Mathematical Modelling and Analysis
Volume 12 Number 2, 2007, pages 157–178
 ISSN 1392-6292 print, ISSN 1648-3510 online

A FINITE DIFFERENCE METHOD FOR PIECEWISE DETERMINISTIC PROCESSES WITH MEMORY

M. ANNUNZIATO

Dipartimento di Matematica ed Informatica,
Università degli Studi di Salerno
Via Ponte Don Melillo, 84084 Fisciano (SA), Italy
E-mail: mannunzi@unisa.it

Received October 10, 2006; revised February 24, 2007; published online May 1, 2007

Abstract. In this paper the numerical approximation of solutions of Liouville-Master Equation for time-dependent distribution functions of Piecewise Deterministic Processes with memory is considered. These equations are linear hyperbolic PDEs with non-constant coefficients, and boundary conditions that depend on integrals over the interior of the integration domain. We construct a finite difference method of the first order, by a combination of the upwind method, for PDEs, and by a direct quadrature, for the boundary condition. We analyse convergence of the numerical solution for distribution functions evolving towards an equilibrium. Numerical results for two problems, whose analytical solutions are known in closed form, illustrate the theoretical finding.

Key words: Piecewise-deterministic process, dichotomic noise, random telegraph process, binary noise, upwind method, conservative systems, non local boundary conditions

1. Introduction

We deal with the following system of PDEs:

$$\partial_t F_s(x,y,t) + A_s(x)\,\partial_x F_s(x,y,t) + \partial_y F_s(x,y,t) = -\lambda_s(y)\,F_s(x,y,t) \quad (1.1)$$

with Cauchy initial conditions:

$$F_s(x,y,t_0) = F_{0,s}(x)\delta(y) \quad (1.2)$$

and boundary conditions:

$$F_s(x,0,t) = \sum_{j=1}^{S} q_{sj} \int_0^{t-t_0} F_j(x,y,t)\,\lambda_j(y)\,dy \tag{1.3}$$

for $s = \{1,\dots,S\}$ unknowns $F_s : \mathcal{D} \to \mathbb{R}$, with

$$(x,y,t) \in \mathcal{D} := (\Omega \times [0, T-t_0] \times [t_0, T]) \subset \mathbb{R}^3,$$

where $\Omega = [\Omega_a, \Omega_b] \subset \mathbb{R}$, q_{sj} are the elements of a stochastic matrix having the following fundamental properties: $0 \le q_{sj} \le 1$ and $\sum_s q_{sj} = 1$. The known functions $F_{0,s}(x)$, $A_s(x)$ and $\lambda_s(y) \ge 0$, will be discussed later.

Eq. (1.1), jointly with boundary conditions, is Liouville-Master Equation [1] for the probability distribution functions $F_s(x,y,t)$ of a continuous *piecewise-deterministic process* (PDP), that has been introduced by Davis [12, 13] (see also [24]). Indeed, here we deal with a simplified version of Davis' PDPs, but still enough general to cover many interesting models. The definition of PDP is more popular between researchers working on operations research and probability calculus (see, e.g., [10]), rather than others outside these fields, even though the latter unknowingly use it, at least in a simplified form. Before to proceed with the discussion of the numerical solution of our problem, we give a short introduction of the underling PDP process we are considering here. [2]

DEFINITION 1. We name $X(t)$, $X : \mathbb{R} \to \mathbb{R}$, be a continuous PDP if:

(a) $X(t)$ satisfies the equation:

$$\dot{X}(t) = A_s(X), \quad s = 1,\dots,S, \tag{1.4}$$

where $A_s : \mathbb{R} \to \mathbb{R}$ is a function chosen randomly on a set of $\{A_1,\dots,A_S\}$ known functions. Given A_s, we say that the dynamics is in the (deterministic) state s. We require that $A_s(x)$ be Lipschitz continuous, so that, for fixed s, $X(t)$ exists, is unique and non-explosive solution.

(b) The initial condition is settled by the Cauchy problem to Eq. (1.4), i.e. $X(t_0) = X_0$, and by the initial state $s = s_0$ of the same equation.

(c) Each state s is characterised by an its own probability density function (PDF) $\psi_s : \mathbb{R}^+ \to \mathbb{R}^+$ of "transition events":

$$\psi_s(t), \quad \text{with} \int_0^\infty \psi_s(t)\,dt = 1. \tag{1.5}$$

[1] In general, equations for density probability of random processes are derived from Chapman-Kolmogorov equation. As discussed in Ref. [15] the same equation turns into a Liouville equation in absence of randomness, and into a Master Equation, if only jump processes are involved. We use both terms in order to stress the deterministic and the random character of the processes considered here.

[2] The author acknowledges Prof. M.H.A. Davis for some explanations about the definition of PDP.

