Mathematical Modelling and Analysis
Volume 13 Number 1, 2008, pages 99–112
 ISSN 1392-6292 print, ISSN 1648-3510 online

OPTIMAL DIFFERENCE SCHEMES ON PIECEWISE-UNIFORM MESHES FOR A SINGULARLY PERTURBED PARABOLIC CONVECTION–DIFFUSION EQUATION

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Received October 8, 2007; revised November 29, 2007; published online February 15, 2008

Abstract. A grid approximation of a boundary value problem is considered for a singularly perturbed parabolic convection–diffusion equation. For this problem, upwind difference schemes on the well–known piecewise-uniform meshes converge ε-uniformly in the maximum discrete norm at the rate $O(N^{-1}\ln N + N_0^{-1})$, where $N+1$ and N_0+1 are the number of mesh points in x and t respectively; the number of nodes in the x-mesh before the transition point (the point where the step-size changes) and after it are the same. Under the condition $N \approx N_0$ this scheme converges at the rate $O(P^{-1/2}\ln P)$; here $P=(N+1)(N_0+1)$ is the total number of nodes in the piecewise-uniform mesh. Schemes on piecewise-uniform meshes are constructed that are optimal with respect to the convergence rate. These schemes converge ε-uniformly at the rate $O(P^{-1/2}\ln^{1/2} P)$. In optimal meshes based on widths that are similar to Kolmogorov's widths, the ratio of mesh points in x and t is of $O((\varepsilon+\ln^{-1} P)^{-1})$. Under the condition $\varepsilon = o(1)$, most nodes in such a mesh in x are placed before the transition point.

Key words: boundary value problem, perturbation parameter ε, parabolic convection–diffusion equation, finite difference approximation, optimal meshes, boundary layer, Kolmogorov's widths, ε-uniform convergence.

1 Introduction

At present, methods for the construction of ε-uniformly convergent difference schemes on special piecewise-uniform meshes that condense in boundary layers are sufficiently well developed; e.g., see [2, 4, 9, 10, 12] for partial and [7] for ordinary differential equations. These methods are widespread because of their simplicity and convenience (e.g., see [4, 9, 10, 12] and the references therein). The well-known upwind finite difference scheme on a piecewise-uniform mesh converges ε-uniformly in the maximum discrete norm at the rate $O(N^{-1}\ln N + N_0^{-1})$, where $N+1$ and N_0+1 are the number of mesh points in x and t respectively (e.g., see [6, 12, 14]). But the convergence rate of such a scheme is, in general, not optimal with respect to P, where P is the total number of nodes in this piecewise-uniform mesh and $P = (N+1)(N_0+1)$.

Thus, it is of intrinsic interest to construct difference schemes on piecewise-uniform meshes that converge with an *optimal convergence rate*; from a theoretical and practical point of view, we shall investigate schemes that converge ε-uniformly in the maximum norm.

In the present paper, a finite difference approximation for a boundary value problem is considered for a singularly perturbed parabolic convection–diffusion equation. Using a technique of widths that are similar to Kolmogorov's widths (e.g., see [1, 3, 13]), an ε-uniformly convergent difference scheme on a piecewise-uniform mesh is constructed and studied. This scheme is optimal (for given P) with respect to their convergence rate. Also optimal difference schemes are considered that are constructed using known piecewise-uniform meshes that condense in the boundary layer.

2 Problem Formulation. The Aim of the Research

2.1. On the set $\overline{G}$

$$\overline{G} = G \cup S, \quad G = D \times (0,T], \tag{2.1}$$

where $D = (0,d)$, we consider the boundary value problem for the singularly perturbed parabolic convection–diffusion equation

$$\begin{aligned} L\,u(x,t) &= f(x,t), \quad (x,t) \in G, \\ u(x,t) &= \varphi(x,t), \quad (x,t) \in S. \end{aligned} \tag{2.2}$$

Here

$$L = \varepsilon\, a(x,t)\frac{\partial^2}{\partial x^2} + b(x,t)\frac{\partial}{\partial x} - c(x,t) - p(x,t)\frac{\partial}{\partial t}, \quad (x,t) \in G,$$

the functions $a(x,t)$, $b(x,t)$, $c(x,t)$, $p(x,t)$, $f(x,t)$ are assumed to be sufficiently smooth on the set $\overline{G}$, while $\varphi(x,t)$ is assumed to be sufficiently smooth on the smooth parts of S, moreover[1]

$$\begin{aligned} & a_0 \le a(x,t) \le a^0, \quad b_0 \le b(x,t) \le b^0, \quad 0 \le c(x,t) \le c^0, \\ & p_0 \le p(x,t) \le p^0, \quad (x,t) \in \overline{G}; \\ & |f(x,t)| \le M, \quad (x,t) \in \overline{G}; \quad |\varphi(x,t)| \le M, \quad (x,t) \in S; \end{aligned} \tag{2.3}$$

where a_0, b_0, $p_0 > 0$; the parameter ε takes arbitrary values in the opened-closed interval $(0,1]$.

For small values of ε, a regular boundary layer appears in a neighbourhood of the set $S_1^L = \{(x,t):\ x=0,\ 0 < t \le T\}$. Here S_1^L and S_2^L are the left and

[1] Here and below, M, M_i (or m) denote sufficiently large (small) positive constants that are independent of the parameter ε and of the discretization parameters.

right parts of the lateral boundary S^L; $S = S^L \cup S_0$, $S^L = S_1^L \cup S_2^L$, $S_0 = \overline{S}_0$ is the lower part of the boundary.

