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NUMERICAL INVESTIGATION OF THE FINITE SUPERELEMENT METHOD FOR THE 3D ELASTICITY PROBLEMS

M. GALANIN[1], S. LAZAREVA[2] and E. SAVENKOV[3]

[1,3]*Keldysh Institute of Applied Mathematics of RAS*
Russia, 125047, Moscow, Miusskaya sq., 4
[2]*Bauman Moscow State Technical University*
Russia, 105005, Moscow, 2nd Baumanskaya st., 5
E-mail: galan@keldysh.ru; lazarevas@gmail.com; savenkov@keldysh.ru

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Abstract. The results of numerical investigation of the Finite Superelement Method (FSEM) for the solution of 3D elasticity problems are given. A definition of FSEM is proposed, and the general theory is briefly explained. Then the variants of FSEM are considered for the model problem. Their comparative analysis is being carried out. These variants are based on the finite element interpolation techniques on superelements boundaries. FSEM and FEM efficiency comparison is presented for the model problem. Quantative error data are obtained. A certain example of a 3D elasticity problem is considered in conclusion. A notable advantage of a higher degree FSEM approximation technique is illustrated.

Key words: Finite superelement method, special decomposition, 3D elasticity problem

1. Introduction

Finite superelement method (FSEM) was suggested by R. Fedorenko and L. Strakhovskaya in [3]. It has been investigated in our papers [4, 5, 7, 8]. This paper presents a continuation of the investigation. FSEM is considered as a numerical method the main purpose of which is to solve problems with local sharp domain singularities.

There exists a wide range of problems, the solution of which is characterized with sharp and local singularities. They reveal themselves as small ones compared with the entire domain. The complexity of these problems is in the fact that these local singularities are small not only with respect to the entire domain itself, but even to a suitable mesh step. Some possible solutions of this problem are known. One can use an adaptive mesh. Alternative approach is to use a special approximation on a coarse mesh. Finite superelement method is a special approximation technique on a coarse (superelement) mesh. Each domain singularity can be resolved independently inside one of the subdomains – superelements.

FSEM considers decomposition of the computational domain into a number of superelements. It has to be noted, that all singularities are situated inside superelements, but not at their boundaries. The trace of the solution at superelements boundaries is smooth enough. According to the FSEM, numerical solution is searched within a linear combinations of a special basis. These basis functions are the solutions of the given differential equation inside superelements. They are finite, the support of them is in connection with superelements. A variety of solution interpolators can be used on superelements boundaries. In this paper different variants of FSEM are investigated, they are based on different solution interpolation techniques on the boundaries.

This work presents numerical investigation of FSEM variants for the solution of 3D problems of the elasticity theory. Different FSEM approximations require different interpolators on superelements boundaries. Here we use conventional finite element approaches to construct them.

Our objectives are the following. We carry out a comparative analysis of FSEM approximations to the 3D model problem solution. Numerical solution errors are obtained. Then the comparison of the FSEM and FEM efficiency is given. Finally, we present the computational results of a test problem of the elasticity theory. Thus, the practical efficiency of the method is illustrated. When choosing the correct way of constructing the FSEM calculation scheme, the investigated method is of high efficiency. Higher degree approximation leads to more accurate numerical results.

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2. FSEM Theoretical Background

This chapter introduces a brief explanation of the theory of FSEM. For more details, see works [3, 4, 5, 7, 8]. Let Ω be some bounded domain, $\partial\Omega$ denotes its boundary. Let A be some linear elliptic operator on Ω.

The Green formula and the Poincaré-Steklov operator are the basis of the FSEM theoretical background. The Green formula is well known and has the following structure:

$$(\mathrm{A}u, v)_{L_2(\Omega)} = \mathrm{a}_{L_2(\Omega)}(u, v) - \langle \delta u, \gamma v \rangle_{L_2(\partial\Omega)},$$

where $\mathrm{a}_{L_2(\Omega)}(\cdot,\cdot)$ denotes some bilinear form corresponding to operator A. Operator γ is a trace operator on $\partial\Omega$, and δ denotes the normal derivative operator in a general sense. Particular form of operators γ, δ and bilinear form $a_{L_2(\Omega)}(\cdot,\cdot)$ depends on the form of A.

