\psi}[a,b]. \tag{3.8}$$

It follows from (3.7) and (3.8) and the definition of $C_{1-\gamma;\psi}^{\gamma}[a,b]$, that

$$I_{a^+}^{1-\beta(1-\alpha);\psi}h \in C_{1-\gamma;\psi}^{1}[a,b].$$

Thus, $h$ and $I_{a^+}^{1-\beta(1-\alpha);\psi}h$ satisfy the conditions of Theorem 1. Hence, by applying the operator $I_{a^+}^{\beta(1-\alpha);\psi}$ on both sides of (3.6) with using Theorem 1 and Lemma 2, we obtain

$$\begin{aligned} I_{a^+}^{\beta(1-\alpha);\psi}D_{a^+}^{\gamma;\psi}u(t) &= I_{a^+}^{\beta(1-\alpha);\psi}D_{a^+}^{\beta(1-\alpha);\psi}h(t) \\ &= h(t) - \frac{I_{a^+}^{1-\beta(1-\alpha);\psi}h(a)}{\Gamma(\beta(1-\alpha))}\left[\psi(t)-\psi(a)\right]^{\beta(1-\alpha)-1} = h(t), \end{aligned}$$

where $I_{a^+}^{1-\beta(1-\alpha);\psi}h(a) = 0$ is implied by Lemma 5. Comparing the last equality with (2.2), we get $D_{a^+}^{\alpha,\beta;\psi}u(t) = h(t)$, which means that (1.1) holds. Next, we show that if $u \in C_{1-\gamma;\psi}^{\gamma}[a,b]$ satisfies (3.1), it also satisfies the condition (1.2). To this end, we multiply both sides of (3.1) by $I_{a^+}^{1-\gamma;\psi}$ and use Proposition 1, Definition 1 and Lemma 1, we have

$$\begin{aligned} I_{a^+}^{1-\gamma;\psi}u(t) &= \frac{\Gamma(\gamma)}{B}\left[\sum_{k=1}^{m}\frac{c_k}{\Gamma(\alpha)}\int_a^{\tau_k}\ominus_{\psi}^{\alpha}(\tau_k,s)h(s)ds + u_a\right] \\ &\quad + \frac{1}{\Gamma(\alpha+1-\gamma)}\int_a^t \ominus_{\psi}^{\alpha-\gamma+1}(t,s)h(s)ds. \end{aligned}$$

Since $1-\gamma < \alpha+1-\gamma$, Lemma 5 can be used when taking the limit as $t \to a$,

$$I_{a^+}^{1-\gamma;\psi}u(a) = \frac{\Gamma(\gamma)}{B}\left[\sum_{k=1}^{m}\frac{c_k}{\Gamma(\alpha)}\int_a^{\tau_k}\ominus_{\psi}^{\alpha}(\tau_k,s)h(s)ds + u_a\right]. \tag{3.9}$$

Substituting $t = \tau_k$ into (3.1), we have

$$\begin{aligned} u(\tau_k) &= \frac{\left[\psi(\tau_k)-\psi(a)\right]^{\gamma-1}}{B}\left[\sum_{k=1}^{m}\frac{c_k}{\Gamma(\alpha)}\int_a^{\tau_k}\ominus_{\psi}^{\alpha}(\tau_k,s)h(s)ds + u_a\right] \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_a^{\tau_k}\ominus_{\psi}^{\alpha}(\tau_k,s)h(s)ds. \end{aligned}$$

Then, we derive

$$\sum_{k=1}^{m} c_k u(\tau_k) = \sum_{k=1}^{m} \frac{c_k}{\Gamma(\alpha)} \int_a^{\tau_k} \ominus_\psi^\alpha(\tau_k, s)h(s)ds \times \left[1 + \sum_{k=1}^{m} \frac{c_k}{B} \left[\psi(\tau_k) - \psi(a)\right]^{\gamma-1}\right] + \sum_{k=1}^{m} \frac{c_k}{B} \left[\psi(\tau_k) - \psi(a)\right]^{\gamma-1} u_a$$
$$= \frac{\Gamma(\gamma)}{B} \sum_{k=1}^{m} \frac{c_k}{\Gamma(\alpha)} \int_a^{\tau_k} \ominus_\psi^\alpha(\tau_k, s)h(s)ds + \left[\frac{\Gamma(\gamma)}{B} - 1\right] u_a.$$

Which gives

$$\sum_{k=1}^{m} c_k u(\tau_k) + u_a = \frac{\Gamma(\gamma)}{B} \Big[\sum_{k=1}^{m} \frac{c_k}{\Gamma(\alpha)} \int_a^{\tau_k} \ominus_\psi^\alpha(\tau_k, s)h(s)ds + u_a\Big]. \tag{3.10}$$

It follows (3.9) and (3.10) that $I_{a^+}^{1-\gamma;\psi} u(a) = \sum_{k=1}^{m} c_k u(\tau_k) + u_a$. The Theorem is proved completely. □

Now, we need to the following hypotheses:

**(A1)** Let $f : (a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function such that $f(\cdot, u(\cdot), Hu(\cdot)) \in C_{1-\gamma;\psi}^{\beta(1-\alpha)}[a, b]$ for any $u \in C_{1-\gamma;\psi}[a, b]$, and there exists $M > 0$ such that

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \le M \left[|u_1 - u_2| + |v_1 - v_2|\right],$$

for all $t \in (a, b]$ and $u_i, v_i \in \mathbb{R}$ $(i = 1, 2)$.

**(A2)** Let $\mathcal{K} : \mathbb{D} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous on $\mathbb{D}$ and there exists $L^* > 0$ such that

$$|Hu_1 - Hu_2| \le L^* |u_1 - u_2|, \text{ where } Hu(t) = \int_0^t \mathcal{K}(t, s, u(s))ds,$$

for all $t \in (a, b]$ and $u_1, u_2 \in \mathbb{R}$.