2.2. In the case of the boundary value problem (2.2), (2.1), we are interested in numerical methods whose solutions converge uniformly with respect to the parameter ε (or, briefly, converge ε-uniformly) in the maximum discrete norm. Our main interest is in approximations constructed on the set $\overline{G}_h$ based on grid solutions that converge ε-uniformly in the *maximum norm* to the solution of the boundary value problem. But in the case of singularly perturbed problems, the convergence of the discrete solution on $\overline{G}_h$ does not imply the convergence of its interpolants on $\overline{G}$. Thus, the ε-uniform convergence of the discrete solution $z(x,t)$ at the nodes of the grid $\overline{G}_h$ is, in general, inadequate for a description of the ε-uniform convergence for approximations constructed on the set $\overline{G}$. For example, the solution of the difference scheme constructed by the classical approximation of the boundary value problem (2.2), (2.1) on the uniform mesh $\overline{G}_{h(3.3)} = \overline{G}_h^{\,u}$, converges on the grid $\overline{G}_h$ as $\varepsilon^{-1}\, h \to \infty$ for $h \to 0$, when the characteristic width of the boundary layer defined by the parameter ε is much less than the step-size of the mesh in x. The simplest *interpolant*

$$\overline{z}(x,t) = \overline{z}\big(x,t;\ z(\cdot),\, \overline{G}_h\big), \qquad (x,t) \in \overline{G}, \tag{2.4}$$

i.e., the linear interpolant on triangular elements (partitions of elementary rectangles from $\overline{G}$, produced by nodes of the grid $\overline{G}_h$; it does not matter which triangulation one constructs from these mesh points) that is constructed from the grid function does not converge on $\overline{G}$ to the solution of the boundary value problem. Under the prescribed condition for the grid $\overline{G}_{h(3.3)}$, the interpolant $\overline{u}^h(x,t)\overline{z}_{(2.4)}\big(x,t;\ u^h(\cdot),\, \overline{G}_h\big)$, $(x,t) \in \overline{G}$, constructed from the grid function $u^h(x,t) = u(x,t)$, $(x,t) \in \overline{G}_h$, where $u(x,t)$ is the solution of the problem (2.2), (2.1), also does not converge on $\overline{G}$. Here and below, we consider a convergence in the maximum norm.

Let us give some definitions. In the case when the interpolant $\overline{z}_{(2.4)}(x,t)$, $(x,t) \in \overline{G}$, converges on $\overline{G}$, we say that the difference scheme *resolves* the boundary value problem (for some values of the parameter ε); otherwise, we say that the difference scheme does not resolve the boundary value problem. In the case when the interpolant $\overline{z}(x,t)$, $(x,t) \in \overline{G}$ converges on $\overline{G}$ ε-uniformly, we say that the difference scheme *resolves* the boundary value problem *ε-uniformly*.

We say that the solution of the difference scheme converges (or, briefly, the difference scheme converges), if the discrete solution converges on $\overline{G}_h$ and if the difference scheme resolves the boundary value problem. Here the convergence (*solvability*) of the difference scheme can be ε-uniform or be valid only under some restriction on the parameter.

We are interested in ε-uniformly convergent difference schemes on piecewise-uniform meshes that have *optimal convergence rate with respect to* P, where P is the total number of nodes in such a mesh. For brevity, we say that these schemes and the associated meshes are optimal.

2.3. Let us give the aim of this research. For the problem (2.2), (2.1), known difference schemes on piecewise-uniform meshes from [12] (see also [6, 14]) converge ε-uniformly (see the scheme (3.2), (3.7) and the estimate (3.8) in Section 3). These schemes are not optimal with respect to P.

Our aim for the boundary value problem (2.2), (2.1) is, in the class of difference schemes on piecewise-uniform meshes, to construct meshes that are optimal with respect to P and to study convergence on such meshes in the maximum norm. Thus, we shall consider optimal meshes that are constructed based on widths and known piecewise-uniform meshes.

2.4. For the following considerations, we need some estimates. We give estimates for the solution of the boundary value problem and its derivatives; the derivation of these estimates is similar to [5, 6, 14]. We write the solution of problem (2.2) as the decomposition

$$u(x,t) = U(x,t) + V(x,t), \qquad (x,t) \in \overline{G}, \tag{2.5}$$

where $U(x,t)$ and $V(x,t)$ are the regular and singular parts of the solution. The functions $U(x,t)$ and $V(x,t)$ satisfy the estimates

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} U(x,t) \right| \le M\big[1 + \varepsilon^{2-k}\big], \tag{2.6}$$

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} V(x,t) \right| \le M\varepsilon^{-k} \exp\big(-m\varepsilon^{-1} r(x,\Gamma_1)\big), \quad (x,t) \in \overline{G},$$

where $k+2\,k_0 \le 4$, $k \le 3$, and m is an arbitrary number in the interval $(0, m_0)$, $m_0 = \min_{\overline{G}} \big[a^{-1}(x,t)\, b(x,t)\big]$, and $r(x,\Gamma_1)$ is the distance from the point x to the left boundary Γ_1 of the set D.