(d) When a transition event occurs, the dynamics switches instantaneously from a state j (A_j) to a new state i (A_i), given randomly according to the transition probability matrix (or transition measure) $\{q_{ij}\}$. The position $X(t)$ of the process is not affected when the state switches.

Assumptions (a), (c) and (d) define the three *local characteristics* of the PDP. We see that Eq. (1.4) can be integrated as an ordinary differential equation, provided that no switching event happen inside the integration interval. Therefore, with the exceptions of switching times, the process is deterministic, continuous and composed of pieces of solutions of Eq. (1.4). Anyway, the whole resulting process $X(t)$ is not deterministic, it represents a random sample path on a probability space. [3]

The statistical description of the process is given by the unknown functions $F_s(x, y, t)$: each represents the probability to find the process $X(t)$, in the state s, at time t in a position less than x, being past the time Y since the last switching event. Formally we write:

$$F_s(x, y, t) := \mathbb{P}(X(t) \leq x,\ y \leq Y < y + dy, \text{state} = s)$$

where $\mathbb{P}$ is a probability measure of a probability space for the process. If we are interested only in the position x of the process, we can integrate over all values of y, and the distribution function for the process, regardless the time y, reads as:

$$\mathcal{F}_s(x, t) := \int_0^{t-t_0} dy\ F_s(x, y, t). \tag{1.6}$$

With the further hypothesis $X(t) \in \Omega$, we have:

$$\partial_x F_s(x, y, t)\Big|_{x=\partial\Omega} = 0, \tag{1.7}$$

since there is a null probability for the process to be outside the interval Ω. Besides, the probability measure have to be conserved during the evolution, so that:

$$\lim_{x \to \Omega_b} \sum_{s=1}^{S} \mathcal{F}_s(x, t) = 1, \qquad \forall t \tag{1.8}$$

and

$$\lim_{x \to \Omega_a} \sum_{s=1}^{S} \mathcal{F}_s(x, t) = 0, \qquad \forall t \tag{1.9}$$

have to be satisfied. This three last equations are boundary conditions for (1.1), that complete the definition of the problem we approach to treat here.

The function $\lambda(y)$, named *hazard function* (or *hazard rate*), is related to the statistics of the PDF switching times (1.5) by:

$$\lambda_s(y) := \frac{\psi_s(y)}{\int_y^\infty \psi_s(\tau)\, d\tau}. \tag{1.10}$$

[3] For our purposes, we do not need to specify what the abstract probability space is.

It represents the probability per unit of time that a transition event will occur, i.e. a transition rate, having past the time y since the last event. The explicit dependence of λ on y makes both the statistics of the switching events and the process $X(t)$ be non-Markovian, so that y plays the role of *memory.*

The main aim of this article is to solve Eq. (1.1), jointly to all the above mentioned boundary conditions, by a finite difference scheme of the first order and prove convergence of the numerical solution. We note that numerical methods for solving linear hyperbolic PDEs with non-constant coefficient, such as (1.1), are well known in literature [2, 3, 4, 9, 14, 20, 32], but what makes this problem a little special is the non local boundary condition (1.3): the value of the unknowns F_s, on the boundary $y = 0$, depends on an integration of the F_s over the interior of the domain. This means that the numerical scheme for Eq. (1.1) have to be supported by one for (1.3), so that conditions for convergence of the numerical solution have to be investigated again. As a result we found a Courant-Friedrichs-Lewy (CFL) condition for ensuring linear convergence.

The secondary, but not of minor importance, aim of this article is to provide a connection bridge between PDPs as known by experts of the field and, as above mentioned, the same processes as known by others, who apply them to modelling in several areas of science and engineering. Here we give a sample of quotas, for which PDPs can be concerned by others, grouped in two categories: diffusive processes and systems having an equilibrium. We mention: anomalous diffusion [6], reaction-diffusion [17], scattering of radiation [18], biological dispersal [28], for the former category, and non-Maxwellian equilibriums [2, 3, 32, 9, 20, 14], diagnostic techniques for semiconductor lasers [19], filtered telegraph signals [19, 30], harmonic oscillators [23], ecological systems [22], for the latter. Many of the models involved in such references, concern the application of a two-state noise to a dynamical equation. [4]

The common end of all these researches, consists in extracting statistical properties from processes governed by that equation. Generally, an approximation method can be applied to the original model: such as by the projector technique [3, 25, 34], by a "coarse-grain" technique, (see, e.g., [16]), an asymptotic analysis (see, e.g., [6]). However, not always these techniques provide a satisfactory description. In some cases an exact analytical result can obtained as in Refs. [19, 26, 30], and more recently by the characteristic functional method [8]. Obviously, computations can also be performed by Monte Carlo's simulations, but, at the best of our knowledge, few or nothing has been devoted to a direct calculation of the time-dependent distribution function including an explicit memory variable. Concerning this, we remark that the main alternative is based on the inclusion of *supplementary variables* [11, 13], that turns PDP into Markovian, i.e. a memoryless process.