**Theorem 5.** *Assume that the hypotheses (A1) and (A2) are fulfilled. Then there exists at least one solution for the $\psi-$Hilfer problem (1.1)–(1.2) in the space $C_{1-\gamma;\psi}^{\gamma}[a, b] \subset C_{1-\gamma;\psi}^{\alpha,\beta}[a, b]$, provided that*

$$\sigma := \left[\sum_{k=1}^{m} \frac{c_k}{B} \left[\psi(\tau_k) - \psi(a)\right]^\alpha + \left[\psi(b) - \psi(a)\right]^\alpha\right] \frac{\mathcal{B}(\gamma, \alpha)}{\Gamma(\alpha)} (M + ML^*) < 1, \tag{3.11}$$

*where $0 \neq B$ is defined as in Theorem 4, and $\mathcal{B}(\cdot, \cdot)$ is a Beta function.*

*Proof.* We use the Krasnoselskii's fixed point theorem to prove the existence of solution $u$ in the weighted space $C_{1-\gamma;\psi}^{\gamma}[a, b]$. Define the operator $T : C_{1-\gamma;\psi}[a, b] \to C_{1-\gamma;\psi}[a, b]$ by

$$(Tu)(t) = \frac{\left[\psi(t) - \psi(a)\right]^{\gamma-1}}{B} \Big[\sum_{k=1}^{m} \frac{c_k}{\Gamma(\alpha)} \int_a^{\tau_k} \ominus_\psi^\alpha(\tau_k, s)\mathcal{F}_u(s)ds + u_a\Big] + \frac{1}{\Gamma(\alpha)} \int_a^t \ominus_\psi^\alpha(t, s)\mathcal{F}_u(s)ds, \tag{3.12}$$

where $\mathcal{F}_u(s) := f(s,u(s),Hu(s))$. Consider the ball $\mathbb{B}_r = \{u \in C_{1-\gamma;\psi}([a,b] : \|u\|_{C_{1-\gamma;\psi}} \leq r\}$, $\widetilde{f}(s) = \mathcal{F}_0(s) := f(s,0,0)$, $M^* = |H(0)|$ and $r \geq \frac{\rho}{1-\sigma}$, where $\sigma < 1$ and

$$\rho \ : \ = \Big[ \sum_{k=1}^{m} \frac{c_k}{B} \left[\psi(\tau_k) - \psi(a)\right]^{\alpha} + \left[\psi(b) - \psi(a)\right]^{\alpha} \Big] \times \Big[ \|\widetilde{f}\|_{C_{1-\gamma;\psi}} + MM^*/\Gamma(\alpha+1) \Big] + \frac{1}{B} |u_a| .$$

Now, we need to analyze the operator $T$ into sum two operators $T_1 + T_2$ as follows

$$T_1 u(t) = \frac{\left[\psi(t) - \psi(a)\right]^{\gamma-1}}{B} \Big[ \sum_{k=1}^{m} \frac{c_k}{\Gamma(\alpha)} \int_a^{\tau_k} \ominus_{\psi}^{\alpha}(\tau_k, s) \mathcal{F}_u(s) ds + u_a \Big],$$

$$T_2 u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \ominus_{\psi}^{\alpha}(t,s) \mathcal{F}_u(s) ds.$$

The proof will be given in several steps.

**Step 1:** We prove that $T_1 u + T_2 v \in \mathbb{B}_r$ for every $u, v \in \mathbb{B}_r$.

For operator $T_1$, by our hypotheses, we have

$$\begin{aligned} \Big| \Big[ \psi(t) - \psi(a) \Big]^{1-\gamma} T_1 u(t) \Big| &\leq \frac{1}{B} \sum_{k=1}^{m} \frac{c_k}{\Gamma(\alpha)} \int_a^{\tau_k} \ominus_{\psi}^{\alpha}(\tau_k, s) \Big[ |\mathcal{F}_u(s) - \mathcal{F}_0(s)| \\ &+ |\mathcal{F}_0(s)| \Big] ds + \frac{1}{B} |u_a| \leq \frac{1}{B} \sum_{k=1}^{m} \frac{c_k}{\Gamma(\alpha)} \int_a^{\tau_k} \ominus_{\psi}^{\alpha}(\tau_k, s) \left[\psi(s) - \psi(a)\right]^{\gamma-1} \\ &\times \Big[ (M + ML^*) \|u\|_{C_{1-\gamma;\psi}} + \|\widetilde{f}\|_{C_{1-\gamma;\psi}} \Big] ds + \frac{1}{B} |u_a| \\ &+ \frac{1}{B} \sum_{k=1}^{m} \frac{c_k}{\Gamma(\alpha)} \int_a^{\tau_k} \ominus_{\psi}^{\alpha}(\tau_k, s) MM^* ds, \end{aligned}$$

where we used the formula

$$|\mathcal{F}_0(s)| = |\widetilde{f}(s)| \leq \left[\psi(s) - \psi(a)\right]^{\gamma-1} \|\widetilde{f}\|_{C_{1-\gamma;\psi}}.$$

From Definition 1 and Proposition 1, we get

$$\begin{aligned} \Big| \left[\psi(t) - \psi(a)\right]^{1-\gamma} T_1 u(t) \Big| \ &\leq \ \sum_{k=1}^{m} \frac{c_k}{B} \frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \left[\psi(\tau_k) - \psi(a)\right]^{\alpha+\gamma-1} \\ &\times \Big[ (M + ML^*) \|u\|_{C_{1-\gamma;\psi}} + \Big\| \widetilde{f} \Big\|_{C_{1-\gamma;\psi}} \Big] \\ &+ \frac{1}{B} |u_a| + \sum_{k=1}^{m} \frac{c_k}{B} \frac{\left[\psi(\tau_k) - \psi(a)\right]^{\alpha}}{\Gamma(\alpha+1)} MM^* ds. \end{aligned}$$

As $0 < \gamma < 1$, then

$$\frac{\left[\psi(\tau_k) - \psi(a)\right]^{\gamma}}{\left[\psi(\tau_k) - \psi(a)\right]} < 1, \tag{3.13}$$

hence, for every $u \in \mathbb{B}_r$, we find that

$$\begin{aligned}\|T_1 u\|_{C_{1-\gamma;\psi}} \quad \leq \quad & \sum_{k=1}^{m} \frac{c_k}{B}\left[\frac{\mathcal{B}(\gamma,\alpha)}{\Gamma(\alpha)}(M+ML^*)r + \left\|\widetilde{f}\right\|_{C_{1-\gamma;\psi}} \right.\\ & \left. +\frac{MM^*}{\Gamma(\alpha+1)}\right]\left[\psi(\tau_k)-\psi(a)\right]^{\alpha} + \frac{1}{B}\left|u_a\right|. \qquad (3.14)\end{aligned}$$

As for operator $T_2$, by using the previous hypotheses, we have

$$\begin{aligned}&\left|\left[\psi(t)-\psi(a)\right]^{1-\gamma} T_2 v(t)\right| \leq \frac{\left[\psi(t)-\psi(a)\right]^{1-\gamma}}{\Gamma(\alpha)}\int_a^t \ominus_\psi^\alpha(t,s)\\ &\quad\times\left[\,\left|\mathcal{F}_v(s)-\mathcal{F}_0(s)\right|+\left|\mathcal{F}_0(s)\right|\,\right]ds \leq \frac{\left[\psi(t)-\psi(a)\right]^{1-\gamma}}{\Gamma(\alpha)}\int_a^t \ominus_\psi^\alpha(t,s)\\ &\quad\times\left[\psi(s)-\psi(a)\right]^{\gamma-1}\left[M(1+L^*)\left\|v\right\|_{C_{1-\gamma;\psi}} + \left\|\widetilde{f}\right\|_{C_{1-\gamma;\psi}}\right]ds\\ &\quad+\frac{\left[\psi(t)-\psi(a)\right]^{1-\gamma}}{\Gamma(\alpha)}\int_a^t \ominus_\psi^\alpha(t,s)MM^*ds.\end{aligned}$$