Theorem 1. *Let the data of the boundary value problem* (2.2), (2.1) *satisfy condition* (2.3). *Let also the following conditions be fulfilled:*

$$a,\ ,b,\ ,c,\ p,\ f \in C^{6+\alpha}(\overline{G}), \quad \varphi \in C^{6+\alpha}(S), \quad \alpha > 0,$$

$$\varphi(x,t) = 0, \quad (x,t) \in S_0;$$

$$\frac{\partial^{k_0}}{\partial t^{k_0}}\,\varphi(x,t) = 0, \quad \frac{\partial^{k+k_0}}{\partial x^k\, \partial t^{k_0}}\, f(x,t) = 0, \quad (x,t) \in S^c,$$

where $k,\, k_0 \le 6$, $S^c = \overline{S}^L \bigcap S_0$. *Then the solution of the boundary value problem and its components in the representation* (2.5) *satisfy the estimates* (2.6).

3 Schemes on Uniform and Piecewise-Uniform Meshes

Let us consider monotone difference schemes.

3.1. On the set $\overline{G}$, we introduce the rectangular grid

$$\overline{G}_h = \overline{\omega} \times \overline{\omega}_0, \tag{3.1}$$

where $\overline{\omega}$ and $\overline{\omega}_0$ are, in general, arbitrary nonuniform meshes on the intervals $[0,d]$ and $[0,T]$ respectively. Let

$$h^i = x^{i+1} - x^i,\ \ x^i,\, x^{i+1} \in \overline{\omega},\ \ h = \max_i h^i,\ \ h_t^k = t^{k+1} - t^k,$$
$$t^k,\, t^{k+1} \in \overline{\omega}_0,\ \ h_t = \max_k h_t^k.$$

Assume that the conditions $h \le M\,N^{-1}$, $h_t \le M\,N_0^{-1}$ are satisfied, where $N+1$ and N_0+1 are the number of nodes in the meshes $\overline{\omega}$ and $\overline{\omega}_0$ respectively.

We approximate problem (2.2), (2.1) by the finite difference scheme [11]

$$\begin{aligned} \Lambda\, z(x,t) &= f(x,t), \qquad (x,t) \in G_h, \\ z(x,t) &= \varphi(x,t), \qquad (x,t) \in S_h. \end{aligned} \tag{3.2}$$

Here $G_h = G \cap \overline{G}_h$, $S_h = S \cap \overline{G}_h$,

$$\Lambda \equiv \varepsilon\, a(x,t)\, \delta_{\overline{x}\widehat{x}} + b(x,t)\, \delta_x - c(x,t) - p(x,t)\, \delta_{\overline{t}}, \qquad (x,t) \in G_h,$$

and $\delta_{\overline{x}\widehat{x}}\, z(x,t)$ is the second-order central difference derivative on the nonuniform mesh,

$$\delta_{\overline{x}\widehat{x}}\, z(x,t) = 2(h^i + h^{i-1})^{-1}[\delta_x\, z(x,t) - \delta_{\overline{x}}\, z(x,t)], \quad (x,t) = (x^i,t) \in G_h;$$

$\delta_x\, z(x,t)$ and $\delta_{\overline{x}}\, z(x,t)$ are the first-order forward and backward difference derivatives. The scheme (3.2), (3.1) is ε-uniformly monotone [11].

3.2. In the case when the grid is uniform in both variables:

$$\overline{G}_h = \overline{G}_h^{\,u} \overline{\omega} \times \overline{\omega}_0, \tag{3.3}$$

then using the maximum principle, we obtain the estimate

$$|u(x,t) - z(x,t)| \le M\left[\left(\varepsilon + N^{-1}\right)^{-1} N^{-1} + N_0^{-1}\right], \quad (x,t) \in \overline{G}_h; \tag{3.4}$$

which is unimprovable with respect to N, N_0, ε.

The interpolant $\overline{z}(x,t) = \overline{z}_{(2.4)}\big(x,t;\, z_{(3.2),\,(3.3)}(\cdot),\, \overline{G}_h^{\,u}\big)$ satisfies the estimate

$$|u(x,t) - \overline{z}(x,t)| \le M\left[\left(\varepsilon + N^{-1}\right)^{-1} N^{-1} + N_0^{-1}\right], \quad (x,t) \in \overline{G}. \tag{3.5}$$

The scheme (3.2), (3.3) converges under the condition $N^{-1} \ll \varepsilon$; more precisely,

$$\varepsilon^{-1} = o(N), \quad N \to \infty, \quad \varepsilon \in (0,1]. \tag{3.6}$$

3.3. Let us construct the ε-uniformly convergent scheme (see, e.g., [9, 12]). On the set $\overline{G}$, we introduce the grid

$$\overline{G}_h = \overline{G}_h^{\,s} \equiv \overline{\omega}^{\,s} \times \overline{\omega}_0, \tag{3.7a}$$

where $\overline{\omega}_0 = \overline{\omega}_{0(3.3)}$ and $\overline{\omega}^{\,s}$ is a piecewise uniform mesh constructed as follows. The interval $[0, d]$ is divided into two parts $[0, \sigma]$ and $[\sigma, d]$, where the step-sizes in each part are constant and respectively equal to $h^{(1)} = 2\,\sigma\,N^{-1}$ and $h^{(2)} = 2(d - \sigma)N^{-1}$. The parameter σ is defined by

$$\sigma = \sigma(\varepsilon,\, N) = \min\left[2^{-1}\,d,\; m^{-1}\,\varepsilon\,\ln N\right], \tag{3.7b}$$

where m is an arbitrary number in the interval $(0, m_0)$, and $m_0 = m_{0(2.6)}$.