In the next section we provide an example that emphasizes the connection between PDPs and models with dichotomic noise, and a conjecture that ensures the existence of a stationary solution of Eq. (1.1). In Sect. 3 we establish the numerical scheme. We introduce definitions in Sect. 4 and in Sect. 5 give

[4] For more related citations, the reader can search the following key words: dichotomic/binary noise/process, random telegraph process, colored noise.

some theoretical results about the related convergence. In Sect. 6 we present numerical results to two problems for which an analytical stationary solution is known in closed form, and verify the stated convergence properties.

2. Explanatory Example

Let us consider a dissipative process $X(t)$ subject to a noised input $\xi(t)$, described by the equation:

$$\frac{dX}{dt} = -X(t) + \xi(t). \tag{2.1}$$

If $\xi(t)$ is taken as the random telegraph signal, Eq. (2.1) acts as filter, and $X(t)$ is referred as filtered random telegraph process [18, 30]. The same equation is elsewhere referred as Langevin equation [2, 9] subject to a dichotomous noise. $\xi(t)$ alternately takes on values ± 1, with an exponential (or Poisson) statistics for the transition events (1.5): $\psi(\tau) = \mu\, e^{-\mu\tau}$, where μ^{-1} is the expectation time between transitions. The process $X(t)$ results composed of pieces of increasing and decreasing exponentials. The statistical properties of the process $X(t)$ can be found by the associated probability density distributions $p^{\pm}(x,t)$, governed by a Liouville-Master Equation [2, 19, 24, 33]:

$$\begin{cases} \partial_t p^+ - (x-1)\partial_x p^+ = (1-\mu)p^+ + \mu p^-, \\ \partial_t p^- - (x+1)\partial_x p^- = \mu p^+ + (1-\mu)p^-. \end{cases} \tag{2.2}$$

Now let us see the same process from the point of view of PDPs. The exponential statistics for $\psi(t)$ makes the process of transitions be Markovian and the hazard function constant: $\lambda(t) = \mu$. Eq. (1.1) turns into:

$$\partial_t F_s(x,y,t) + A_s(x)\,\partial_x F_s(x,y,t) + \partial_y F_s(x,y,t) = -\mu\, F_s(x,y,t).$$

By integrating this equation over all the values of y, we get:

$$\partial_t \mathcal{F}_s(x,t) + A_s(x)\,\partial_x \mathcal{F}_s(x,t) - F_s(x,0,t) = -\mu\, \mathcal{F}_s(x,t),$$

having used the property $F_s(x,y,t) = 0$ if $y > 0$, since the process is memoryless. From Eq.(1.3) we have:

$$F_s(x,0,t) = \mu \sum_{j=1}^{S} q_{sj} \int_0^t dy\; F_j(x,y,t) = \mu \sum_{j=1}^{S} q_{sj} \mathcal{F}_j(x,t)$$

and inserting it into the previous equation we get:

$$\partial_t \mathcal{F}_s(x,t) + A_s(x)\,\partial_x \mathcal{F}_s(x,t) = \mu \sum_{j=1}^{S} (q_{sj} - \delta_{sj}) \mathcal{F}_j(x,t)$$

If $S = 2$, with the known functions:

$$A_1(x) = (1-x), \quad A_2(x) = -(1+x), \tag{2.3}$$

and transition measure:

$$q_{11} = q_{22} = 0, \quad q_{12} = q_{21} = 1, \tag{2.4}$$

provided that $p^{\pm}(x,t) = \partial_x \mathcal{F}_{1,2}(x,t)$, we obtain just the equation (2.2). This shows the connection between PDPs and processes driven by dichotomous noise.

2.1. Remarks on equilibrium solutions

In what follows we focus our attention on solutions $F(x,y,t)$ having an equilibrium, but we presume that the numerical scheme can be extended to diffusion processes too. Conditions for the existence of equilibrium solution can be conjectured by using simple dynamical arguments [5, 26]. If all dynamical equations (1.4) own only attraction points and all these are contained into the intersection of the basin of attraction of each, then a process starting from this region will never escape. Hence, there should exist a region Ω where the process is confined and a stationary distribution $\mathcal{F}_{eq}(x) = \lim_{t\to\infty} \mathcal{F}(x,t)$ exists.