In view of Proposition 1 and Equation (3.13), then for every $v \in \mathbb{B}_r$, we get

$$\begin{aligned}\|T_2 v\|_{C_{1-\gamma;\psi}} \quad \leq \quad & \left[\frac{\mathcal{B}(\gamma,\alpha)}{\Gamma(\alpha)}(M+ML^*)r + \left\|\widetilde{f}\right\|_{C_{1-\gamma;\psi}} \right.\\ & \left. +\frac{MM^*}{\Gamma(\alpha+1)}\right]\left[\psi(b)-\psi(a)\right]^{\alpha}. \qquad (3.15)\end{aligned}$$

By definitions of $\sigma$ and $r$ with (3.14) and (3.15), we get

$$\|T_1 u + T_2 v\|_{C_{1-\gamma;\psi}} \leq \|T_1 u\|_{C_{1-\gamma;\psi}} + \|T_2 v\|_{C_{1-\gamma;\psi}} \leq \sigma r + \rho \leq r.$$

This proves that $T_1 u + T_2 v \in \mathbb{B}_r$ for every $u, v \in \mathbb{B}_r$.

**Step 2:** We prove that the operator $T_1$ is a contration mapping on $\mathbb{B}_r$. By the preceding assumptions, then for any $u, v \in \mathbb{B}_r$, and for $t \in (a,b]$, we have

$$\begin{aligned}&\left|\left[\psi(t)-\psi(a)\right]^{1-\gamma} T_1 u(t) - \left[\psi(t)-\psi(a)\right]^{1-\gamma} T_1 v(t)\right| \leq \sum_{k=1}^{m}\frac{c_k}{B}\frac{1}{\Gamma(\alpha)}\\ &\quad\times\int_a^{\tau_k}\ominus_\psi^\alpha(t,s)\left|\mathcal{F}_u(s)-\mathcal{F}_v(s)\right|ds \leq \sum_{k=1}^{m}\frac{c_k}{B}\frac{1}{\Gamma(\alpha)}\int_a^{\tau_k}\ominus_\psi^\alpha(t,s)\\ &\quad\times\left[\psi(s)-\psi(a)\right]^{\gamma-1} M\left[\left\|u-v\right\|_{C_{1-\gamma;\psi}} + L^*\left\|u-v\right\|_{C_{1-\gamma;\psi}}\right]ds\\ &\leq \frac{\mathcal{B}(\gamma,\alpha)}{\Gamma(\alpha)}\left(M+ML^*\right)\sum_{k=1}^{m}\frac{c_k}{B}\left[\psi(\tau_k)-\psi(a)\right]^{\gamma-1+\alpha}\left\|u-v\right\|_{C_{1-\gamma;\psi}}.\end{aligned}$$

This gives with (3.13) that

$$\|T_1 u - T_1 v\|_{C_{1-\gamma;\psi}} \leq \frac{\mathcal{B}(\gamma,\alpha)}{\Gamma(\alpha)}\left(M+ML^*\right)\sum_{k=1}^{m}\frac{c_k}{B}\left[\psi(\tau_k)-\psi(a)\right]^{\alpha}\left\|u-v\right\|_{C_{1-\gamma;\psi}}.$$

The operator $T_1$ is contraction mapping due to assumption (3.11).

**Step 3:** We show that the operator $T_2$ is completely continuous on $\mathbb{B}_r$.

From the continuity of $\mathcal{F}_u$, we conclude that the operator $T_2$ is continuous. Now, we prove that $(T_2\mathbb{B}_r)$ is uniformly bounded. Indeed, it is enough to show that for some $r > 0$, there exists a positive constant $\ell$ such that $\|T_2u\|_{C_{1-\gamma;\psi}} \leq \ell$. According to Step 1, for $u \in \mathbb{B}_r$, we know that

$$\begin{aligned} \|T_2u\|_{C_{1-\gamma;\psi}} \leq & \left[\frac{\mathcal{B}(\gamma,\alpha)}{\Gamma(\alpha)}(M+ML^*)\|u\|_{C_{1-\gamma;\psi}} + \left\|\widetilde{f}\right\|_{C_{1-\gamma;\psi}}\right. \\ & \left.+\frac{MM^*}{\Gamma(\alpha+1)}\right][\psi(b)-\psi(a)]^{\alpha} := \ell, \end{aligned}$$

which is independent of $t$ and $u$. This means that $\|T_2u\|_{C_{1-\gamma;\psi}} \leq \ell$ i.e. $T_2$ is uniformly bounded on $\mathbb{B}_r$. Moreover, we prove that $(T_2\mathbb{B}_r)$ is equicontinuous in $\mathbb{B}_r$. Let $u \in \mathbb{B}_r$ and $t_1, t_2 \in [a,b]$ with $t_1 < t_2$, we have

$$\begin{aligned} & \left|[\psi(t_2)-\psi(a)]^{1-\gamma} T_2u(t_2) - [\psi(t_1)-\psi(a)]^{1-\gamma} T_2u(t_1)\right| \\ & \leq \left|\frac{[\psi(t_2)-\psi(a)]^{1-\gamma}}{\Gamma(\alpha)}\int_a^{t_2} \ominus_{\psi}^{\alpha}(t_2,s)[\psi(s)-\psi(a)]^{\gamma-1}\right. \\ & \times \max_{s\in[a,b]}\left|[\psi(s)-\psi(a)]^{1-\gamma}\mathcal{F}_u(s)\right| ds \\ & - \frac{[\psi(t_1)-\psi(a)]^{1-\gamma}}{\Gamma(\alpha)}\int_a^{t_1} \ominus_{\psi}^{\alpha}(t_1,s)[\psi(s)-\psi(a)]^{\gamma-1} \\ & \left.\times \max_{s\in[a,b]}\left|[\psi(s)-\psi(a)]^{1-\gamma}\mathcal{F}_u(s)\right| ds\right| \\ & \leq \|\mathcal{F}_u\|_{C_{1-\gamma;\psi}[a,b]}\frac{\mathcal{B}(\gamma,\alpha)}{\Gamma(\alpha)}\left|[\psi(t_2)-\psi(a)]^{\alpha} - [\psi(t_1)-\psi(a)]^{\alpha}\right|. \end{aligned}$$

Observe that the right-hand side of the above inequality is independent of $u$. Thus, using the continuity of $\psi$, $|t_2 - t_1| \to 0$ implies that $\left|[\psi(t_2)-\psi(a)]^{1-\gamma} T_2u(t_2) - [\psi(t_1)-\psi(a)]^{1-\gamma} T_2u(t_1)\right| \to 0$. This proves that $(T_2\mathbb{B}_r)$ is equicontinuous. In view of Arzela-Ascoli Theorem, it follows that $(T_2\mathbb{B}_r)$ is relatively compact. As a consequence of Theorem 3, we conclude that the problem (1.1)–(1.2) has at least one solution in $C_{1-\gamma;\psi}[a,b]$.