The parameter σ in the grid (3.7) of the scheme (3.2), (3.7) is chosen to satisfy the condition that the singular component of the discrete solution converges ε-uniformly in the σ-neighbourhood of the set S_1^L and its majorant converges to zero ε-uniformly outside that neighbourhood [12]. Thus, the difference scheme (3.2), (3.7) is the scheme on the piecewise-uniform grid that is *apriori* adapted with respect to an indicator, i.e., the grid boundary layer and its majorant (in a neighbourhood of the boundary layer and outside it, respectively).

We now deduce the error bound for the solution of the difference scheme (3.2), (3.7). Write the solution of the difference scheme as the decomposition

$$z(x,\, t) = z_U(x,\, t) + z_V(x,\, t), \quad (x,\, t) \in \overline{G}_h,$$

that corresponds to (2.5). Here $z_U(x,\, t)$ and $z_V(x,\, t)$ are the regular and singular components of the discrete solution, and they are solutions of the problems

$$\begin{cases} \Lambda\, z_U(x,\, t) = f(x,\, t), & (x,\, t) \in G_h, \\ z_U(x,\, t) = U(x,\, t), & (x,\, t) \in S_h; \end{cases} \qquad \begin{cases} \Lambda\, z_V(x,\, t) = 0, & (x,\, t) \in G_h, \\ z_V(x,\, t) = V(x,\, t), & (x,\, t) \in S_h, \end{cases}$$

where $U(x,t)$ and $V(x,t)$ are the components in the representation (2.5). Taking into account the estimates of derivatives of the function $U(x,\, t)$, we find

$$|U(x,\, t) - z_U(x,\, t)| \le M\left[N^{-1} + N_0^{-1}\right], \quad (x,\, t) \in \overline{G}_h.$$

Estimating $V(x,\, t) - z_V(x,\, t)$ outside a neighbourhood of the set $\overline{S}_1^{\,L}$, we take into account that the functions $V(x,\, t)$ and $z_V(x,\, t)$ decay towards zero as (x, t) moves away from the set $\overline{S}_1^{\,L}$. As a majorant for $V(x,\, t)$ and $z_V(x,\, t)$, we use the function $v(x)$, $x \in \overline{\omega}^s$, that is the solution of the problem

$$\begin{aligned} \Lambda\, v(x_1) \equiv \{\varepsilon\,\delta_{\overline{x}\widehat{x}} + m_1\,\delta_x\}\, v(x) &= 0, \quad x \in \omega^{\,se}, \\ v(x) &= 1, \quad x = 0, \end{aligned}$$

where the function $v(x)$ tends to zero as $x \to \infty$, and m_1 is an arbitrary number in the interval $(0,\, m_0)$ for $m_0 = m_{0(2.6)}$. The mesh $\overline{\omega}_1^{se}$ on the set $[0,\, \infty)$ is the extension of the mesh $\overline{\omega}^{s}_{1(3.7)}$ for $x > d$; the step-size in $\overline{\omega}_1^{se}$ for $x \ge \sigma$ is $h_{(2)(3.7)}$.

With these estimates for the functions $V(x,\, t)$ and $z_V(x,\, t)$ for $x \ge \sigma$, we estimate $V(x,\, t) - z_V(x,\, t)$ for $x \le \sigma$. For $x \le \sigma$ the grid $\overline{G}_h$ is uniform with step-size in x equal to $h_{(1)(3.7)}$.

For the component $z_V(x,\, t)$, we obtain the estimate

$$|V(x,\, t) - z_V(x,\, t)| \le M \left[N^{-1} \ln N + N_0^{-1}\right], \quad (x,\, t) \in \overline{G}_h.$$

Thus, for the solution of the scheme (3.2), (3.7), we obtain the ε-dependent estimate

$$|u(x,t) - z(x,t)| \le M \,\left[(\varepsilon + \ln^{-1} N)^{-1}\, N^{-1} + N_0^{-1}\right], \quad (x,t) \in \overline{G}_h \qquad (3.8a)$$

and the ε-uniform estimate

$$|u(x,t) - z(x,t)| \le M \,\left[N^{-1} \ln N + N_0^{-1}\right], \quad (x,t) \in \overline{G}_h, \qquad (3.8b)$$

which are unimprovable with respect to N, N_0, ε and N, N_0 respectively.

The interpolant $\overline{z}(x,t) = \overline{z}_{(2.4)}\big(x,t;\, z_{(3.2),\,(3.7)}(\cdot),\, \overline{G}_h^{\,s}\big)$ satisfies the estimate

$$|u(x,t) - \overline{z}(x,t)| \le M \left[N^{-1} \ln N + N_0^{-1}\right], \quad (x,t) \in \overline{G}. \qquad (3.9)$$

Theorem 2. *Let the components in the representation* (2.5) *of the solution* $u(x,t)$ *of the boundary value problem* (2.2), (2.1) *satisfy the estimates of Theorem 1. Then the difference scheme* (3.2), (3.7) *converges* ε*-uniformly, while the scheme* (3.2), (3.3) *converges under the condition* (3.6). *The discrete solutions and their interpolants satisfy the respective estimates* (3.4), (3.8) *and* (3.5), (3.9).