Poincaré-Steklov operator maps some function, given on the domain boundary, to another one specified at this boundary. It is calculated in the following way:

$$\mathrm{P}\varphi := \delta u = \delta \mathrm{G}\varphi,$$

where u is the weak solution of a problem investigated, G denotes the Green operator, and the equality $\gamma u = \varphi$ is fulfilled at any point on $\partial\Omega$. The result is that Poincaré-Steklov operator maps Dirichlet boundary condition to the corresponding Neumann boundary condition.

Now we consider the following problem:

$$\begin{cases} \mathrm{A}u = f \text{ in } \Omega, \\ u = g \text{ on } \partial\Omega_\gamma, \quad \delta u = \mu \text{ on } \partial\Omega_\delta, \end{cases} \tag{2.1}$$

where $\partial\Omega = \partial\Omega_\gamma \cup \partial\Omega_\delta$ and $\partial\Omega_\gamma \cap \partial\Omega_\delta = \emptyset$, and unknown $u \in V$.

Consider non-overlapping domain Ω decomposition into a number of subdomains-superelements Ω_k with boundaries $\partial\Omega_k$. In the sequel operators $\mathrm{P_k}$, $\mathrm{G_k}$, γ_k,... are operators P, G, γ,... for subdomain Ω_k. Furthermore we use the notation v_k for the restriction of an arbitrary function v at the region Ω_k.

The Green formula and Poincaré-Steklov operator allow us to exclude the problem solution inside superelements and to consider only boundary values. On this basis the initial problem can be reduced to the problem for boundary traces of superelements (see [4, 5, 7]).

Let $\varphi_k = \gamma_\mathrm{k} w_k$ be a trace of some function $w \in V$ on $\partial\Omega_k$, which is defined in Ω and is sufficiently smooth. Consider also, the function $\tilde{w}$ is defined in the following way

$$\gamma_\mathrm{k}\tilde{\omega}_k = \mathrm{G_k}\varphi_k + u_{f,k} \tag{2.2}$$

in every superelement Ω_k. Here $u_{f,k}$ is a solution of the following problem:

$$\begin{cases} \mathrm{A}u_{f,k} = f \text{ in } \Omega_k, \\ u_{f,k} = 0 \text{ at } \partial\Omega_k. \end{cases}$$

Let us suppose that $\tilde{w} \in V$ and it is continued across superelements boundaries due to the concrete form of the function φ.

Note, that $\tilde{w}$ is an exact solution of equation under consideration in every superelement w_k separately, but it doesn't satisfy the original problem (2.1) as a whole. To be the solution of this problem in Ω it has to fit some additional conditions across superelements boundaries. These conditions can be derived as follows. Function $\tilde{w}$ is the solution of the problem, if it satisfies a conventional weak equation for the solution u. Thus, recovering that (2.2) holds, one can obtain (see [4, 5, 7] for more details):

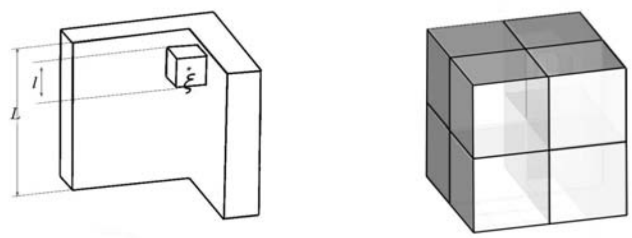
$$\begin{cases} \sum\limits_k \langle \mathrm{P_k}\varphi_k, \psi_k \rangle_{L_2(\partial\Omega_k)} = \sum\limits_k \langle \delta_\mathrm{k} u_{f,k}, \gamma_\mathrm{k} v_k \rangle_{L_2(\partial\Omega_k)} + \langle \mu, \psi \rangle_{L_2(\partial\Omega)}, \quad \forall \psi \in V_{\Gamma,0}, \\ \gamma\varphi = g \text{ at } \partial\Omega_\gamma. \end{cases} \tag{2.3}$$