Finally, we show that such a solution is indeed in $C^{\gamma}_{1-\gamma;\psi}[a,b]$. By applying $D^{\gamma;\psi}_{a^+}$ on both sides of (3.1), we get

$$D^{\gamma;\psi}_{a^+}u(t) = D^{\gamma;\psi}_{a^+} I^{\alpha;\psi}_{a^+}h(t) = D^{\gamma-\alpha;\psi}_{a^+} f(t,u(t),Hu(t) = D^{\beta(1-\alpha);\psi}_{a^+} f(t,u(t),Hu(t).$$

Since $f(\cdot,u(\cdot),Hu(\cdot)) \in C^{\beta(1-\alpha)}_{1-\gamma;\psi}[a,b]$, it follows by definition of the space $C^{\beta(1-\alpha)}_{1-\gamma;\psi}[a,b]$ that $D^{\gamma;\psi}_{a^+}u(t) \in C_{1-\gamma;\psi}[a,b]$ which implies that $u(t) \in C^{\gamma}_{1-\gamma;\psi}[a,b]$. □

# 4 The Ulam-Hyers-Rassias stability

In this section, we will investigate the various types of stability results of the problem (1.1)–(1.2). The stability results are based on the Banach fixed point theorem.

**Theorem 6.** *Assume that hypotheses (A1) and (A2) are fulfilled. If $\sigma < 1$. Then, the problem (1.1)–(1.2) has a unique solution, where $\sigma$ is defined as in Theorem 5.*

*Proof.* *Consider the operator $T : C_{1-\gamma;\psi}[a,b] \to C_{1-\gamma;\psi}[a,b]$ defined as in Equation (3.12). In view of Theorem 5, we know that the fixed points of $T$ are solutions of problem (1.1)–(1.2). Now, we prove that $T$ has a unique fixed point, which is a solution of problem (1.1)–(1.2). Indeed, by hypotheses (A1)–(A2), Proposition 1 and Equation (3.13), then for $u, v \in C_{1-\gamma;\psi}[a,b]$, $t \in (a,b]$, we have*

$$\begin{aligned}
&\left|[\psi(t)-\psi(a)]^{1-\gamma}Tu(t)-[\psi(t)-\psi(a)]^{1-\gamma}Tv(t)\right| \le \sum_{k=1}^{m}\frac{c_k}{B}\frac{1}{\Gamma(\alpha)}\\
&\times\int_a^{\tau_k}\ominus_\psi^\alpha(\tau_k,s)\left|\mathcal{F}_u(s)-\mathcal{F}_v(s)\right|ds+\frac{[\psi(t)-\psi(a)]^{1-\gamma}}{\Gamma(\alpha)}\\
&\times\int_a^{t}\ominus_\psi^\alpha(t,s)\left|\mathcal{F}_u(s)-\mathcal{F}_v(s)\right|ds \le \sum_{k=1}^{m}\frac{c_k}{B}\frac{(M+ML^*)}{\Gamma(\alpha)}\\
&\times\int_a^{\tau_k}\ominus_\psi^\alpha(\tau_k,s)\left[\psi(s)-\psi(a)\right]^{\gamma-1}\|u-v\|_{C_{1-\gamma;\psi}}ds\\
&+\frac{(M+ML^*)\left[\psi(t)-\psi(a)\right]^{1-\gamma}}{\Gamma(\alpha)}\int_a^{t}\ominus_\psi^\alpha(t,s)\left[\psi(s)-\psi(a)\right]^{\gamma-1}\\
&\times\|u-v\|_{C_{1-\gamma;\psi}}ds \le \Big[\sum_{k=1}^{m}\frac{c_k}{B}\psi(\tau_k)-\psi(a))^{\alpha}+\left[\psi(b)-\psi(a)\right]^{\alpha}\Big]\\
&\times\frac{\mathcal{B}(\gamma,\alpha)}{\Gamma(\alpha)}(M+ML^*)\|u-v\|_{C_{1-\gamma;\psi}}.
\end{aligned}$$

*This gives, $\|Tu-Tv\|_{C_{1-\gamma;\psi}} \le \sigma\|u-v\|_{C_{1-\gamma;\psi}}$. Since $\sigma < 1$, the operator $T : C_{1-\gamma;\psi}[a,b] \to C_{1-\gamma;\psi}[a,b]$ is a contraction mapping. Hence by Banach fixed point theorem, it follows that $T$ has a unique fixed point, which is a solution of problem (1.1)–(1.2).* □

Now, we study the Ulam-Hyers stability, and Ulam-Hyers-Rassias stability. Let $\epsilon > 0$, $\widetilde{u} \in C^{\gamma}_{1-\gamma;\psi}[a,b]$ and $\Phi \in C_{1-\gamma;\psi}[a,b]$, such that

$$\left|D_{a^+}^{\alpha,\beta,\psi}\widetilde{u}(t)-\mathcal{F}_{\widetilde{u}}(s)\right| \le \epsilon,\ t\in(a,b], \tag{4.1}$$

$$\left|D_{a^+}^{\alpha,\beta,\psi}\widetilde{u}(t)-\mathcal{F}_{\widetilde{u}}(s)\right| \le \epsilon\Phi(t),\ t\in(a,b], \tag{4.2}$$

$$\left|D_{a^+}^{\alpha,\beta,\psi}\widetilde{u}(t)-\mathcal{F}_{\widetilde{u}}(s)\right| \le \Phi(t),\ t\in(a,b], \tag{4.3}$$

where $\mathcal{F}_{\widetilde{u}}(s) := \mathcal{F}(t, \widetilde{u}(t), H\widetilde{u}(t))$.