4 Optimal Approximations of Problem Solutions

Consider approximations of solutions of the problem (2.2), (2.1), using an analog of Kolmogorov's widths (e.g., see [1, 3]) and the references therein).

4.1. We are interested in approximations of the set $\mathcal{U}$ of solutions of the class of the boundary value problems (2.2), (2.1) (defined by the conditions (2.3)) in the space X of continuous functions on $\overline{G}$ with the discrete maximum norm. We assume that the solutions are sufficiently smooth on $\overline{G}$ for fixed values of the parameter ε and also the solutions and their components in the representation (2.5) satisfy the estimates (2.6) from Section 2.

Let $\overline{G}^{\,h}$ be the set of points (we say the grid) on $\overline{G}$. The number of nodes in the grid $\overline{G}^{\,h}$ on $\overline{G}$ is denoted by P and $\overline{G}^{\,h} = \overline{G}^{\,h}(P)$. Let T_P be a triangulation (partition) of the set $\overline{G}$, generated by $\overline{G}^{\,h}$ (e.g., see [8]); assume that the nodes of $\overline{G}^{\,h}$ are the vertices of triangular elements are formed by the straight line segments that pass through the nodes of $\overline{G}^{\,h}$.

Let $\rho_1(T_P^j)$ and $\rho_2(T_P^j)$, where T_P^j is a triangular element in the partition T_P, be the radii of the inscribed and circumscribed circles for the element T_P^j, $j = 1, \dots, J$ and $J = J(P)$ is the number of triangular elements in the partition T_P (assume that the condition $J \approx P$ holds). The value $\eta(T_P^j) = \rho_1^{-1}(T_P^j)\, \rho_2(T_P^j)$ is the *anisotropy* coefficient for the element T_P^j. Let a discrete function $u^h(x,t)$, $(x,t) \in \overline{G}^{\,h}$ be defined on the set $\overline{G}^{\,h}$ and we denote by

$\overline{u}^{\,h}(x,t),\ (x,t)\in\overline{G}$ its linear interpolant constructed from the values of $u^h(x,t)$ at the vertices of the triangular elements. The set of the interpolants for the fixed triangulation T_P is denoted by U_P^h. For a fixed number of nodes P in $\overline{G}$, the set of all admissible grid sets $\overline{G}^{\,h}$ and of triangulations T_P based on them will be denoted by $\mathcal{T}_P$ (we say that $\mathcal{T}_P$ is *the set of partitions* of the domain $\overline{G}$). The set of partitions $\mathcal{T}_P$ and the set of interpolants U_P^h (for each triangulation in $\mathcal{T}_P$) define *the space* X. We define the width $d_P^*(\mathcal{U},X)$ (*the optimal width*) by the relation

$$d_P^*(\mathcal{U},\,X)=\inf_{\mathcal{T}_P}\ \sup_{u\in\mathcal{U}}\ \inf_{\overline{u}^{\,h}\in U_P^h}\ ||\,u-\overline{u}^{\,h}\,||, \tag{4.1}$$

$||\cdot||$ is the norm in C. A definition of Kolmogorov's width can be found, for example, in [1], Chapter 3. The quantity $d_P^*(\mathcal{U},\,X)$ is interpreted as the error of the optimal approximation of the set $\mathcal{U}$ in the space X (on a grid with P nodes) or, briefly, the error of the optimal approximation.

Let

$$d_P^{1*}\Big(\mathcal{U},\,X^\wedge\Big) \tag{4.2a}$$

be the width (4.1), where the set $\overline{G}^{\,h}$ is a rectangular grid $\overline{G}_h^\wedge$ in $\overline{G}$:

$$\overline{G}_h^\wedge=\overline{\omega}_1^\wedge\times\overline{\omega}_0^\wedge \tag{4.2b}$$

with an arbitrary distribution of nodes in the meshes $\overline{\omega}_1^\wedge$ and $\overline{\omega}_0^\wedge$ in x and t respectively.

4.2. Consider the width $d_{P(4.2)}^{1*}$ on a class of grids $\overline{G}_{h(4.2)}^\wedge$ which are piecewise uniform in x:

$$\overline{G}_h^\wedge=\overline{G}_h^{\wedge S}\equiv\overline{\omega}_1^{\wedge S}\times\overline{\omega}_0^\wedge, \tag{4.3}$$

where $\overline{\omega}_1^{\wedge S}$ is a piecewise uniform mesh, i.e., uniform on the sets $[0,\,\sigma^\wedge]$ and $[\sigma^\wedge,\,d]$. In that case we obtain the (unimprovable with respect to P, ε) estimate

$$d_P^{1*}\big(\mathcal{U},\,X^\wedge;\,\overline{G}_{h(4.3)}^\wedge\big)\le\begin{cases} M\ P^{-1}\,\ln P & \textit{for}\quad \varepsilon\ln P\le m_1,\\ M\,\varepsilon^{-1}\,P^{-1} & \textit{for}\quad \varepsilon\ln P> m_1,\end{cases} \tag{4.4a}$$

and also the ε–uniform (unimprovable) estimate

$$d_P^{1*}\big(\mathcal{U},\,X^\wedge;\,\overline{G}_{h(4.3)}^\wedge\big)\le M\ P^{-1}\,\ln P. \tag{4.4b}$$

This means that the width converges ε–uniformly.