3. The Finite-Difference Scheme

In this section we show the numerical scheme to solve Eqs. (1.1) and (1.3) based on a finite difference method of first order. For the sake of simplicity we take $t_0 = 0$ and the domain of F_s becomes $\mathcal{D} := \{\Omega \times [0,T] \times [0,T]\}$. It is convenient to perform the numerical integration along the characteristic lines $\xi = t - y$. With this new variable, we define the unknowns $\phi_s(x,y,\xi) = \phi_s(x,y,t-y) := F_s(x,y,t)$, so that Eq. (1.1) transforms as:

$$A_s(x)\, \partial_x \phi_s(x,y,\xi) + \partial_y \phi_s(x,y,\xi) = -\lambda_s(y)\phi_s(x,y,\xi). \tag{3.1}$$

This equation is valid for $0 < \xi < t$ and $0 < y < t$. The initial condition is given on

$$\phi_s(x,y,\xi)|_{\xi=-y} = F_{0,s}(x)\delta(y). \tag{3.2}$$

With the new variable we get

$$\phi_s(x,0,\xi)|_{\xi=t} = F_s(x,0,t),$$

and the boundary condition Eq. (1.3) becomes:

$$\phi_s(x,0,t) = \sum_{l=1}^{S} q_{sl} \int_0^t \phi_l(x,y,\xi)|_{\xi=t-y} \lambda_l(y) dy. \tag{3.3}$$

We will assume that similar conditions of Eqs. (1.7), (1.8) and (1.9) are satisfied for $\phi_s(x,y,\xi)$, and a stationary solution exists.

On the domain $\mathcal{D}$, we introduce a uniform mesh:

$$(x_k, y_j, t_n) \quad \begin{cases} k = 0, \dots, N_k \\ j, n = 0, \dots, N, \qquad N = T/\Delta t, \end{cases} \tag{3.4}$$

with step size Δx and $\Delta y = \Delta t$, so that we define the discrete known functions as ${}^lA_k := A_l(x_k)$ and ${}^l\lambda_j := \lambda_l(y_j)$, and the discrete solution:

$${}^lF^n_{kj}, \qquad n = 0, \dots, N, \quad j < n$$

as an approximation of $F_l(x_k, y_j, t_n)$ at the mesh points.

The change of variable $\xi = t - y$ corresponds to the following discrete mapping on the mesh:

$$(k, j, n) = (k, j, i)\Big|_{i=n-j}, \tag{3.5}$$

therefore we get the following relation:

$$F_l(x_k, y_j, t_n) = \phi_l(x_k, y_j, t_n - y_j) = \phi_l(x_k, y_j, \xi_i) \Rightarrow {}^lF^n_{kj} = {}^l\phi^{n-j}_{kj} = {}^l\phi^i_{kj}$$

between the discrete solutions. Here the index i identifies the set of mesh points lying on the characteristics lines.

The numerical scheme is obtained by discretizing both equations (3.1) and (3.3). We apply upwind discretization to the first equation, and get:

$${}^l\phi^i_{k,j+1} = {}^l\phi^i_{kj} - {}^lA_k \frac{\Delta y}{\Delta x} \left({}^l\phi^i_{k+\nu,j} - {}^l\phi^i_{k+\nu-1,j} \right) - {}^l\lambda_j \; {}^l\phi^i_{kj} \Delta y, \qquad i = 0, \dots, N, \tag{3.6}$$

where $\nu = 1$ if ${}^lA_k < 0$, and $\nu = 0$ if ${}^lA_k > 0$. The boundary condition (1.7) is included by requiring that ${}^l\phi^i_{0j} = {}^l\phi^i_{1j}$ and ${}^l\phi^i_{N_k-1,j} = {}^l\phi^i_{N_k,j}$.

For the second equation, we substitute integral with a quadrature scheme:

$${}^s\phi^i_{k,0} = \Delta y \sum_{l=1}^{S} q_{sl} \sum_{j=0}^{i} w^{(i)}_j \; {}^l\phi^{i-j}_{kj} \; {}^l\lambda_j, \qquad i > 0, \tag{3.7}$$

where $w^{(i)}_j \geq 0$ is a sequence of weights.

The integration proceeds as follows. Given the initial condition

$${}^l\phi^0_{k,0} = \phi_l(x_k, 0, 0) = F_l(x_k, 0, 0),$$

Eq. (3.6) allows us to calculate all the ${}^l\phi^i_{kj}$ starting from $j = 1$ up to N, for the fixed characteristic line $i = 0$. In general given the values on the boundary $j = 0$: ${}^l\phi^i_{k,0}$, we can find all ${}^l\phi^i_{kj}$ starting from $j = 1$ up to $N - i$, for a fixed characteristic line i (see curved arrows of Fig. (1)). But the starting values ${}^l\phi^i_{kj}$ for upwind are unknown and have to be estimated by using the boundary integration (3.7) (see vertical dot-dashed arrows of Fig. (1)). This is a system of equations for the unknowns ${}^l\phi^i_{k,0}$. When all ${}^l\phi^i_{kj}$ are known, the discrete