Definition 5. The problem (1.1)–(1.2) is Ulam-Hyers stable if there exists a real number $C_{\mathcal{F}} > 0$ such that, for each $\epsilon > 0$ and for each solution $\widetilde{u} \in C^{\gamma}_{1-\gamma;\psi}[a,b]$ of Inequality (4.1), there exists a solution $u \in C^{\gamma}_{1-\gamma;\psi}[a,b]$ of problem (1.1)–(1.2) with

$$|\widetilde{u}(t) - u(t)| \leq C_{\mathcal{F}}\epsilon, \quad t \in (a,b].$$

Definition 6. The problem (1.1)–(1.2) is generalized Ulam-Hyers stable if there exists $\varphi \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\varphi(0) = 0$ such that, for each $\epsilon > 0$ and for each solution $\widetilde{u} \in C^{\gamma}_{1-\gamma;\psi}[a,b]$ of Inequality (4.1), there exists a solution $u \in C^{\gamma}_{1-\gamma;\psi}[a,b]$ of problem (1.1)–(1.2) with

$$|\widetilde{u}(t) - u(t)| \leq \varphi(\epsilon) \quad t \in (a,b].$$

Definition 7. The problem (1.1)–(1.2) is Ulam-Hyers-Rassias stable with respect to $\Phi \in C_{1-\gamma;\psi}[a,b]$, if there exists a real number $C_{\mathcal{F},\Phi} > 0$ such that, for each $\epsilon > 0$ and for each solution $\widetilde{u} \in C^{\gamma}_{1-\gamma;\psi}[a,b]$ of Inequality (4.2), there exists a solution $u \in C^{\gamma}_{1-\gamma;\psi}[a,b]$ of problem (1.1)–(1.2) with

$$|\widetilde{u}(t) - u(t)| \leq C_{\mathcal{F},\Phi}\epsilon\Phi(t), \quad t \in (a,b].$$

Definition 8. The problem (1.1)–(1.2) is generalized Ulam-Hyers-Rassias stable with respect to $\Phi \in C_{1-\gamma;\psi}[a,b]$, if there exists a real number $C_{\mathcal{F},\Phi} > 0$ such that, for each solution $\widetilde{u} \in C^{\gamma}_{1-\gamma;\psi}[a,b]$ of Inequality (4.3), there exists a solution $u \in C^{\gamma}_{1-\gamma;\psi}[a,b]$ of problem (1.1)–(1.2) with

$$|\widetilde{u}(t) - u(t)| \leq C_{\mathcal{F},\Phi}\Phi(t), \quad t \in (a,b].$$

**Theorem 7.** *Assume that the conditions (A1) and (A2) are satisfied. Then the problem (1.1)–(1.2) is Ulam-Hyers stable.*

*Proof.* Let $\epsilon > 0$ and let $\widetilde{u} \in C^{\gamma}_{1-\gamma,\psi}[a,b]$ be a function which satisfies the inequality (4.1) and let $u \in C^{\gamma}_{1-\gamma,\psi}[a,b]$ the unique solution of the following integro-differential equation

$$D^{\alpha,\beta;\psi}_{a^+}u(t) = \mathcal{F}_u(s), \quad 0 < \alpha < 1, \ 0 \leq \beta \leq 1, \ t \in (a,b], \tag{4.4}$$

$$I^{1-\gamma;\psi}_{a^+}u(t) \mid_{t=a} = I^{1-\gamma;\psi}_{a^+}\widetilde{u}(t) \mid_{t=a} = u_a + \sum_{k=1}^{m} c_k u(\tau_k), \tau_k \in (a,b], \quad \gamma = \alpha+\beta-\alpha\beta. \tag{4.5}$$

In view of Theorem 4, we get

$$u(t) = A_u + \frac{1}{\Gamma(\alpha)}\int_a^t \ominus^{\alpha}_{\psi}(t,s)\mathcal{F}_u(s)ds, \tag{4.6}$$

where

$$A_u = \frac{[\psi(t) - \psi(a)]^{\gamma-1}}{B}\Big[\sum_{k=1}^{m} \frac{c_k}{\Gamma(\alpha)}\int_a^{\tau_k} \ominus^{\alpha}_{\psi}(\tau_k,s)\mathcal{F}_u(s)ds + u_a\Big].$$

On the other hand, if $u(\tau_k) = \widetilde{u}(\tau_k)$ and $I_{a^+}^{1-\gamma;\psi}u(a) = I_{a^+}^{1-\gamma;\psi}\widetilde{u}(a)$, it follows that $A_u = A_{\widetilde{u}}$. Now, by integration of the inequality (4.1), we obtain

$$\left|\widetilde{u}(t) - A_{\widetilde{u}} - \frac{1}{\Gamma(\alpha)}\int_a^t \ominus_\psi^\alpha(t,s)\mathcal{F}_{\widetilde{u}}(s)ds\right| \leq \frac{\epsilon\left[\psi(b)-\psi(a)\right]^\alpha}{\Gamma(\alpha+1)}, \ \forall t \in (a,b].$$

From the above, it follows

$$\begin{aligned}
|\widetilde{u}(t) - u(t)| &\leq \left|\widetilde{u}(t) - A_{\widetilde{u}} - \frac{1}{\Gamma(\alpha)}\int_a^t \ominus_\psi^\alpha(t,s)\mathcal{F}_{\widetilde{u}}(s)ds\right| \\
&\quad + \left|A_{\widetilde{u}} - A_u + \frac{1}{\Gamma(\alpha)}\int_a^t \ominus_\psi^\alpha(t,s)\left|\mathcal{F}_{\widetilde{u}}(s) - \mathcal{F}_u(s)\right|ds\right| \\
&\leq \frac{\epsilon\left[\psi(b)-\psi(a)\right]^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)}\int_a^t \ominus_\psi^\alpha(t,s)\left|\mathcal{F}_{\widetilde{u}}(s) - \mathcal{F}_u(s)\right|ds \\
&\leq \frac{\epsilon\left[\psi(b)-\psi(a)\right]^\alpha}{\Gamma(\alpha+1)} + \frac{(M+ML^*)}{\Gamma(\alpha)}\int_a^t \ominus_\psi^\alpha(t,s)\left|\widetilde{u}(s) - u(s)\right|.
\end{aligned}$$

In view of Lemma 6, we conclude that

$$\begin{aligned}
|\widetilde{u}(t) - u(t)| &\leq \frac{\epsilon\left[\psi(b)-\psi(a)\right]^\alpha}{\Gamma(\alpha+1)} + \int_a^t \left[\sum_{k=1}^{\infty}\frac{\left[(M+ML^*)\right]^k}{\Gamma(\alpha k)}\right. \\
&\times \psi'(s)(\psi(t)-\psi(s))^{\alpha k-1}\left.\frac{\epsilon\left[\psi(b)-\psi(a)\right]^\alpha}{\Gamma(\alpha+1)}\right]ds \leq \epsilon\frac{\left[\psi(b)-\psi(a)\right]^\alpha}{\Gamma(\alpha+1)} \\
&\times \left[1 + \frac{1}{\left[\psi(b)-\psi(a)\right]}\Big(E_{\alpha,1}\left(M+ML^*\right)\left[\psi(b)-\psi(a)\right]^\alpha - 1\Big)\right].
\end{aligned}$$

