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DIFFERENCE SEQUENCE SPACE $m(M, \phi, \Delta_m^n, p)^F$ OF FUZZY REAL NUMBERS*

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Abstract. The difference sequence space $m(M, \phi, \Delta_m^n, p)^F$ of fuzzy real numbers for both $1 \le p < \infty$ and $0 < p < 1$, is introduced. Some properties of this sequence space like solidness, symmetricity, convergence-free are studied. Some inclusion relations involving this sequence space are obtained.

Key words: Orlicz function; symmetric space; solid space; convergence-free; metric space; completeness.

1 Introduction

The concept of fuzzy set theory was introduced by Zadeh [18]. Later on sequences of fuzzy numbers have been discussed by Tripathy and Nanda [17], Nuray and Savas [7], Kwon [5], Esi [1], Tripathy and Dutta [12], Et, Altin and Altinok [2] and many others.

Kizmaz [4] studied the notion of difference sequence spaces at the initial stage. He investigated the difference sequence spaces $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ of crisp sets. The notion is defined as follows, $Z(\Delta) = \{x = (x_k) : (\Delta x_k) \in Z\}$, for $Z = \ell_\infty$, c and c_0, where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$, for all $k \in N$.

The above spaces are Banach spaces, normed by, $\| x \|_\Delta = |x_1| + \sup_k |\Delta_k|$.

The idea of Kizmaz [4] was applied to introduce different type of difference sequence spaces and studied their different properties by Tripathy [11], Tripathy and Esi [13] and many others.

Tripathy and Esi [12] introduced the new type of difference sequence spaces, for fixed $m \in N$, $Z(\Delta_m) = \{x = (x_k) : (\Delta_m x_k) \in Z\}$, for $Z = \ell_\infty, c$ and c_0 where $\Delta_m x = (\Delta_m x_k) = (x_k - x_{k+m})$, for all $k \in N$. This generalizes the notion of difference sequence spaces studied by Kizmaz [4].

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The above spaces are Banach spaces, normed by,

$$\| x \|_{\Delta} = \sum_{r=1}^{m} |x_r| + \sup_k |\Delta_m x_k|.$$

Triapthy, Esi and Triapthy [14] further generalized this notion and introduced the following notion. For $m, n \geq 1$, $Z(\Delta_m^n) = \{x = (x_k) : (\Delta_m^n x_k) \in Z\}$ for $Z = \ell_\infty, c$ and c_0. This generalized difference has the following binomial representation,

$$\Delta_m^n x_k = \sum_{r=0}^{n} (-1)^r \binom{n}{r} x_{k+rm}. \tag{1.1}$$

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$ for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$. If the convexity of the Orlicz function is replaced by sub-additivity, then this function is called a modulus function.

Remark 1. An Orlicz function satisfies the inequality $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Sargent [9] introduced the crisp set sequence space $m(\phi)$ and studied some properties of this space. Later on it was studied from the sequence space point of view and some matrix classes were characterized with one member as $m(\phi)$ by Rath and Tripathy [8], Tripathy [10], Tripathy and Sen [16] and others. In this article we introduce the space $m(M, \phi, \Delta_m^n, p)^F$ of sequences of fuzzy real numbers defined by Orlicz function.

Throughout the article $w^F, \ell^F, \ell_\infty^F$ represent the classes of *all, absolutely summable and bounded* sequences of fuzzy real numbers, respectively.

2 Definitions and Background

A fuzzy real number X is a fuzzy set on R, i.e. a mapping $X : R \to I(= [0, 1])$ associating each real number t with its grade of membership $X(t)$.

A fuzzy real number X is called *convex* if

$$X(t) \geq X(s) \wedge X(r) = \min\big(X(s), X(r)\big),$$

where $s < t < r$. If there exists $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number X is called *normal*.

A fuzzy real number X is said to be *upper semi continuous* if for each $\varepsilon > 0$ and for all $a \in I$ the mapping $X^{-1}([0, a + \varepsilon))$ is open in the usual topology of R. The class of all upper semi continuous, normal, convex fuzzy real numbers is denoted by $R(I)$.