The parameters of the grid (4.3) related to the width $d_P^{1*}\big(\mathcal{U},\,X^\wedge;\,\overline{G}_{h(4.3)}^\wedge\big)$ (which we call the *optimal piecewise-uniform grid for the width*) satisfy the estimates

$$N^\wedge\le\begin{cases} M\ P^{1/2}\,\ln^{1/2}P & \textit{for}\quad \varepsilon\ln P\le m_1,\\ M\,P^{1/2}\,\varepsilon^{-1/2} & \textit{for}\quad \varepsilon\ln P> m_1;\end{cases} \tag{4.5a}$$

$$N_0^\wedge\le\begin{cases} M\ P^{1/2}\,\ln^{-1/2}P & \textit{for}\quad \varepsilon\ln P\le m_1,\\ M\,P^{1/2}\,\varepsilon^{1/2} & \textit{for}\quad \varepsilon\ln P> m_1;\end{cases}$$

$$\eta(P,\,\varepsilon)\le M\,\varepsilon^{-1},$$

where $N^\wedge + 1$ and $N_0^\wedge + 1$ are the number of nodes in the meshes $\overline{\omega}_1^{\wedge S}$ and $\overline{\omega}_0^\wedge$ respectively and $P = (N^\wedge + 1)(N_0^\wedge + 1)$. The estimates (4.5a) are unimprovable with respect to P and ε. Thus, for the optimal grid, the anisotropy of the grid is of $O(\varepsilon^{-1})$ for $\varepsilon = o(1)$.

The value $N_\sigma^\wedge$, where $N_\sigma^\wedge + 1$ is the number of nodes in the set $[\sigma^\wedge, d]$, satisfies the estimate

$$N_\sigma^\wedge \le \begin{cases} M\ P^{1/2}\ln^{-1/2} P & \text{for} \quad \varepsilon \ln P \le m_1, \\ M\,P^{1/2}\,\varepsilon^{1/2} & \text{for} \quad \varepsilon \ln P > m_1; \end{cases} \tag{4.5b}$$

moreover,

$$m^1\,\varepsilon \ln P \le \sigma^\wedge = \sigma^\wedge(P,\,\varepsilon) \le M^1\,\varepsilon \ln P \quad \textit{for} \quad \varepsilon \ln P \le m_1. \tag{4.5c}$$

The estimate (4.5a) is unimprovable with respect to P and ε. For $\varepsilon \ln P \ge M_1$, the value σ satisfies the relation $\sigma = d$.

Under condition (4.5c), we have

$$N_\sigma^\wedge (N^\wedge)^{-1} \approx \ln^{-1} P. \tag{4.5d}$$

Thus, in the optimal piecewise-uniform grid for the width, most of nodes in the mesh in x are placed before the transition point, i.e., in the part of the mesh with the smallest step-size. It is quite obvious that one would get a similar result for an ordinary differential equation.

4.3. For the width d_P^{1*} in the case of the grids

$$\overline{G}_h^\wedge = \overline{G}_h^{\wedge u} \equiv \overline{\omega}_1^{\wedge u} \times \overline{\omega}_0^\wedge, \tag{4.6}$$

where $\overline{\omega}_1^{\wedge u}$ is the uniform mesh and $\overline{\omega}_0^\wedge \overline{\omega}_{0(4.2)}^\wedge$, we obtain the estimate

$$d_P^{1*}\big(\mathcal{U},\,X^\wedge;\,\overline{G}_{h(4.6)}^\wedge\big) \le M\,P^{-1}\,(\varepsilon + P^{-1})^{-1}. \tag{4.7}$$

The parameters of the optimal grid satisfy the estimates

$$N^\wedge \le M\,P^{1/2}\,(\varepsilon + P^{-1})^{-1/2}, \quad N_0^\wedge \le M\,P^{1/2}\,(\varepsilon + P^{-1})^{1/2}, \tag{4.8}$$
$$\eta \le M\,(\varepsilon + P^{-1})^{-1}.$$

The estimates (4.7), (4.8) are unimprovable.

The width d_P^{1*} on the grid (4.6) converges under the (unimprovable) condition $P^{-1} \ll \varepsilon$; more precisely,

$$P^{-1} = o(\varepsilon), \quad P \to \infty, \quad \varepsilon \in (0,\,1]. \tag{4.9}$$

Theorem 3. *Let the components in the representation* (2.5) *of the solution to the boundary value problem* (2.2), (2.1) *satisfy the estimates* (2.6) *for* $K = 2$. *The width* $d_P^{1*}(\mathcal{U},\,X^\wedge)$ *on the grid* (4.3) *converges* ε*-uniformly, while on the grid* (4.6) *it converges under the condition* (4.9). *The widths* $d_P^{1*}(\mathcal{U},\,X^\wedge)$ *on the grids* (4.3) *and* (4.6) *satisfy the estimates* (4.4) *and* (4.7) *respectively; the parameters of the optimal grids for the widths satisfy the estimates* (4.5) *and* (4.8).