Take $C_{\mathcal{F}} = \frac{[\psi(b)-\psi(a)]^\alpha}{\Gamma(\alpha+1)}\left[1 + \frac{1}{[\psi(b)-\psi(a)]}\Big(E_{\alpha,1}\left(M+ML^*\right)\left[\psi(b)-\psi(a)\right]^\alpha - 1\Big)\right]$, we get $|\widetilde{u}(t) - u(t)| \leq C_{\mathcal{F}}\epsilon$. □

**Theorem 8.** *Let the hypotheses of Theorem 7 hold. If there exists $\varphi \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\varphi(0) = 0$. Then the problem (1.1)–(1.2) has generalized Ulam-Hyers stability.*

*Proof.* In a manner similar to above Theorem 7, with putting $\varphi(\epsilon) = C_{\mathcal{F}}\epsilon$ and $\varphi(0) = 0$, we get $\|\widetilde{u} - u\|_{C_{1-\gamma;\psi}} \leq \varphi(\epsilon)$. □

**Theorem 9.** *Under the hypotheses (A1) and (A2). If the following condition is satisfied:*

**(A3)** *There exists an increasing function $\Phi \in C_{1-\gamma,\psi}[a,b]$ and there exists $\lambda_\Phi > 0$ such that, for any $t \in (a,b]$, $I_{a^+}^{\alpha;\psi}\Phi(t) \leq \lambda_\Phi\Phi(t)$.*

*Then the problem (1.1)–(1.2) is Ulam-Hyers-Rassias stable.*

*Proof.* Let $\epsilon > 0$ and $\widetilde{u} \in C_{1-\gamma,\psi}^{\gamma}[a,b]$ be the unique solution of the problem (4.4)–(4.5) that satisfies the Inequality (4.2). By integration of (4.2) and using hypothesis (A3), we get

$$\left|\widetilde{u}(t)-A_{\widetilde{u}}-\frac{1}{\Gamma(\alpha)}\int_a^t \ominus_\psi^\alpha(t,s)\mathcal{F}_{\widetilde{u}}(s)ds\right| \le \epsilon\lambda_\Phi\Phi(t),\ \forall t\in(a,b].$$

Now, let $u\in C^{\gamma}_{1-\gamma,\psi}[a,b]$ be the unique solution of problem (1.1)–(1.2) that is defined as in Equation (4.6) due to Theorem 4. From the above, it follows

$$\begin{aligned}|\widetilde{u}(t)-u(t)| &\le \left|\widetilde{u}(t)-A_{\widetilde{u}}-\frac{1}{\Gamma(\alpha)}\int_a^t \ominus_\psi^\alpha(t,s)\mathcal{F}_{\widetilde{u}}(s)ds\right|\\ &+\left|A_{\widetilde{u}}-A_u+\frac{1}{\Gamma(\alpha)}\int_a^t \ominus_\psi^\alpha(t,s)\left|\mathcal{F}_{\widetilde{u}}(s)-\mathcal{F}_u(s)\right|ds\right|\\ &\le \epsilon\lambda_\Phi\Phi(t)+\frac{1}{\Gamma(\alpha)}\int_a^t \ominus_\psi^\alpha(t,s)\left|\mathcal{F}_{\widetilde{u}}(s)-\mathcal{F}_u(s)\right|ds\\ &\le \epsilon\lambda_\Phi\Phi(t)+\frac{(M+ML^*)}{\Gamma(\alpha)}\int_a^t \ominus_\psi^\alpha(t,s)\left|\widetilde{u}(s)-u(s)\right|.\end{aligned}$$

In view of Lemma 6, we conclude that

$$\begin{aligned}|\widetilde{u}(t)-u(t)| &\le \epsilon\lambda_\Phi\Phi(t)+\int_a^t\sum_{k=1}^{\infty}\frac{[(M+ML^*)]^k}{\Gamma(\alpha k)}\ominus_\psi^{\alpha k}(t,s)\epsilon\lambda_\Phi\Phi(s)ds\\ &=\epsilon\lambda_\Phi\Phi(t)+\epsilon\lambda_\Phi\Big[(M+ML^*)\,I_{a^+}^{\alpha,\psi}\Phi(t)+(M+ML^*)^2\,I_{a^+}^{2\alpha,\psi}\Phi(t)+\cdots\Big]\\ &\le \epsilon\lambda_\Phi\Phi(t)+\epsilon\lambda_\Phi\Big[(M+ML^*)\,\lambda_\Phi\Phi(t)+(M+ML^*)^2\,(\lambda_\Phi)^2\,\Phi(t)+\cdots\Big]\\ &=\epsilon\lambda_\Phi\Phi(t)\Big[1+\sum_{k=1}^{\infty}(M+ML^*)^k\,(\lambda_\Phi)^k\Big].\end{aligned}$$

Take $C_{\mathcal{F},\Phi}=\lambda_\Phi\left[1+\sum_{k=1}^{\infty}(M+ML^*)^k(\lambda_\Phi)^k\right]$, then we obtain $|\widetilde{u}(t)-u(t)|\le C_{\mathcal{F},\Phi}\epsilon\Phi(t)$. □

*Corollary 1.* Let the conditions of Theorem 9 hold. Then the problem (1.1)–(1.2) is generalized Ulam-Hyers-Rassias stable.

*Proof.* Set $\epsilon=1$ in the proof of Theorem 9, we get $|\widetilde{u}(t)-u(t)|\le C_{\mathcal{F},\Phi}\Phi(t)$. □

## 5 Mittag-Leffler function approximation of integrals

In this section, we shall investigate a new procedure for approximate integrals depending on Mittag-Leffler function, firstly we define the first $k$ terms in the series representiaion of Mittag-Leffler function in (2.1) to be $E_k(z,\mu,\nu)$, so, we can approximate any function $f(z)$ as

$$f(z)=\sum_{k=0}^{n}a_kE_k(z,\mu,\nu). \tag{5.1}$$

The coefficients $a_k$ can be written as

$$a_k = \sum_{k=0}^{n} u_k \theta_{lk}, \tag{5.2}$$

Writting (5.1)–(5.2) in matrix form, we have

$$F = A^T Œ(z), \quad A = F^T \Theta, \tag{5.3}$$

where $Œ(z) = [E_0(z,\mu,\nu), E_1(z,\mu,\nu), ..., E_n(z,\mu,\nu)]^T$,
$F = [f_n(z_0), f_n(z_1), ..., f_n(z_n)]^T$, $\Theta = \{\theta_{lk}\}_{l,k=0}^{n}$ and $A = \{a_i\}_{i=0}^{n}$ is the unknowns vector. Combining the two equations in (5.3) we conclude that

$$F = [F^T \Theta]^T Œ(z) = F\Theta^T Œ(z), \tag{5.4}$$

or

$$Œ(z)\Theta = I, \text{ so } \quad \Theta = [Œ(z)]^{-1}. \tag{5.5}$$

Thus (5.4) in view of (5.1) can be written as:

$$f(z) = \sum_{k=0}^{n} f(z_k)\Theta_k E_k(z,\mu,\nu). \tag{5.6}$$

For an approximation of integrals, integrating (5.6), we obtain

$$\int_0^t f(z)dt = \sum_{k=0}^{n} f(z_k) M_k(t), \tag{5.7}$$

where $M_k(t) = \Theta_k \int_0^t E_k(z,\mu,\nu)dz$, is the Mittag operational matrix of integration. This approximation of integrals is accurate and highly efficient depending on the error analysis and results of Mittag-Leffler approximation [23].