For $X \in R(I)$, the α-level set X^α for $0 < \alpha \leq 1$ is defined by $X^\alpha = \{t \in R : X(t) \geq \alpha\}$. The 0-level, i.e. the set X^0, is the closure of strong 0-cut, thus we have that $\overline{\{t \in R : X(t) > 0\}}$ is compact.

The absolute value of $X \in R(I)$, i.e. $|X|$, is defined as (see, Kaleva and Seikkala [3])

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\}, & \text{for } t \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

For $r \in R, \overline{r} \in R(I)$ is defined as,

$$\overline{r}(t) = \begin{cases} 1, & \text{for } t = r, \\ 0, & \text{otherwise.} \end{cases}$$

The additive identity and multiplicative identity of $R(I)$ are denoted by $\overline{0}$ and $\overline{1}$ respectively. The zero sequence of fuzzy real numbers is denoted by $\overline{\theta}$.

Let D be the set of all closed bounded intervals $X = [X^L, X^R]$. Define $d : D \times D \to R$ by $d(X, Y) = \max\{|X^L - Y^L|, |X^R - Y^R|\}$. Then (D, d) is a complete metric space.

Define $\overline{d} : R(I) \times R(I) \to R$ by $\overline{d}(X, Y) = \sup_{0<\alpha\leq 1} d(X^\alpha, Y^\alpha)$, for $X, Y \in R(I)$. Then it is well known that $(R(I), \overline{d})$ is a complete metric space.

A sequence $X = (X_k)$ of fuzzy real numbers is said to be convergent to the fuzzy number X_0, if for every $\varepsilon > 0$, there exists $k_0 \in N$ such that $\overline{d}(X_k, X_0) < \varepsilon$, for all $k \geq k_0$. A sequence space E is said to be *solid* if $(Y_n) \in E$, whenever $(X_n) \in E$ and $|Y_n| \leq |X_n|$, for all $n \in N$.

Let $X = (X_n)$ be a sequence, then $S(X)$ denotes the set of all permutations of the elements of (X_n) i.e. $S(X) = \{(X_{\pi(n)}) : \pi \text{ is a permutation of } N\}$. A sequence space E is said to be *symmetric* if $S(X) \subset E$ for all $X \in E$.

A sequence space E is said to be *convergence-free* if $(Y_n) \in E$ whenever $(X_n) \in E$ and $X_n = \overline{0}$ implies $Y_n = \overline{0}$.

A sequence space E is said to be *monotone* if E contains the canonical pre-images of all its step spaces.

Lemma 1. *A sequence space E is monotone whenever it is solid.*

Let $\wp_s$ be the class of all subsets of N those do not contain more than s number of elements. Throughout $\{\phi_s\}$ is a non-decreasing sequence of positive real numbers such that $n\phi_{n+1} \leq (n + 1)\phi_n$ for all $n \in N$.

The space $m(\phi)$ introduced by Sargent [9] is defined as,

$$m(\phi) = \Big\{(x_k) \in w : \| x \|_{m(\phi)} = \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} |x_k| < \infty\Big\}.$$

Lindenstrauss and Tzafriri [6] used the notion of Orlicz function and introduced the sequence space:

$$\ell_M = \Big\{(x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \quad \text{for some } \rho > 0\Big\}.$$

The space ℓ_M becomes a Banach space with the norm defined by

$$\| (x_k) \| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\Big\},$$

which is called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p, which is an Orlicz sequence space with $M(x) = x^p$, for $1 \leq p < \infty$.

In the later stage different classes of Orlicz sequence spaces were introduced and studied by Esi [1], Tripathy and Mahanta [15] and many others.

In this article we introduce the following difference sequence space:

$$m(M,\phi,\Delta_m^n,p)^F=\Big\{X=(X_k):\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\Big(M\Big(\frac{\overline{d}(\Delta_m^n X_k,\overline{0})}{\rho}\Big)\Big)^p<\infty,\ \text{for some } \rho>0\Big\},\quad \text{for } 0<p<\infty.$$

3 Main Results

In this section we prove some results involving all these sequence spaces.