5 Schemes on Piecewise-Uniform Grids with Improved Convergence

5.1. Let us consider the difference scheme (3.2) on a grid that corresponds to the optimal grid (4.5) for the width (4.2), (4.3). On the set $\overline{G}$, we construct a grid which is piecewise uniform in x:

$$\overline{G}_h = \overline{G}_h^S \equiv \overline{\omega}_1^S \times \overline{\omega}_0, \tag{5.1a}$$

where $\overline{\omega}_1^S$ and $\overline{\omega}_0$ are piecewise uniform and uniform meshes respectively; $N+1$ and N_0+1 are the number of nodes in the meshes $\overline{\omega}_1^S$ and $\overline{\omega}_0$ respectively, moreover, $(N+1)(N_0+1) = P$. The values $N = N(P, \varepsilon)$ and $N_0 = N_0(P, \varepsilon)$ are defined by the relations (see (4.5a)):

$$N \le \begin{cases} M\,P^{1/2}\,\ln^{1/2} P & \text{for} \quad \varepsilon\,\ln P \le m_1, \\ M\,P^{1/2}\,\varepsilon^{-1/2} & \text{for} \quad \varepsilon\,\ln P > m_1; \end{cases} \tag{5.1b}$$

$$N_0 \le \begin{cases} M\,P^{1/2}\,\ln^{-1/2} P & \text{for} \quad \varepsilon\,\ln P \le m_1, \\ M\,P^{1/2}\,\varepsilon^{1/2} & \text{for} \quad \varepsilon\,\ln P > m_1; \end{cases}$$

these relations are unimprovable with respect to P and ε, while the constant m_1 is defined below. The mesh step-sizes of $\overline{\omega}_1^S$ are constant on the intervals $[0, \sigma]$ and $[\sigma, d]$, and are respectively

$$h_{(1)} = \sigma\,N_{(1)}^{-1}, \quad h_{(2)} = (d-\sigma)\,N_{(2)}^{-1},$$

where $N_{(1)}+1$ and $N_{(2)}+1$ are the number of mesh points on $[0, \sigma]$ and $[\sigma, d]$, and $\sigma \in (0, d]$; $h_{(2)} \approx (N_0)^{-1}$, moreover, $N_{(1)} = N$ for $\sigma = d$. Here $\eta(P, \varepsilon) \le M\,\varepsilon^{-1}$. By virtue of (4.5d), we have $N_{(2)}\,N^{-1} \le M\,\ln^{-1} P$, i.e., most nodes in the mesh in x are placed before the transition point, similarly to the grid (4.3).

The value σ satisfies the condition

$$\sigma = \sigma(\varepsilon,\, P)\min\big[d,\; M_1\,\varepsilon\,\ln P\big], \tag{5.1c}$$

where $M_1 = 2^{-1}\,m_{(2.6)}^{-1}$. Set

$$m_1 = M_1^{-1}\,d; \tag{5.1d}$$

in particular, in (5.1b) one can have $M = 1$. The grid $\overline{G}_h^S$ has now been constructed.

In the grid (5.1), the transition point in the mesh $\overline{\omega}_1^S$ coincides with the right endpoint of the interval $[0, d]$ under the condition $\varepsilon\,\ln P \ge m_1$. In that case, the grid $\overline{G}_h^S$ is uniform. The grid (5.1) is the piecewise uniform grid where the optimal rate (with respect to P) of convergence to the difference scheme (3.2) on piecewise uniform meshes is achieved up to a constant multiplier (it is optimal with respect to the order). This corresponds to the optimal grid (4.3) for the width (4.5). We call the grid (5.1) optimal with respect to the convergence rate of the scheme (3.2) on piecewise-uniform meshes that are constructed based on widths (or, briefly, the optimal piecewise-uniform grid based on widths).

For the solution of the difference scheme (3.2) on the grid (5.1), we have the unimprovable ε-dependent estimate

$$|u(x,t) - z(x,t)| \le M \left\{ \begin{array}{ll} P^{-1/2} \ln^{1/2} P & \textit{for} \quad \varepsilon \ln P \le m_1 \\ P^{-1/2} \varepsilon^{-1/2} & \textit{for} \quad \varepsilon \ln P > m_1 \end{array} \right\} \tag{5.2a}$$

$$\le M\, P^{-1/2} (\varepsilon + \ln^{-1} P)^{-1/2}, \quad (x,t) \in \overline{G}_h,$$

and also the unimprovable ε-uniform estimate

$$|u(x,t) - z(x,t)| \le M P^{-1/2} \ln^{1/2} P, \quad (x,t) \in \overline{G}_h. \tag{5.2b}$$

Theorem 4. *Let the condition of Theorem 2 be satisfied. Then the solution of the difference scheme* (3.2)*,* (5.1) *satisfies the estimate* (5.2)*.*

5.2. Consider the difference scheme on improved piecewise-uniform grids that are constructed based on the grids $\overline{G}^s_{h(3.7)}$.