## 6 An example

Consider the $\psi-$Hilfer fractional integro-differential equation with nonlocal condition

$$D_{0^+}^{\alpha,\beta,\psi} u(t) = f(t,u(t),Hu(t)), \quad t \in (0,1], \tag{6.1}$$

$$I_{0^+}^{1-\gamma,\psi} u(0) = 0.4u(2/3), \tag{6.2}$$

where $u_0 = 0$, $\alpha = \frac{1}{2}, \beta = \frac{1}{3}$, $\gamma = \frac{2}{3}$, $c_1 = \frac{2}{5}$, $\tau_1 = \frac{2}{3}$,

$$f(t,u(t),Hu(t)) = [\psi(t)-\psi(0)]^{\frac{-1}{6}} + \frac{1}{16}\Big[ [\psi(t)-\psi(0)]^{\frac{5}{6}} \sin u(t) + \int_0^t e^{\frac{-1}{2}u(s)} ds \Big],$$

and $Hu(t) = \int_0^t \mathcal{K}(t,s,u(s))ds = \int_0^t e^{\frac{-1}{2}u(s)}ds$. Consider $\psi : [0,1] \to \mathbb{R}$ such that $\psi(t) = t$ for all $t \in [0,1]$, clearly,

$$t^{\frac{1}{3}} f(t,u(t),Hu(t)) = t^{\frac{1}{6}} + \frac{1}{16} t^{\frac{7}{6}} \sin u(t) + \frac{1}{16} t^{\frac{1}{3}} \int_0^t e^{\frac{-1}{2}u(s)} ds \in C[0,1],$$

hence $f(t,u(t),Hu(t)) \in C_{\frac{1}{3},t}[0,1]$. Observe that, for any $u,v \in \mathbb{R}^+$ and $t \in (0,1]$,

$$|f(t,u,Hu) - f(t,v,Hv)| \le \frac{1}{16}\left[|u-v| + |Hu-Hv|\right]$$
$$|Hu-Hv| = \left|\int_0^t e^{\frac{-1}{2}u(s)}ds - \int_0^t e^{\frac{-1}{2}v(s)}ds\right| \le \int_0^t e^{\frac{-1}{2}|u(s)-v(s)|}ds \le \frac{1}{2}\,|u-v|\,.$$

Therefore, the conditions (A1) and (A2) are satisfied with $M = \frac{1}{16}$ and $L^* = \frac{1}{2}$. It is easy to check that the condition (3.11) is satisfied. Indeed,

$$B = \Gamma(\gamma) - c_1\left[\psi(\tau_1) - \psi(a)\right]^{\gamma-1} = \Gamma(\frac{2}{3}) - \frac{2}{5}\left(\frac{2}{3}\right)^{-\frac{1}{3}} \simeq 0.89.$$

Furthermore, by simple computations we get

$$\sigma = \left[\frac{c_1}{B}\psi(\tau_1) - \psi(a))^{\alpha} + \left[\psi(b) - \psi(a)\right]^{\alpha}\right]\frac{\mathcal{B}(\gamma,\alpha)}{\Gamma(\alpha)}\left(M + ML^*\right) \approx 0.19 < 1,$$
$$\frac{\mathcal{B}(\gamma,\alpha)}{\Gamma(\alpha)}\left(M + ML^*\right)\frac{c_1}{B}\left[\psi(\tau_1) - \psi(a)\right]^{\alpha} \simeq 0.05 < 1.$$

It follows from Theorem 5 that the problem (6.1)–(6.2) has a solution on $[0,1]$. Now, to applying Theorem 6, we have previously seen that $\sigma < 1$ and the hypotheses (A1)–(A2) hold. Therefore, Theorem 6 shows that the problem (6.1)–(6.2) has a unique solution on $[0,1]$.

On the other hand, as shown in Theorem 7, for every $\epsilon > 0$ if $\widetilde{u} \in C^{\frac{2}{3}}_{\frac{1}{3};t}[0,1]$ satisfies

$$\left|D^{\alpha,\beta,\psi}_{0^+}\widetilde{u}(t) - f(t,\widetilde{u}(t),H\widetilde{u}(t))\right| \le \epsilon,\ t \in (0,1],$$

there exists a unique solution $u \in C^{\frac{2}{3}}_{\frac{1}{3};t}[0,1]$ such that

$$|\widetilde{u}(t) - u(t)| \le C_{\mathcal{F}}\epsilon,$$

where

$$C_{\mathcal{F}} = \frac{1}{\Gamma(0.5+1)}E_{\frac{1}{2},1}\left(\frac{1}{16} + \frac{1}{32}\right) \simeq 1.258.$$

Hence the problem (6.1)–(6.2) is Ulam-Hyers stable.

Now, we consider $\Phi(t) = [\psi(t) - \psi(0)]$, then $[\psi(t) - \psi(0)]^{1-\frac{2}{3}}[\psi(t) - \psi(0)] = [\psi(t) - \psi(0)]^{\frac{4}{3}} = t^{\frac{4}{3}} \in C[0,1]$ i.e. $\Phi(t) \in C_{\frac{1}{3},t}[0,1]$. To verify the condition (A3), we must to show that $I^{\alpha,\psi}_{a^+}\Phi(t) \le \lambda_{\Phi}\Phi(t)$, $\lambda_{\Phi} > 0$. Indeed, by Defintion of $I^{\alpha,\psi}_{0^+}$, equation (2.3) and simple computation, we have

$$I^{\frac{1}{2},\psi}_{0^+}\Phi(t) = \frac{1}{\Gamma(0.5)}\int_0^t \ominus^{\frac{1}{2}}_{\psi}(t,s)\left[\psi(s) - \psi(0)\right]ds$$
$$\le \frac{[\psi(t) - \psi(0)]}{\Gamma(0.5)}\int_0^t \ominus^{-\frac{1}{2}}_{\psi}(t,s)ds \le \frac{[\psi(1) - \psi(0)]^{\frac{1}{2}}}{\Gamma(1.5)}\Phi(t) = \frac{1}{\Gamma(1.5)}\Phi(t).$$