The following spaces are proposed here:

$$V_\Gamma = \{\varphi = \{\varphi_k\} : \exists w \in V : \varphi_k = \gamma_k w_k\},$$
$$V_{\Gamma,0} = \{\varphi = \{\varphi_k\} : \exists w \in V : \varphi_k = \gamma_k w_k, \text{ and } \gamma w = 0 \text{ at } \partial\Omega\}.$$

Here the unknown function $\varphi \in V_\Gamma$. Problem (2.3) is the required problem for superelements boundary traces. There are various FSEM calculation schemes. We consider precise definitions of these variants in the example of Section 3.

3. FSEM for the 3D Model Problem of the Elasticity Theory



a) computational domain Ω b) decomposition of the domain Ω into $K^3 = 8$ superelements

Figure 1. Computational domain for the model problem.

3.1. A statement of the model problem

We consider the 3D model problem of the elasticity theory. Computational domain Ω is a cube having a cubic hole. Hole side size l is much smaller than the domain side size L. The center of the hole is at some $\boldsymbol{\xi}$ point (Fig. 1). Dirichlet boundary condition is given through the whole boundary including the boundary of the hole. We choose it in such a way that the exact solution of the model problem is known. It coincides with the restriction of the problem solution in R^3 when placing a point-source of body forces at $\boldsymbol{\xi}$. The following problem is under consideration:

$$\begin{cases} \mathrm{A}\boldsymbol{u} = (\lambda + 2\mu)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} - \mu\,\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{u} = 0, \quad \forall \boldsymbol{x} \in \Omega, \\ u_i(\boldsymbol{x}) = \psi(\boldsymbol{x}) = \dfrac{\rho}{4\pi\,(\lambda + 2\mu)} \dfrac{x_i - \xi_i}{r^3}, \quad \forall \boldsymbol{x} \in \partial\Omega, \quad i = \overline{1,3}. \end{cases} \tag{3.1}$$

Here λ, μ are Lame coefficients, ρ is a mass density, $\boldsymbol{u}$ is the unknown displacement field. Vector $\boldsymbol{x}$ defines point coordinates in the computational domain, and r is the distance from $\boldsymbol{\xi}$:

$$r = \left[(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2\right]^{1/2}.$$

Function ψ defines a boundary condition on $\partial\Omega$. The exact solution $\boldsymbol{u}^{(ex)}$ of the model problem has the form:

$$u_i^{(ex)}(\boldsymbol{x}) = \frac{\rho}{4\pi\,(\lambda + 2\mu)} \frac{x_i - \xi_i}{r^3} \quad \forall \boldsymbol{x} \in \Omega, \quad i = \overline{1,3}. \tag{3.2}$$

Thus, the problem singularity is situated near the hole.

3.2. FSEM variants for the model problem

The construction of variants of FSEM is based on different interpolation techniques on superelements boundaries. We use here the conventional finite element interpolators. Let us describe these variants for the model problem considered. Given domain Ω is decomposed into the $K \times K \times K$ cubic-shaped superelements Ω_k:

$$\Omega = \bigcup_{k=1}^{K^3} \Omega_k, \quad \Omega_k \cap \Omega_l = \emptyset, \quad \text{if } k \neq l.$$

We assume that the hole is in the center of one of the superelement (Fig. 1).

Problem solution is approximated independently inside each superelement. Moreover, the approximating basis functions are the same for different superelements. We can consider the FSEM approximation in one separate superelement. The others are similar.

Let us assume that superelement Ω_k is fixed. Displacement field $\boldsymbol{u}$ has three components. Each component u_j can be approximated independently. It is determined as a linear combination of functions $\Phi_i^{(j)}$, $j = 1, 2, 3$:

$$u_j(\boldsymbol{x}) = \sum_{i=0}^{n} a_i^{(j)} \Phi_i^{(j)}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega_k, \quad j = 1, 2, 3. \tag{3.3}$$

Here $\boldsymbol{a}_i = \left\{a_i^{(j)}\right\}_{j=1}^{3}$ are vectors of unknown expansion coefficients, $\Phi_i^{(j)}$ denotes j-th component of the i-th basis function, and $a_i^{(j)}$ – j-th component of the i-th coefficients vector. The expansion term number n defines the number of superelement degrees of freedom (regarding to the one component of the displacement field).