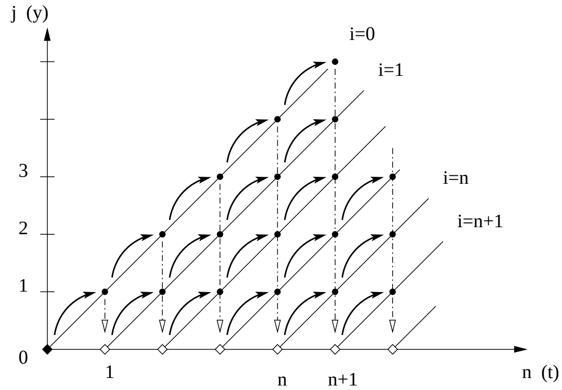

Figure 1. Representation of the integration scheme on the regular mesh of the (t, y) plane. Full curved arrows: upwind step of Eq. (3.6). Dot-dashed vertical arrows: quadrature of Eq. (3.7).

distribution function can be retrieved by ${}^lF^n_{k,j} = {}^l\phi^i_{kj}|_{i=n-j}$, and also the discrete counterpart of (1.6):

$$ {}^l\mathcal{F}^n_k := \sum_{j=0}^{n} v_j^{(n)}\ {}^lF^n_{k,j}\ \Delta y, \tag{3.8} $$

can be estimated by a quadrature formula of weights $v_j^{(n)}$.

4. Preliminary Definitions

4.1. Global errors and convergence

We are interested in how well ${}^l\phi^i_{kj}$, ${}^lF^n_{k,j}$ and ${}^l\mathcal{F}^n_k$ approximate the corresponding analytical solutions. We consider the global pointwise error:

$$ {}^le^i_{k,j} := \phi_l(x_k, y_j, \xi_i) - {}^l\phi^i_{kj} \tag{4.1} $$

for the transformed solution. The same value defines the global error for ${}^lF^n_{k,j}$ under the discrete mapping (3.5).

In order to prove convergence, we introduce norms for measuring errors. For spatial x_k and memory variable y_j the $\infty-$norm is used. The discrete 1-norm for the states s of the system is the natural choice, because of the conservation of the probability of Eq. (1.8). For convenience of notation we define the global error for the state l at time t_i and time memory y_j as:

$$ {}^lE^i_j := \max_k |\ {}^le^i_{k,j}| \tag{4.2} $$

and the global error regardless states as:

$$\|E_j^i\|_1 := \sum_{l=1}^{S} {}^l E_j^i. \tag{4.3}$$

We say that ${}^l F_{k,j}^n$ converges to $F_l(x_k, y_j, t_n)$ in the norm $\|\cdot\|$ if:

$$\|E^i\| = \max_j \|E_j^i\|_1 \to 0, \quad \text{as } \Delta x, \Delta y \to 0. \tag{4.4}$$

The global error for the distribution function (3.8) is defined as:

$$\hat{E}_k^n := \mathcal{F}(x_k, t_n) - \mathcal{F}_k^n = \int_0^{t_n} dy \sum_l F_l(x_k, y, t_n) - \Delta y \sum_{j=0}^{n} v_j^{(n)} \sum_l {}^l F_{k,j}^n$$

and the associated convergence is stated by:

$$\|\hat{E}^n\|_\infty = \max_k |\hat{E}_k^n| \to 0, \quad \text{as } \Delta x, \Delta y \to 0. \tag{4.5}$$

4.2. Local truncation error and quadrature error

As usual [21, 27], the local truncation error is defined by inserting the true solution $\phi_l(x, y, \xi)$ into the discrete scheme of Eq. (3.6), i.e.:

$$\begin{aligned} {}^l\varepsilon_{kj}^i := \; & \frac{\phi_l(x_k, y_i + \Delta y, \xi_i) - \phi_l(x_k, y_i, \xi_i)}{\Delta y} + \lambda_l(y_j)\phi_l(x_k, y_i, \xi_i) \\ & + A_l(x_k)\frac{\phi_l(x_k + \nu\Delta x, y_i, \xi_i) - \phi_l(x_k + (\nu - 1)\Delta x, y_i, \xi_i)}{\Delta x}. \end{aligned} \tag{4.6}$$

By evaluating the remainder term of the Taylor's expansion with respect to ${}^l\phi_{kj}^i$, we get:

$${}^l\varepsilon_{kj}^i = \frac{1}{2}\Delta x \left(\alpha \partial_y^2 \phi_l(x_k, \eta_j, \xi_i) - |A_l(x_k)| \partial_x^2 \phi_l(\eta_k, y_j, \xi_i)\right),$$

where $\alpha := \Delta y / \Delta x$, and η_k, η_j are unknown points.