Thus, the hypothesis (A3) is satisfied with $\lambda_\Phi = 1/\Gamma(\frac{3}{2}) > 0$. And for every $\epsilon > 0$ if $\widetilde{u} \in C^{\frac{2}{3}}_{\frac{1}{3};t}[0,1]$ satisfies

$$\left| D^{\alpha,\beta,\psi}_{0^+}\widetilde{u}(t) - f(t,\widetilde{u}(t),H\widetilde{u}(t)) \right| \leq \epsilon\left[\psi(t)-\psi(0)\right],\ t\in(0,1],$$

there exists a unique solution $u \in C^{\frac{2}{3}}_{\frac{1}{3};t}[0,1]$ such that

$$|\widetilde{u}(t) - u(t)| \leq C_{\mathcal{F},\Phi}\epsilon\left[\psi(t)-\psi(0)\right] = C_{\mathcal{F},\Phi}\epsilon t,$$

where $C_{\mathcal{F},\Phi} = \dfrac{1}{\Gamma(\frac{3}{2})}\left[1+\sum_{k=1}^{\infty}\left(\frac{3}{32}\right)^k\left(\dfrac{1}{\Gamma(\frac{3}{2})}\right)^k\right]$. Hence problem (6.1)–(6.2) is Ulam-Hyers-Rassias stable. Finaly, take $\epsilon = 1$, we get

$$|\widetilde{u}(t) - u(t)| \leq C_{\mathcal{F},\Phi}t.$$

Therefore, the problem (6.1)–(6.2) is generalized Ulam-Hyers-Rassias stable.

## 7 Numerical solution

Making use of the analytic technique discussed in Section 3, the example described by (6.1)–(6.2) has the solution

$$\begin{aligned}u(t) = {}& \frac{\frac{2}{5}t^{\gamma-1}}{\Gamma(\alpha)[\Gamma(\gamma)-\frac{2}{5}(\frac{2}{3})^{\gamma-1}]}\int_0^{\frac{2}{3}}\left(\frac{2}{3}-s\right)^{\alpha-1}\left[s^{\frac{-1}{6}}+\frac{1}{16}s^5\sin(u(s))\right.\\ &\left.+\frac{1}{16}\int_0^s e^{\frac{-1}{2}u(\tau)}d\tau\right]ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[s^{\frac{-1}{6}}+\frac{1}{16}s^5\sin(u(s))\right.\\ &\left.+\frac{1}{16}\int_0^s e^{\frac{-1}{2}u(\tau)}d\tau\right]ds.\end{aligned} \tag{7.1}$$

This is an implicit nonlinear formula for the solution. We can't find the value of it at any point directly. We must solve it numerically to catch the solution values. Evaluating (7.1) at some points $t_i, i = 0,1,\dots,n$ and approximating all integrations making use of Equation (5.7), we obtain

$$\begin{aligned}L(i) = {}& u(t_i) - \frac{\frac{2}{5}t_i^{\gamma-1}}{\Gamma(\alpha)[\Gamma(\gamma)-\frac{2}{5}(\frac{2}{3})^{\gamma-1}]}\sum_{k=0}^{n}(\frac{2}{3}-t_k)^{\alpha-1}M_k(\frac{2}{3})\\ &\times\left[t_k^{\frac{-1}{6}}+\frac{1}{16}t_k^5 sin(u(t_k))+\frac{1}{16}\sum_{j=0}^{n}e^{\frac{-1}{2}u(t_j)}M_j(t_k)\right]+\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{i}(t_i-t_k)^{\alpha-1}M_k(t_i)\\ &\times\left[t_k^{\frac{-1}{6}}+\frac{1}{16}t_k^5 sin(u(t_k))+\frac{1}{16}\sum_{j=0}^{n}e^{\frac{-1}{2}u(t_j)}M_j(t_k)\right]=0.\end{aligned} \tag{7.2}$$

To avoid the singularity at the upper bound of integrals, we take $t_n$ to be close to (but not equal) the upper bound of each integral. Now, the problem in Equation (7.2) is a nonlinear system of algebraic equations in the unknowns

$\{u(t_0), u(t_1), ..., u(t_n)\}$. Applying least squares method, the problem become: minimize $E = \sum_{i=0}^{n} \left[L(i)\right]^2$. This is an unconstrained optimization problem. Which can be solved easily to catch the unknowns.

In Table 1, we present the errors for the least squares method $n = 5$. The error is very small. This ensures that the minimum value and then the solution is evaluated accurately. Figure 1 presents the solution curves with some values of $\alpha$, $\gamma$ and $n = 5$.

**Table 1.** Errors for the least squares method, $n = 5$.

| $\alpha$ | $\gamma$ | $E$ |
|---|---|---|
| 1/2 | 2/3 | 2.00054E-22 |
| 1/3 | 1/2 | 1.32462E-21 |
| 1/4 | 2/5 | 1.92318E-21 |
| 1/5 | 1/3 | 6.64266E-22 |

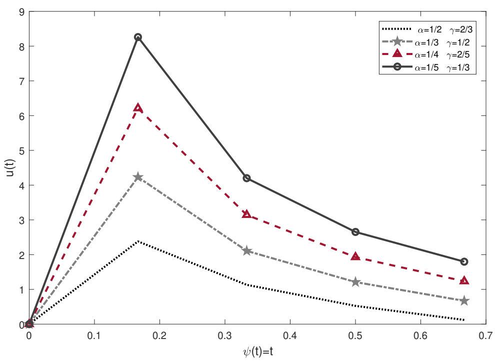


**Figure 1.** Solution of illustrated example with some values of $\alpha$, $\gamma$ and $n = 5$.

## 8 Conclusions

We can conclude that the main results of this article have been successfully achieved, that is, through of Banach fixed point theorem and Krasnoselskii's fixed point theorem, we have investigated the existence and uniqueness of solutions of a nonlinear fractional integro-differential equation introduced by $\psi$-Hilfer fractional derivative. Further, we discussed stabilities of Ulam-Hyers, generalized Ulam-Hyers, Ulam-Hyers-Rassias and generalized Ulam-Hyers-Rassias. This paper contributes to the growth of the fractional calculus, especially in the case fractional differential equations involving a general formulation of Hilfer fractional derivative with respect to another function. The numerical results are evaluated by investigating Mittag operational matrix of integration. The resulting nonlinear system of algebraic equations is constructed as an unconstrained optimization problem which is solved easily to obtain the unknowns. Table 1 and Figure 1 introduced in the results shows that the numerical method success in approximating the solution. Further, this paper may be carried forward to higher fractional order of differential equation of $\psi-$Hilfer fractional derivative.

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