Theorem 1. (a) *The sequence space $m(M,\phi,\Delta_m^n,p)^F$ is a complete metric space with the metric*

$$\eta(X,Y)=\sum_{r=1}^{mn}\overline{d}(X_r,Y_r)+\inf\Big\{\rho>0\colon \sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\Big(M\Big(\frac{\overline{d}(\Delta_m^n X_k,\Delta_m^n Y_k)}{\rho}\Big)\Big)^p\leq 1\Big\},$$

for $X,Y\in m(M,\phi,\Delta_m^n,p)^F, m\geq 1, n\geq 1$ and $0<p<1$.

(b) *The sequence space $m(M,\phi,\Delta_m^n,p)^F$ is a complete metric space with the metric*

$$\eta(X,Y)=\sum_{r=1}^{mn}\overline{d}(X_r,Y_r)+\inf\Big\{\rho>0:\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\times\Big(\sum_{k\in\sigma}\Big(M\Big(\frac{\overline{d}(\Delta_m^n X_k,\Delta_m^n Y_k)}{\rho}\Big)\Big)^p\Big)^{\frac{1}{p}}\leq 1\Big\},$$

for $X,Y\in m(M,\phi,\Delta_m^n,p)^F, m\geq 1, n\geq 1$ and $1\leq p<\infty$.

Proof. **(a)**. Clearly, $m(M,\phi,\Delta_m^n,p)^F$ is a metric space with the metric η, defined above. We have to prove that it is a complete metric space. Let $(X^{(i)})$ be a Cauchy sequence in $m(M,\phi,\Delta_m^n,p)^F$ such that $X^{(i)}=(X_n^{(i)})_{n=1}^{\infty}$. Let $\varepsilon>0$ be given. For a fixed $x_0>0$, choose $r>0$ such that $M(\frac{rx_0}{2})\geq 1$. Then there exists a positive integer $n_0=n_0(\varepsilon)$ such that $\eta(X^{(i)},X^{(j)})<\varepsilon/(rx_0)$, for all $i,j\geq n_0$. By the definition of η, we get:

$$\sum_{r=1}^{mn}\overline{d}(X_r^{(i)},X_r^{(j)})+\inf\Big\{\rho>0:\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\times\sum_{k\in\sigma}\Big(M\Big(\frac{\overline{d}(\Delta_m^n X_k^{(i)},\Delta_m^n X_k^{(j)})}{\rho}\Big)\Big)^p\leq 1\Big\}<\varepsilon,\quad \text{for all } i,j\geq n_0,\quad (3.1)$$

which implies that, $\sum\limits_{r=1}^{mn}\overline{d}(X_r^{(i)},X_r^{(j)})<\varepsilon$, for all $i,j\geq n_0$ and finally we get

$$\overline{d}(X_r^{(i)},X_r^{(j)})<\varepsilon,\quad \text{for all } i,j\geq n_0,\ r=1,2,3,\ldots,mn.$$

Hence $(X_r^{(i)})$ is a Cauchy sequence in $R(I)$, so it is convergent in $R(I)$, by the completeness property of $R(I)$, for $r = 1,2,3,\dots mn$. Let,

$$\lim_{i\to\infty} X_r^{(i)} = X_r, \quad \text{for } r = 1,2,3,\dots mn. \tag{3.2}$$

$$\sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma}\Big(M\Big(\frac{\overline{d}(\Delta_m^n X_k^{(i)},\Delta_m^n X_k^{(j)})}{\rho}\Big)\Big)^p \leq 1, \quad \text{for all } i,j\geq n_0. \tag{3.3}$$

For $s = 1$ and σ varying over $\wp_s$, we get,

$$\sum_{k\in\sigma}\Big(M\Big(\frac{\overline{d}(\Delta_m^n X_k^{(i)},\Delta_m^n X_k^{(j)})}{\eta(X^{(i)},X^{(j)})}\Big)\Big)^p \leq \phi_1, \qquad \text{for all } i,j\geq n_0$$

$$\Rightarrow M\Big(\frac{\overline{d}(\Delta_m^n X_k^{(i)},\Delta_m^n X_k^{(j)})}{\eta(X^{(i)},X^{(j)})}\Big)\Big)^p \leq \phi_1^{\frac{1}{p}} \leq M\Big(\frac{rx_0}{2}\Big), \qquad \text{for all } i,j\geq n_0.$$