The quadrature error committed from (3.7) for the evaluation of the boundary integral (3.3) is defined as:

$${}^l R_k^i := \mathcal{M}_i t_i \Delta y^\beta \partial_y^\beta (\phi_l(x_k, \tilde{\eta}_j, \xi_i)\, \lambda_l(\tilde{\eta}_j)), \tag{4.7}$$

where $\mathcal{M}_i$ are some constants that can depend on i, and $\tilde{\eta}_j$ are unknown points of the local integration interval. β defines the order of repeated quadrature formulas (3.7).

The quadrature error committed from (3.8) for the evaluation of (1.6) is defined as:

$${}^l \hat{R}_k^n := \hat{\mathcal{M}}_n t_n \Delta t^{\hat{\beta}} \partial_y^{\hat{\beta}} F_l(x_k, \hat{\eta}_j, t_n),$$

where, as for the previous error, $\hat{\mathcal{M}}_n$ are some values that can depend on n, and $\hat{\eta}_j$ are unknown points. $\hat{\beta}$ defines the order of (3.8).

5. Analysis of Convergence

In this section we show first a lemma for convergence of the numerical solution ${}^l\phi^i_{kj}$ for the transformed equation (3.6), then prove a theorem for the convergence order of the numerical solution $\mathcal{F}^n_k$ to the distribution function $\mathcal{F}(x_k, t_n)$. The proofs are based on classical arguments by finding bounds for global errors.

Lemma 1. *Let* $\phi_l(x, y, \xi) \in C^{2,\bar{\beta},2}(\mathcal{D})$ *be a solution of Eq.* (3.1) *under the boundary conditions* (3.2), *with* $\bar{\beta} = \max\{\beta, 2\}$, $\max_l \|A_l(x)\|_\infty \leq M$, *and* $\max_l \|\lambda_l(y)\|_\infty \leq L_u$, *for* $x \in \Omega$ *and* $l \in \{1, \ldots, S\}$. *Let* (x_k, y_j, t_n) *be an uniform mesh on* $\mathcal{D}$ *defined in* (3.4), *of step sizes* Δx *and* $\Delta y = \Delta t$. *Let* ${}^l\phi^i_{kj}$ *be the numerical solution resulting from the scheme as defined in Eqs.* (3.6) *and* (3.7), *under the transformed mesh of Eq.* (3.5). *If the Courant-Friedrichs-Lewy (CFL) condition*

$$\Delta y < \left(\frac{M}{\Delta x} + L_u\right)^{-1} \tag{5.1}$$

is satisfied then:

1. *Given the error* $\|E^i_0\|_1$ *at boundary* $y = 0$, *the error* $\|E^i_m\|_1$ *computed at time step* t_{i+m} *along the characteristic* ξ_i, *is bounded by:*

$$\|E^i_m\|_1 \leq \|E^i_0\|_1 (1 - L_u \Delta y)^m + m\Delta y \|\mathcal{E}^i\|_1, \tag{5.2}$$

   where $\|\mathcal{E}^i\|_1 = \mathcal{O}(\Delta x), \ \ \forall i$.

2. *If* $\Delta y^{-1} > \max_i(w^{(i)}_0) \max_l \lambda_l(0)$, *there exist constants* L_d, K, *such that the error computed at time* t_i *along the boundary* $y = 0$ *is bounded by:*

$$\|E^i_0\|_1 \leq \|\bar{R}\|_1 \exp(Kt_i) + \|\bar{\mathcal{E}}\|_1 K \frac{t^2_{i+1}}{2} \exp((K - L_d)t_i), \tag{5.3}$$

   where $\|\bar{R}\|_1 = \mathcal{O}(\Delta x^\beta)$ *and* $\|\bar{\mathcal{E}}\|_1 := \max_i \|\mathcal{E}^i\|_1 = \mathcal{O}(\Delta x)$.

This lemma states that the numerical solution ${}^l\phi^i_{kj}$ converges to analytical with a linear order. Errors calculated for ϕ are the same as for F. Being interested in finding the probability regardless of memory and state at fixed time t_n, we search an estimate for the error $\|\hat{E}^n\|_\infty$ as defined in Eq. (4.5).

Theorem 1. *Let the hypothesis of the Lemma 1 be satisfied, then for the problem of Eq.* (1.1) *the numerical solution* (3.8), *obtained as discussed in Sec. 3, converges to analytical solution of* (1.6) *in the sense* (4.5), *with order* $\mathcal{O}(\Delta x)$.