5.2.1. For the solution of the difference scheme (3.2), (3.7) we have the estimate (3.8). Under the condition

$$N \approx N_0, \quad \textit{i.e.,} \quad N\, N_0 \approx P^{1/2}, \tag{5.3}$$

which is natural for regular problems on the grid (3.7), we obtain the unimprovable ε-dependent estimate

$$|u(x,t) - z(x,t)| \le M\, P^{-1/2} (\varepsilon + \ln^{-1} P)^{-1}, \quad (x,t) \in \overline{G}_h, \tag{5.4a}$$

and also the unimprovable ε-uniform estimate

$$|u(x,t) - z(x,t)| \le M\, P^{-1/2} \ln P, \quad (x,t) \in \overline{G}_h. \tag{5.4b}$$

In the case of the grid (3.7), (5.3), we have

$$\eta(P,\, \varepsilon) \le M\, \varepsilon^{-1} (\varepsilon + \ln^{-1} P).$$

We say that the grid (3.7), (5.3) is *standard piecewise uniform.*

5.2.2. Optimizing the estimate (3.8) for the fixed value P, we find quantities N, N_0 that define the grid (3.7):

$$\begin{array}{lllll} N & = & N\,(P,\, \varepsilon) & \le & M\, P^{1/2} (\varepsilon + \ln^{-1} P)^{-1/2}, \\ N_0 & = & N_0(P,\, \varepsilon) & \le & M\, P^{1/2} (\varepsilon + \ln^{-1} P)^{1/2}. \end{array} \tag{5.5}$$

Here $P = (N+1)(N_0+1)$ and $\eta(P,\, \varepsilon) \le M\, \varepsilon^{-1}$.

For the solution of the difference scheme (3.2), (3.7), (5.5), we obtain the unimprovable ε-dependent estimate

$$|u(x,t) - z(x,t)| \le M\, P^{-1/2}\, (\varepsilon + \ln^{-1} P)^{-1/2}, \quad (x,t) \in \overline{G}_h, \tag{5.6a}$$

and also the ε-uniform estimate

$$|u(x,t) - z(x,t)| \le M\, P^{-1/2}\, \ln^{1/2} P, \quad (x,t) \in \overline{G}_h. \tag{5.6b}$$

We say that the grid (3.7), (5.5) is the optimal grid constructed based on the piecewise-uniform grids $\overline{G}^{s}_{h(3.7)}$ (or, the optimal grid based on piecewise-uniform grids).

5.2.3. The values N and N_0 in the optimal grids (5.1) and (3.7), (5.5) depend on P and ε. For the scheme (3.2), we consider the piecewise-uniform grid in which N and N_0which define the grid (3.7), are independent of ε, and they are chosen to satisfy the optimality condition with respect to P (with respect to the ε–uniform convergence rate of this scheme):

$$N = N(P) \approx P^{1/2}\, \ln^{1/2} P, \quad N_0 = N_0(P) \approx P^{1/2}\, \ln^{-1/2} P. \tag{5.7}$$

In this case we have $\eta(P,\, \varepsilon) \le M\, \varepsilon^{-1}\, (\varepsilon + \ln^{-1} P)\, \ln P$.

For the solution of the difference scheme (3.2), (3.7), (5.7), we have the unimprovable (both ε-dependent and ε-uniform) estimate

$$|u(x,t) - z(x,t)| \le M\, P^{-1/2}\, \ln^{1/2} P, \quad (x,t) \in \overline{G}_h, \tag{5.8}$$

i.e., this scheme is not optimal. We say that the grid (3.7), (5.7) is an improved grid (compare with the standard one) constructed based on the piecewise-uniform grids $\overline{G}^{s}_{h(3.7)}$ under the condition (5.7) (or, briefly, an improved piecewise-uniform grid).

Theorem 5. *Let the condition of Theorem 2 be satisfied. Then the solution of the difference scheme* (3.2) *on the grid* (3.7), (5.3) *(on the grids* (3.7), (5.5) *and* (3.7), (5.7)*) satisfies the estimate* (5.4) *(respectively, the estimates* (5.6) *and* (5.8)*).*

Remark 1. If the conditions of Theorem 2 are fulfilled, then interpolants, constructed from the solutions of the schemes under consideration, will satisfy the estimates of Theorems 4, 5, where $z(x,t)$ and $\overline{G}_h$ are $\overline{z}(x,t)$ and $\overline{G}$ respectively.

6 Conclusion

From comparison of estimates (5.2) and (5.4) it follows that the scheme (3.2), (5.1) related to the optimal piecewise-uniform grid based on widths has best convergence (under the condition $\varepsilon + \ln^{-1} P = o(1)$) than the scheme (3.2), (3.7), (5.3) related to the standard piecewise-uniform grid under the condition (5.3). By virtue of estimates (5.2) and (5.6), the scheme (3.2), (3.7), (5.5) related to the optimal grid based on piecewise-uniform meshes, has the same ε-uniform convergence rate as the scheme (3.2), (5.1). The scheme (3.2), (3.7), (5.7) related to the improved piecewise-uniform grid, has the same ε-uniform convergence rate as the scheme (3.2), (5.1), but under the condition $\varepsilon + \ln^{-1} P = o(1)$ the scheme (3.2), (3.7), (5.7) has a lower convergence rate than the scheme (3.2), (5.1).

Acknowledgment

The author is grateful to the participants of the 12th International Conference MMA2007 (Trakai, Lithuania, 2007), and to Prof. Martin Stynes for interesting and helpful discussions.

This research was supported by the Russian Foundation for Basic Research under grants Nos. 07–01–00729, the Boole Centre for Research in Informatics at the National University of Ireland in Cork, and by the Mathematics Applications Consortium for Science and Industry in Ireland (MACSI) under the Science Foundation Ireland (SFI) Mathematics Initiative.

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