Using the continuity of M, we get,

$$\overline{d}\left(\Delta_m^n X_k^{(i)},\Delta_m^n X_k^{(j)}\right) \leq \frac{rx_0}{2}\eta\left(X^{(i)},X^{(j)}\right), \qquad \text{for all } i,j\geq n_0$$

$$\Rightarrow \overline{d}\left(\Delta_m^n X_k^{(i)},\Delta_m^n X_k^{(j)}\right) < \frac{rx_0}{2}\frac{\varepsilon}{rx_0} = \frac{\varepsilon}{2}, \qquad \text{for all } i,j\geq n_0,$$

which implies that $(\Delta_m^n X_k^{(i)})$ is a Cauchy sequence in $R(I)$ and so it is convergent in $R(I)$ by the completeness property of $R(I)$.

Let, $\lim_i \Delta_m^n X_k^{(i)} = Y_k \in R(I)$, for each $k\in N$. We have to prove that

$$\lim_i X^{(i)} = X \quad \text{and} \quad X\in m(M,\phi,\Delta_m^n,p)^F.$$

For $k = 1$, we get from (1.1) and (3.2) that

$$\lim_i X_{mn+1}^{(i)} = X_{mn+1}, \quad \text{for } m\geq 1, n\geq 1.$$

Hence we get that $\lim_i X_k^{(i)} = X_k$, for each $k\in N$. Also, $\lim_i \Delta_m^n X_k^{(i)} = \Delta_m^n X_k$, for each $k\in N$. Using the continuity of M, we get, from (3.3),

$$\sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma}\Big(M\Big(\frac{\overline{d}(\Delta_m^n X_k^{(i)},\Delta_m^n X_k)}{\rho}\Big)\Big)^p \leq 1,$$

for some $\rho > 0$ and $i\geq n_0$. Now on taking the infimum of such ρ's and using (3.1), we get

$$\inf\Big\{\rho>0: \sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma}\Big(M\Big(\frac{\overline{d}(\Delta_m^n X_k^{(i)},\Delta_m^n X_k)}{\rho}\Big)\Big)^p \leq 1\Big\} < \varepsilon,$$

for all $i\geq n_0$. Hence we get,

$$\sum_{r=1}^{mn}\overline{d}(X_r^{(i)},X_r) + \inf\Big\{\rho>0: \sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{\phi_s} \times \sum_{k\in\sigma}\Big(M\Big(\frac{\overline{d}(\Delta_m^n X_k^{(i)},\Delta_m^n X_k)}{\rho}\Big)\Big)^p \leq 1\Big\} < \varepsilon+\varepsilon = 2\varepsilon,$$

for all $i \geq n_0$, which implies that,

$$\eta(X^{(i)}, X) < 2\varepsilon, \quad \forall i \geq n_0, \quad \Rightarrow \quad \lim_i X^{(i)} = X.$$

Now, we shall prove that $X \in m(M, \phi, \Delta_m^n, p)^F$. We know that,

$$\eta(X, \overline{\theta}) \leq \eta(X^{(n)}, X) + \eta(X^{(n)}, \overline{\theta}) < \varepsilon + M, \text{ for all } n \geq n_0(\varepsilon),$$

i.e. $\eta(X, \overline{\theta})$ is finite, which implies that, $X \in m(M, \phi, \Delta_m^n, p)^F$. Hence the space $m(M, \phi, \Delta_m^n, p)^F$ is a complete metric space. This completes the proof of (a) part of the theorem. The (b) part can be proved by following similar techniques. □

Theorem 2. *The sequence space $m(M, \phi, \Delta_m^n, p)^F$ is not solid in general, for $0 < p < \infty$.*

Proof. The result follows from the following example.