FSEM basis functions $\boldsymbol{\Phi}_i = \left\{\Phi_i^{(j)}\right\}_{j=1}^{3}$ are solutions of the considered system of equations (3.1) in superelement Ω_k (precisely $\boldsymbol{\Phi}_i$ are vector-functions). These functions are certain solution interpolators $\boldsymbol{\phi}_i$ at superelement boundaries. Thus, we deal with the following problem:

$$\begin{cases} \mathrm{A}\boldsymbol{\Phi}_i(\boldsymbol{x}) = 0 \quad \forall \boldsymbol{x} \in \Omega_k / \partial\Omega_k, \\ \boldsymbol{\Phi}_i \boldsymbol{x}\,|_{\boldsymbol{x} \in \partial\Omega_k} = \boldsymbol{\phi}_i(\boldsymbol{x})\,|_{\boldsymbol{x} \in \partial\Omega_k}. \end{cases} \tag{3.4}$$

Further we use the notation *boundary basis functions* for the functions $\boldsymbol{\phi}_i$.

Next we choose a set of boundary basis functions $\boldsymbol{\phi}_i$. They are specified at superelement boundaries only and serve as traces of the approximating solution. We choose finite element basis functions on triangles for this purpose. An accountable number of nodes $\boldsymbol{P}_i$, $i = \overline{1, n}$, is placed on the external boundary of the superelement (not at the hole boundary). They include superelement apexes and some additional nodes, if needed. Boundary basis functions possess the following values at these nodes:

$$\varphi^{(j)}_{3i+k}(\boldsymbol{P}_m) = \delta_{im}\delta_{kj}, \quad k = \overline{1,3}, \quad j = \overline{1,3}, \quad i = \overline{0,n}, \quad m = \overline{0,n},$$

where $\varphi^{(j)}_i$ is j-th component of the i-th boundary basis function $\boldsymbol{\phi}_i$, δ_{kj} is the Kronecker delta, and symbol $\boldsymbol{P}_0$ denotes not a node, but the entire boundary of the hole. These functions are continued to the external superelement boundary with the use of one of the conventional finite element approaches (see [1, 13, 16, 17]). The following six interpolations are tested: lagrangian linear, lagrangian quadratic, lagrangian cubic, lagrangian "reduced" cubic, hermitian cubic and hermitian "reduced" cubic basis functions [1, 4, 13, 16, 17].

Let N be the total number of superelements nodes throughout the entire domain Ω. The problem singularity is taken into account by force of "basis" function $\boldsymbol{\Phi}_0$ having a zero subscript. We have to underline its description. Function $\boldsymbol{\Phi}_0$ coincides with the boundary condition at the hole boundary. And it turns to zero at superelement external boundary. If a superelement doesn't contain a hole, then $\boldsymbol{\Phi}_0 \equiv 0$. In addition we note, that the other basis functions $\boldsymbol{\Phi}_i$, with $i = \overline{1, n}$, turn to zero at the boundary of a hole since no nodes are situated there.

According to the FSEM, the unknown coefficients $\boldsymbol{a}_i$ in expansion (3.3) can be found with the help of a standard Bubnov-Galerkin scheme. We choose $\boldsymbol{\Phi}_i$ as both basis functions and trial functions. Then we get the following system of linear algebraic equations:

$$\mathrm{a}\left(\boldsymbol{u}, \boldsymbol{\Phi}_i\right) = 0, \quad i = \overline{1, 3n},$$

or in the coordinate form:

$$\sum_{i=1}^{n} \sum_{m=1}^{3} a^{(m)}_i \mathrm{a}\left(\Phi^{(m)}_i, \Phi^{(l)}_j\right) = -\sum_{m=1}^{3} \mathrm{a}\left(\Phi^{(m)}_0, \Phi^{(l)}_j\right), \quad j = \overline{1, 3n}, \quad l = \overline{1, 3},$$

where $\mathrm{a}\left(\cdot, \cdot\right)$ is the bilinear form of the problem (3.1).