Proof. We substitute the integration of Eq. (1.6) over the memory state with a sequence of quadrature formulas of weights $v^{(n)}_j$:

$$\int_0^{t_n} dy \sum_l F_l(x_k, y, t_n) = \Delta y \sum_{j=0}^{n} v^{(n)}_j \sum_l F_l(x_k, y_j, t_n) + \sum_l {}^l\hat{R}^n_k, \tag{5.4}$$

so that:

$$\|\hat{E}^n\|_\infty = \max_k \left| \Delta y \sum_j v_j^{(n)} \sum_l F_l(x_k, y_j, t_n) + \sum_l {}^l\hat{R}_k^n - \Delta y \sum_j v_j^{(n)} \sum_l {}^lF_{k,j}^n \right|.$$

Let $\bar{v}^{(n)} := \max_j |v_j^{(n)}|$ and $\hat{R}^n := \sum_l \max_k |\,{}^l\hat{R}_k^n|$, from the triangle inequality:

$$\|\hat{E}^n\|_\infty \leq \bar{v}^{(n)} \Delta y \sum_j \sum_l \max_k |F_l(x_k, y_j, t_n) - {}^lF_{k,j}^n| + \hat{R}^n,$$

by using the discrete mapping (3.5) and the definition of the error (4.3), we get:

$$\|\hat{E}^n\|_\infty \leq \bar{v}^{(n)} \Delta y \sum_{j=0}^{n} \|E_j^{n-j}\|_1 + \hat{R}^n.$$

By inserting Eq. (5.2) we find:

$$\|\hat{E}^n\|_\infty \leq \bar{v}^{(n)} \Delta y \sum_{j=0}^{n} (\|E_0^{n-j}\|_1 (1 - L_u \Delta y)^j + j \Delta y \|\mathcal{E}^i\|_1) + \hat{R}^n$$

and

$$\|\hat{E}^n\|_\infty \leq \bar{v}^{(n)} \Delta y \sum_{m=0}^{n} \|E_0^m\|_1 (1 - L_u \Delta y)^{n-m} + \bar{v}^{(n)} \|\bar{\mathcal{E}}\|_1 \frac{t_{n+1}^2}{2} + \hat{R}^n,$$

where $\|\bar{\mathcal{E}}\|_1 := \max_j \|\mathcal{E}^{n-j}\|_1$. This inequality relates the searched probability error to errors along the boundary $j = 0$. Now we insert the second result of the previous lemma stated by Eq. (5.3) and find the order of the error for vanishing Δy. From the summation we get:

$$\sum_{i=0}^{n} (1 - L_u \Delta y)^{n-i} \left(\|\bar{R}\|_1 e^{K t_i} + \|\bar{\mathcal{E}}\|_1 K \frac{t_{i+1}^2}{2} e^{(K - L_d) t_i} \right)$$

that is of order:

$$\|\bar{R}\|_1 \frac{e^{K t_n} - 1}{(K + L_u) \Delta y} + \|\bar{\mathcal{E}}\|_1 K \frac{t_n^2 e^{(K - L_d) t_n}}{2(K + L_u - L_d) \Delta y}.$$

Finally, we get an order of convergence for the error:

$$\begin{aligned} \|\hat{E}^n\|_\infty \lesssim\ & \bar{v}^{(n)} \|\bar{R}\|_1 \frac{\exp(K t_n) - 1}{K + L_u} + \hat{R}^n \\ & + \bar{v}^{(n)} \|\bar{\mathcal{E}}\|_1 K \frac{t_n^2 \exp((K - L_d) t_n)}{2(K + L_u - L_d)} + \bar{v}^{(n)} \|\bar{\mathcal{E}}\|_1 \frac{t_n^2}{2}. \end{aligned} \quad (5.5)$$

■

6. Computational Results

In what follows we present results of the numerical scheme, applied to the examples considered in [29]. For quadrature of Eqs. (3.7) and (3.8), we adopt the rectangle scheme:

$$w_j^{(i)} = v_j^{(i)} = \begin{cases} 0 & \text{for } j = 0 \\ 1 & \text{for } j > 0 \end{cases} \qquad i > 0, \tag{6.1}$$

whose quadrature error (4.7) is:

$${}^l R_k^i = \frac{1}{2} t_i \Delta y \partial_y \big(\phi_l(x_k, \tilde{\eta}_j, \xi_i)\, \lambda_l(\tilde{\eta}_j) \big).$$