Example 1. Let $m = 3,\ n = 2,\ p = 2$. Let $X_k = \overline{k}$, for all $k \in N$ and $\phi_s = s$ for all $s \in N$. Let $M(x) = |x|$, for all $x \in [0, \infty)$. Then, we have, $\overline{d}(\Delta_3^2 X_k, \overline{0}) = 0$, for all $k \in N$. Hence, we have,

$$\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{s} \sum_{k \in \sigma} \Big(M\Big(\frac{\overline{d}(\Delta_3^2 X_k, \overline{0})}{\rho} \Big) \Big)^2 < \infty, \text{ for some } \rho > 0,$$

which implies that, $(X_k) \in m(M, s, \Delta_3^2, 2)^F$. Consider the sequence (α_k) of scalars defined by,

$$\alpha_k = \begin{cases} 1, & \text{for } k \text{ is even,} \\ 0, & \text{otherwise} \end{cases} \quad \Rightarrow \quad \alpha_k X_k = \begin{cases} \overline{k}, & \text{for } k \text{ is even,} \\ \overline{0}, & \text{otherwise,} \end{cases}$$

which implies that,

$$\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{s} \sum_{k \in \sigma} \Big(M\Big(\frac{\overline{d}(\Delta_3^2 \alpha_k X_k, \overline{0})}{\rho} \Big) \Big)^2 = \infty, \text{ for any fixed } \rho > 0,$$

which shows that, $(\alpha_k X_k) \notin m(M, s, \Delta_3^2, 2)^F$. Hence $m(M, \phi, \Delta_m^n, p)^F$ is not solid in general, for $0 < p < \infty$.
□

Theorem 3. *The sequence space $m(M, \phi, \Delta_m^n, p)^F$ is not symmetric in general, for $0 < p < \infty$.*

Proof. Let $m = 1,\ n = 1,\ p = \dfrac{1}{2}$ and $M(x) = x^2$, for all $x \in [0, \infty)$. Let $\phi_s = s$, for all $s \in N$. Let $X_k = \overline{k}$, for all $k \in N$. Then, $\overline{d}(\Delta X_k, \overline{0}) = 1$, for all $k \in N$. Hence $(X_k) \in m(M, s, \Delta, \frac{1}{2})^F$. Let (Y_k) be a rearrangement of (X_k) such that,

$$(Y_k) = (X_1, X_2, X_4, X_3, X_9, X_5, X_{16}, X_6, X_{25} \ldots).$$

Then $\overline{d}(\Delta Y_k,\overline{0}) \approx k-(k-1)^2 \approx k^2$, for all $k \in N$, which shows that,

$$\sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{s}\sum_{k\in\sigma}\left(M\left(\frac{\overline{d}(\Delta Y_k,\overline{0})}{\rho}\right)\right)^{\frac{1}{2}} = \infty,$$

for some $\rho > 0$. Hence, $(Y_k) \notin m\left(M,s,\Delta,\frac{1}{2}\right)^F$. Thus, $m(M,\phi,\Delta_m^n,p)^F$ is not symmetric in general, for $0 < p < \infty$. □

Proposition 1. *The sequence space* $m(M,\phi,\Delta_m^n,p)^F$ *is not convergence-free in general.*

Proof. Let $m = 4, n = 1, p = \frac{1}{2}$. Let $M(x) = x^4$, for all $x \in [0,\infty)$. Let $\phi_s = s$, for all $s \in N$. Consider the sequence (X_k) defined as follows:

$$X_k(t) = \begin{cases} 1+kt & \text{for} \quad t\in[-1/k,0],\\ 1-kt & \text{for} \quad t\in[0,1/k],\\ 0 & \text{otherwise.}\end{cases}$$

Then,

$$\Delta_4 X_k(t) = \begin{cases} 1+\dfrac{k(k+4)}{2k+4}t & \text{for} \quad t\in[-\dfrac{2k+4}{k(k+4)},0],\\ 1-\dfrac{k(k+4)}{2k+4}t & \text{for} \quad t\in[0,\dfrac{2k+4}{k(k+4)}],\\ 0 & \text{otherwise.}\end{cases}$$

Thus we have that, $\overline{d}\left(\Delta_4 X_k,\overline{0}\right) = \dfrac{2k+4}{k(k+4)} = \dfrac{2}{(k+1)} + \dfrac{4}{k(k+4)}$. Then it follows that

$$\sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{s}\sum_{k\in\sigma}\Big(M\Big(\frac{\overline{d}(\Delta_4 X_k,\overline{0})}{\rho}\Big)\Big)^{\frac{1}{2}} < \infty, \quad \forall\rho>0.$$