3.3. Computational results

Let us consider in detail errors of the numerical solution obtained from the model problem. Here we assume

$$L = 2, \;\; l = \frac{4}{9}, \;\; \xi_i = \frac{3}{2}, \;\; i = 1, 2, 3.$$

Material characteristics are the following:

$$\rho = 100, \quad \mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad E = 1, \quad \nu = 0.33.$$

We investigate two variants of domain decomposition. The first one is specified as $2 \times 2 \times 2$, so that $K^3 = 8$. In the second variant we have $5 \times 5 \times 5$ superelements, i.e. $K^3 = 125$. The number of the superelements is fixed and the variants of interpolation formula is changed (n has to be varied at the same time).

All FSEM basis functions (3.4) are computed approximately on a fine mesh inside a superelement. A standard finite element calculation with linear basis functions on tetrahedrons are used to perform this task.

The difference between the exact solution (3.2) and FSEM numerical solution is measured in $C(\Omega)$ space norm. Solution relative error under study is given by

$$\delta = \frac{\left\| u_r(\boldsymbol{x}) - u_r^{(ex)}(\boldsymbol{x}) \right\|}{\left\| u_r^{(ex)}(\boldsymbol{x}) \right\|} = \frac{\max\limits_{\boldsymbol{x}\in\Omega} \left| u_r(\boldsymbol{x}) - u_r^{(ex)}(\boldsymbol{x}) \right|}{\max\limits_{\boldsymbol{x}\in\Omega} \left| u_r^{(ex)}(\boldsymbol{x}) \right|},$$

where u_r is the displacement field radial component: $u_r^2 = \sum\limits_{i=1}^{3} {u_i}^2$

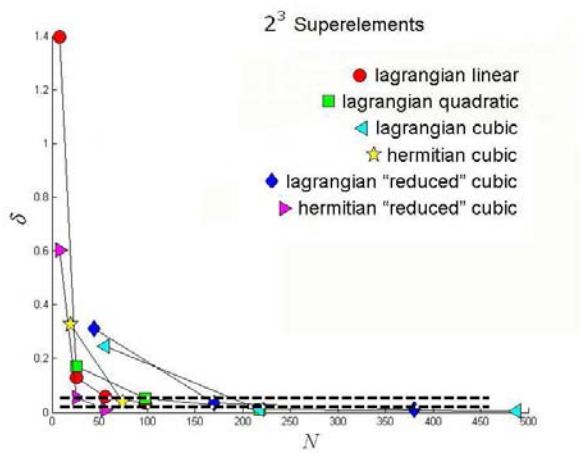


Figure 2. FSEM relative error dependence on the total number of superelement nodes N. FSEM and FEM accuracy comparison. $K^3 = 8$.

Fig. 2 and Fig. 3 show numerical solution error dependence on the FSEM approximation technique. Relative error dependence on the total number of superelement nodes is given. Different graphs present different approximation variants, which are based on various interpolators at superelement boundaries. Tab. 1 shows appropriate rates of convergence. A notable advantage of a higher degree interpolation techniques is evident, but a small coefficient of linear regression is not usual for them. It is caused by the initial closeness of

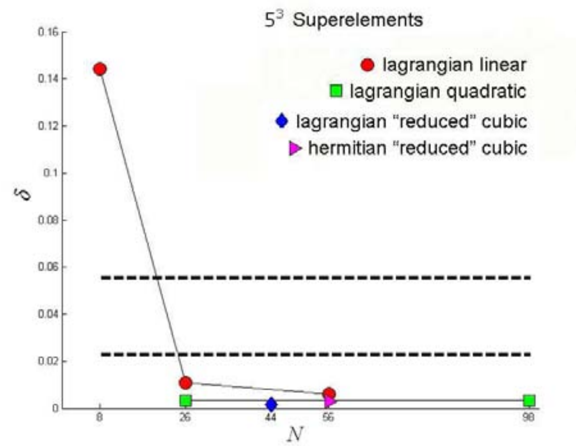


Figure 3. FSEM relative error dependence on the total number of superelement nodes N. FSEM and FEM accuracy comparison. $K^3 = 125$.

the error obtained (even on a few SE nodes) to the "unavoidable" error. Here the presence of the "unavoidable" error is linked to the influence of the basis functions approximate computation and is not the property of the FSEM. The value of error obtained confirms the efficiency of the method (see Tab. 2). One can easily see the notable advantage of higher degree interpolation.

Table 1. Convergence rates.

Interpolator	Convergence Rate
Lagrange linear	1.4745
Lagrange quadratic	1.2849
Lagrange cubic	1.8020
Lagrange "reduced" cubic	1.6357
Hermite cubic	1.6909
Hermite "reduced" cubic	2.0488

Table 2. FSEM and FEM numerical relative error values comparison.

K	FEM δ	Minimal FSEM δ
$K^3 = 8$	$5.5073 \cdot 10^{-2}$, when $h = 1/20$	$5,754 \cdot 10^{-3}$
$K^3 = 125$	$2.2338 \cdot 10^{-2}$, when $h = 1/36$	$1,493 \cdot 10^{-3}$

Our next step is the comparison of the accuracy of FSEM with the conventional method of finite elements (FEM). The model problem is solved with the FEM when finite element mesh sizes (step h): $h = 1/20$, $h = 1/36$, are available. Linear shape functions on tetrahedrons and uniform mesh is used. Numerical solution errors in $C(\Omega)$ space norm are defined. Judging from these results the obtained values are analyzed together with FSEM errors. As it has already been mentioned, Fig. 2 and Fig. 3 present the dependences of the FSEM relative errors on the total number of superelement nodes, when different interpolation techniques are applied. In addition to these dependences one can see the values of FEM relative errors that are marked with horizontal dotted lines. Exact numerical error values for the comparison are given in Tab. 2.

4. Test Problem Computational Results

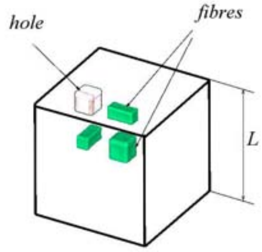
In this section we consider the computational results of a test 3D problem of elasticity theory. Dirichlet boundary conditions are given in this problem:

$$\begin{cases} \mathrm{A}\boldsymbol{u} = (\lambda + 2\mu)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} - \mu\,\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{u} = 0, \quad \forall \boldsymbol{x} \in \Omega, \\ u_i(\boldsymbol{x}) = 1, \text{ when } \boldsymbol{x} \text{ is at the boundary of the hole,} \\ u_i(\boldsymbol{x}) = 0, \text{ when } \boldsymbol{x} \text{ is at the external boundary of domain } \Omega,\ i = \overline{1,3}. \end{cases}$$

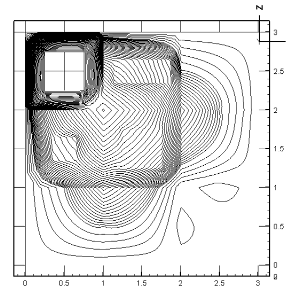
Given domain is elastic media with several fibres, having "brick" form. Fibres sizes are small enough in comparison with the entire domain (Fig. 4). Given domain is decomposed into 27 superelements. The problem is solved with the use of FSEM when lagrange linear, quadratic and "reduced" cubic interpolation techniques are applied at superelements boundaries. The figures below show FSEM solution level lines at a chosen section. Numerical solutions obtained with the help of quadratic and cubic interpolators are similar and physically correct. One can see a notable difference with the linear case.