Thus, $(X_k) \in m\left(M,s,\Delta_4,\frac{1}{2}\right)^F$. Now, let us take another sequence (Y_k) such that,

$$Y_k(t) = \begin{cases} 1+t/k^2 & \text{for} \quad t\in[-k^2,0],\\ 1-t/k^2 & \text{for} \quad t\in[0,k^2].\end{cases}$$

So that,

$$\Delta_4 Y_k(t) = \begin{cases} 1+\dfrac{t}{2k^2+8k+16} & \text{for} \quad t\in[-(2k^2+8k+16),0],\\ 1-\dfrac{t}{2k^2+8k+16} & \text{for} \quad t\in[0,(2k^2+8k+16)],\\ 0 & \text{otherwise.}\end{cases}$$

But, $\overline{d}\left(\Delta_4 Y_k,\overline{0}\right) = (2k^2+8k+16)$, for all $k \in N$, which implies that,

$$\sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{s}\sum_{k\in\sigma}\left(M\left(\frac{\overline{d}(\Delta_4 Y_k,\overline{0})}{\rho}\right)\right)^{1/2} = \infty,$$

for some $\rho > 0$. Thus, $(Y_k) \notin m(M, s, \Delta_4, \frac{1}{2})^F$. Hence $m(M, \phi, \Delta_m^n, p)^F$ is not convergence-free, in general. □

Proposition 2. *$m(M, \phi, \Delta_m^n)^F \subseteqq m(M, \phi, \Delta_m^n, p)^F$, for all $1 \leq p < \infty$.*

Proof. Let $X \in m(M, \phi, \Delta_m^n)^F$, then we have,

$$\sup_{s\geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M\left(\frac{\overline{d}(\Delta_m^n X_k, \overline{0})}{\rho}\right) = K < \infty,$$

for any fixed $\rho > 0$. Hence, for each fixed s and $\sigma \in \wp_s$, we have, for $\rho > 0$:

$$\sum_{k \in \sigma} M\left(\frac{\overline{d}(\Delta_m^n X_k, \overline{0})}{\rho}\right) \leq K\phi_s \quad \Rightarrow \quad \left[\sum_{k \in \sigma} \left\{M\left(\frac{\overline{d}(\Delta_m^n X_k, \overline{0})}{\rho}\right)\right\}^p\right]^{\frac{1}{p}} \leq K\phi_s,$$

$$\sup_{s\geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \left[\sum_{k \in \sigma} \left\{M\left(\frac{\overline{d}(\Delta_m^n X_k, \overline{0})}{\rho}\right)\right\}^p\right]^{\frac{1}{p}} \leq K < \infty,$$

which implies that, $X \in m(M, \phi, \Delta_m^n, p)^F$, for $1 \leq p < \infty$. This completes the proof. □

Proposition 3. *$m(M, \phi, \Delta_m^n, p)^F \subseteq m(M, \psi, \Delta_m^n, p)^F$, if and only if*

$$\sup_{s \geq 1} \left(\frac{\phi_s}{\psi_s}\right) < \infty, \quad \textit{for } 0 < p < \infty.$$

Proof. First, suppose that $\sup_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right) = K < \infty$, then we have, $\phi_s \leq K\psi_s$. Now, if $(X_k) \in m(M, \phi, \Delta_m^n, p)^F$, then

$$\sup_{s\geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} \left\{M\left(\frac{\overline{d}(\Delta_m^n X_k, \overline{0})}{\rho}\right)\right\}^p < \infty,$$

$$\Rightarrow \underset{s \geq 1, \sigma \in \wp_s}{sup} \frac{1}{K\psi_s} \sum_{k \in \sigma} \left\{M\left(\frac{\overline{d}(\Delta_m^n X_k, \overline{0})}{\rho}\right)\right\}^p < \infty,$$

i.e. $(X_k) \in m(M, \psi, \Delta_m^n, p)^F$. Hence, $m(M, \phi, \Delta_m^n, p)^F \subseteq m(M, \psi, \Delta_m^n, p)^F$.