5. The Results Discussion

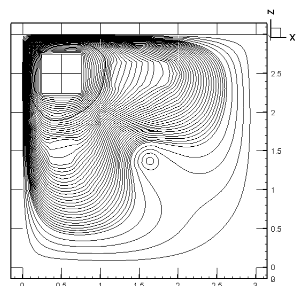
There exist a variety of methods, that are based on some boundary-type solution procedure. Many of them uses regular functions satisfying the governing equations. Such functions serve as a basis for the numerical solution expansion as it is in FSEM. Most famous of them don't propose any decomposition, for example: a sort of boundary element method or classical Trefftz method. The problem begins at the moment domain decomposition appears: indeed, compatibility conditions have to be satisfied at the boundaries (these boundaries are sometimes called "interface" by analogy with a boundary between different materials). The question is in what way the scheme must be constructed to achieve the best results for the given problem? This problem is not solved and it concerns the approximation of boundary equations, presenting the compatibility conditions at the interface. So, the boundary equations of different proposed methods can "match" in some sense, while their approximation scheme, either the solution interpolation technique, have to be rather different. Finite Superelement Method offers more qualitative results in comparison with other methods and by multiple characteristics. It should be noticed, that direct and



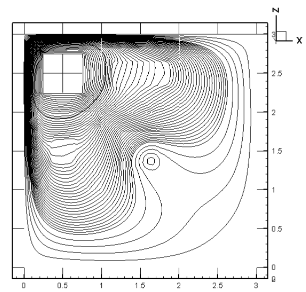
Computational domain Ω
for the test problem



Numerical solution u_r level lines
at the section $x_2 = 3/2$.
Lagrange linear interpolation



Numerical solution u_r level lines
at the section $x_2 = 3/2$.
Lagrange quadratic interpolation



Numerical solution u_r level lines
at the section $x_2 = 3/2$. Lagrange
"reduced" cubic interpolation

Figure 4. Computational results for the test problem solution.

exact quantative analysis of the results with the new works of authors from western countries hasn't been carried out. This task will be done carefully in the future investigations.

Let us discuss some results here. It means some empirical comparison by the example of selected works of different authors, having not only their origins but also a further continuation up to this day. We concentrate on those in the area of elasticity problems. We want to mention the class of methods, that is connected to the so-called "hybrid elements" and the "Trefftz elements". We refer to these approaches only due to the space shortage. This class of methods is of great importance in their origins, has a wide distribution nowadays and is close to the FSEM.

The origins of hybrid elements are close to conventional finite element method. If you want to resolve a sole domain singularity, you can try to construct one large element having some special characteristics near it, instead of using a great number of conventional ones. Some references can be found in the well-known monograph of O.Zienkiewicz ([16], ch.13). Usually, hybrid elements mean such elements, that the variational principles for them are modified for relaxed continuity conditions along elements boundaries. Only a particular cases of them use regular functions, governed by the homogeneous equation. The description of it can be found in [14]. The Trefftz method is a prevalent title for the methods, using solutions of the governing differential equation or its adjoint, defined in subregions. They often use the so-called "T-complete" basis functions firstly introduced by I. Herrera in [9, 10, 11]. A rather full information is given in his paper [12]. One of the most complete and newest description of Trefftz methods is presented in the work of Q.-H. Qin [15]. Besides, one can find a lot of papers devoted to this subject, e.g. works of J. Jirousek, N. Kamiya, E. Kita, T. H. H. Pian, R. Piltner, Q.-H. Qin, J.A. Teixeira de Freitas, A. P. Zielinski, and others.

In contrast to FSEM all basis functions in the referred methods have a simple and analytical structure, it is their advantage. Despite this fact, it is not so evident how to construct the specific (e.g., T-complete) basis, especially in 3D case. One can see much more 2D samples of computations, than 3D samples. Furthermore, one pose a problem when solving a nonhomegeneous equation, for instance, in [2]. On the contrary, it is very easy in the FSEM to construct special basis functions, that are complete in the appropriate space. Moreover, one can keep the accuracy "under control" by varying the FSEM basis functions, as it is described in this paper. The group of methods mentioned doesn't offer such a wide range of variants for the method error to be varied.

The next comment concerns the relaxed continuity conditions along the element boundaries. It is the main feature that the boundary equations are fulfilled in a weak sense. One can see no possibility to solve some strong equation at the interface. FSEM offers a possibility to write down a strong equation as well, it involves some additional conditions at the boundary, but gives further possibilities in the investigation area. There are methods, where the interface coincides with the topological singularity, nevertheless it is possible to solve such problems in FSEM as well (for instance, [6]). There exist a great part of works, that don't emphasize the solution interpolation inside an element, thus, the method under consideration assume to find only nodal values. The FSEM offers both approximation scheme and solution interpolation, interpolation comes in a natural way. The elements, considered by the other authors, use not only assumed displacements, as it does the FSEM, but also assumed stresses along the interface. It is a work for the future in FSEM investigation, but the scheme is, of course, by analogy with the existent schemes.