Conversely, suppose that $m(M, \phi, \Delta_m^n, p)^F \subseteq m(M, \psi, \Delta_m^n, p)^F$. We should prove that $\sup_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right) = \sup_{s\geq 1}(\eta_s) < \infty$. Suppose that $\sup_{s\geq 1}(\eta_s) = \infty$. Then there exists a subsequence (η_{s_i}) of (η_s) such that, $\lim_{i\to\infty}(\eta_{s_i}) = \infty$. Then for $(X_k) \in m(M, \phi, \Delta_m^n, p)^F$, we have,

$$\sup_{s\geq 1, \sigma \in \wp_s} \frac{1}{\psi_s} \sum_{k \in \sigma} \left\{M\left(\frac{\overline{d}(\Delta_m^n X_k, \overline{0})}{\rho}\right)\right\}^p$$

$$\geq \sup_{s\geq 1, \sigma \in \wp_s} \left(\frac{\eta_{s_i}}{\phi_{s_i}}\right) \sum_{k \in \sigma} \left\{M\left(\frac{\overline{d}(\Delta_m^n X_k, \overline{0})}{\rho}\right)\right\}^p = \infty.$$

i.e.

$$\sup_{s\ge1,\sigma\in\wp_s}\frac{1}{\psi_s}\sum_{k\in\sigma}\left\{M\left(\frac{\overline{d}(\Delta_m^nX_k,\overline{0})}{\rho}\right)\right\}^p=\infty,$$

which implies that $(X_k)\notin m(M,\psi,\Delta_m^n,p)^F$, a contradiction. This completes the proof. □

Corollary 1. $m(M,\phi,\Delta_m^n,p)^F=m(M,\psi,\Delta_m^n,p)^F$, if and only if

$$\sup_{s\ge1}(\eta_s)<\infty \quad\text{and}\quad \sup_{s\ge1}(\eta_s^{-1})<\infty,$$

where $\eta_s=\phi_s/\psi_s$, for $0<p<\infty$.

Theorem 4. *$\ell_p(M,\Delta_m^n)^F\subseteq m(M,\phi,\Delta_m^n,p)^F\subseteq\ell_\infty(M,\Delta_m^n)^F$, for $1\le p<\infty$.*

Proof. By taking $M(x)=x^p$, for $1\le p<\infty$ and $\phi_n=1$, for all $n\in N$, we get that $m(M,\phi,\Delta_m^n,p)^F=\ell_p(M,\Delta_m^n)^F$. So, the first inclusion is proved. Next, suppose that, $(X_k)\in m(M,\phi,\Delta_m^n,p)^F$. This implies that,

$$\sup_{s\ge1,\sigma\in\wp_s}\frac{1}{\phi_s}\Big[\sum_{k\in\sigma}\Big\{M\Big(\frac{\overline{d}(\Delta_m^nX_k,\overline{0})}{\rho}\Big)\Big\}^p\Big]^{\frac{1}{p}}=K<\infty.$$

For $s=1$, $M\left(\frac{\overline{d}(\Delta_m^nX_k,\overline{0})}{\rho}\right)\le K\phi_1, k\in\sigma$, which implies that

$$\sup_{k\ge1}M\left(\frac{\overline{d}(\Delta_m^nX_k,\overline{0})}{\rho}\right)<\infty.$$

Thus we have that $X_k\in\ell_\infty(M,\Delta_m^n)^F$. This completes the proof. □

Putting $\psi_n=1$, for all $n\in N$, in Corollary 1, we get

Proposition 4. $m(M,\phi,\Delta_m^n,p)^F=\ell_p(M,\Delta_m^n)^F$ *if and only if*

$$\sup_{s\ge1}(\phi_s)<\infty,\quad \sup_{s\ge1}(\phi_s^{-1})<\infty.$$

Corollary 2. $m(M,\phi,\Delta_m^n,p)^F=\ell_p(M,\Delta_m^n)^F$ if $\lim_{s\to\infty}\left(\frac{\phi_s}{s}\right)>0$, for $0<p<\infty$.

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