One can see a wide use of the collocation method, or the method of least-squares to a smaller extend, when discretized equations. It offers a simplification, but can influence the potential best accuracy. At present the FSEM follows the Bubnov-Galerkin scheme only.

6. Conclusions

The results of numerical investigation of the Finite Superelement Method for the solution of the 3D elasticity problems are given. Different variants of FSEM are being considered, and their comparative analysis is being carried out. These variants are based on the finite element interpolation techniques on superelements boundaries. FSEM and FEM efficiency comparison is presented for the model problem. A certain example of a 3D elasticity problem is considered. The practical efficiency of the method is illustrated, when choosing the correct way of constructing the FSEM approximation. A notable advantage of a higher degree FSEM approximation techniques is shown by the example of the concrete 3D problems of elasticity theory.

References

[1] Ph. Ciarlet. *The finite element method for elliptic problems.* Mir, Moscow, 1980. (In Russian) [English version is available by North-Holland, Amsterdam-New York, 1978]

[2] N. Kamiya E. Kita, Y. Ikeda. Indirect Trefftz method for boundary value problem of Poisson equation. *Engineering Analysis with Boundary Elements*, **27**, 825–833, 2003.

[3] R.P. Fedorenko. *Introduction into computational physics.* MFTI, Moscow, 1994. (In Russian)

[4] M. Galanin, S. Lazareva and E. Savenkov. Finite superelement method for the solution of the 3D elasticity problems. the numerical investigation. *Preprint of KIAM of RAS*, 44, 2006. (In Russian)

[5] M. Galanin and E. Savenkov. On relations of finite superelements and projection methods. *Preprint of KIAM of RAS*, 67, 2001. (In Russian)

[6] M. Galanin and E. Savenkov. Finite superelement method for velocity skin-layer problem. *Preprint of KIAM of RAS*, 3, 2004. (In Russian)

[7] M. Galanin, E. Savenkov and J. Temis. Fedorenko finite superelements method for elasticity problems. *Preprint of KIAM of RAS*, 38, 2004. (In Russian)

[8] M. Galanin, E. Savenkov and J. Temis. Finite superelements method for elasticity problems. *Mathematical Modelling and analysis*, **10**(3), 237–246, 2005.

[9] I. Herrera. General variational principles applicable to the hybrid element method. *Applied Physical and Mathematical Sciences*, **74**(7), 2595–2597, 1977.

[10] I. Herrera. Connectivity as an alternative to boundary integral equations: Construction of bases. *Applied Physical and Mathematical Sciences*, **75**(5), 2059–2063, 1978.

[11] I. Herrera. Boundary methods: A criterion for completeness. *Applied Mathematical and Physical Sciences*, **77**(8), 4395–4398, 1980.

[12] I. Herrera. Trefftz method: A general theory. *Numerical Methods for Partial Differential Equations*, **16**(6), 561–580, 2000.

[13] A. Mitchell and R. Wait. *The finite element method in partial differential equations.* Mir, Moscow, 1981. (In Russian) [English version is available by John Wiley and Sons, London/New York, 1977]

[14] T.H.H. Pian and C.-C. Wu. *Hybrid and incompatible finite element methods.* Chapman and Hall/CRC, London/New York, 2006.

[15] Q.-H. Qin. Trefftz finite element method and its applications. *Transactions of ASME*, 58, 316–337, 2005.

[16] O. Zienkiewicz and R. Taylor. *The finite element method. Vol. 1: The basis. Fifth edition.* Butterworth-Heinemann, Oxford, 2000.

[17] O. Zienkiewicz and R. Taylor. *The finite element method. Vol. 2: Solid mechanics. Fifth edition.* Butterworth-Heinemann, Oxford, 2000.