Note that despite the low order ($\beta = 1$) of approximation for quadrature, the global error of the numerical solution is not degraded, because the same order apply to upwind. The choice $w_0^{(i)} = 0$ makes the equation (3.7) be explicit. This can be consistent, because for vanishing Δy the contribution to integral is also vanishing for a limited ${}^l\phi_{k,0}^i$. For the explicit scheme the computational cost can be evaluated as follows: at time step i all upwinds take $2N_k S i$ operations, the boundary quadrature takes $N_k S^2 i$. By summing over i we get:

$$\text{Computational Cost} \approx N_k S^2 N^2.$$

The Cauchy problem for starting the numerical integration is set, according to Eq. (1.2), as follows:

$${}^s F_{k0}^0 = \begin{cases} 0, & k < 0, \\ 0.5/(S\Delta y), & k = 0, \\ 1/(S\Delta y), & k > 0, \end{cases} \qquad {}^s F_{kj}^0 = 0, \quad j > 0, \tag{6.2}$$

for all $s = 1, \ldots, S$. This choice is an approximation of (1.2) with $F_{0,s}(x) = H(x)$, where $H(x)$ is the Heaviside function. Such Cauchy conditions for the Liouville-Master Equation correspond to having placed the process $X(t)$ at the initial position $X_0 = 0$, to an equiprobable random initial state, having spent no time in it (i.e. $\delta(y)$ in (1.2)). This is a common choice when studying the motion of a particle subject to a random fluctuating force, but is not a good mathematical hypothesis for applying Lemma 1. However, it is well known that the upwind methods tends to regularise the solution (numerical viscosity) around discontinuities [1, 21], and, for the problems we are approaching to solve, a unique stationary solution exists regardless the initial state of the process. In this way we are enabled to use such "non-regular" Cauchy condition for our numerical convergence tests. ${}^l F_{k,j}^n$ are then integrated by (3.8).

We are interested in plotting the density probability function of the process, regardless memory and states, defined as:

$$p(x,t) := \partial_x \mathcal{F}(x,t) := \partial_x \sum_{l=1}^{S} \mathcal{F}_l(x,t). \tag{6.3}$$

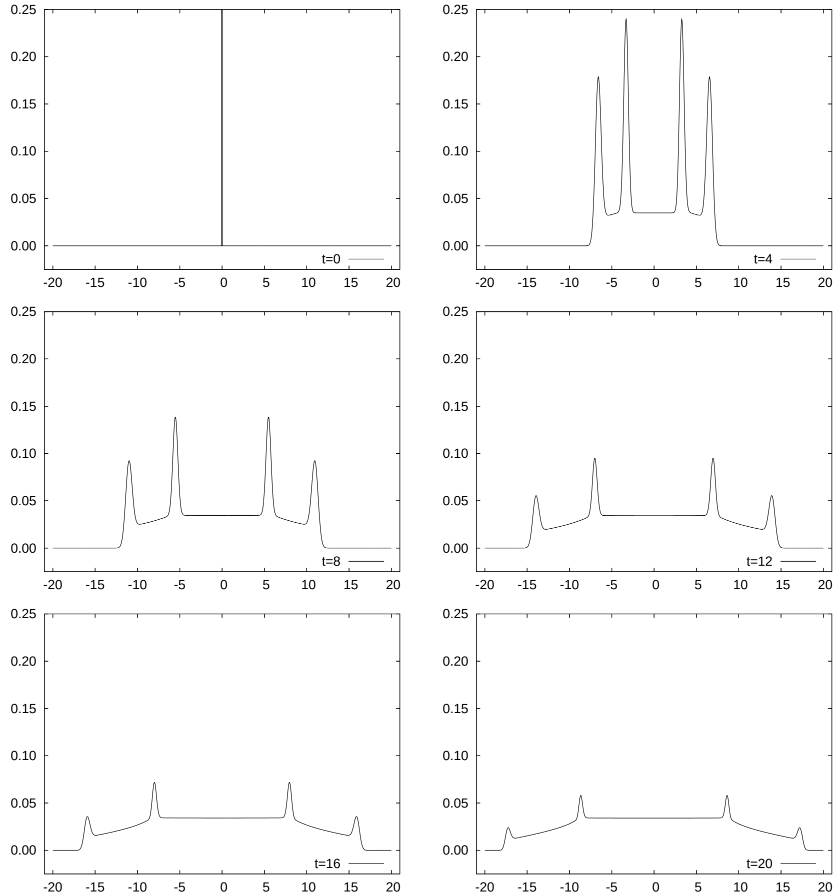

Figure 2. Six snapshots of the temporal evolution of p_k^n for the Langevin equation driven by Poisson distribution time intervals of Sec. 6.1 (horizontal axis: x_k; $t_n = 0, 4, 8, 12, 16, 20$).

The discrete version of this operation is:

$$p_k^n := \frac{1}{\Delta x}(\mathcal{F}_{k+1}^n - \mathcal{F}_k^n) := \frac{1}{\Delta x}\sum_{l=1}^{S}({}^l\mathcal{F}_{k+1}^n - {}^l\mathcal{F}_k^n), \tag{6.4}$$

that is the first order right-derivative of the numerical distribution function.

6.1. RC-filter subject to Markovian process (Poisson PDF)

In Fig. 2 we plot six snapshots of the temporal evolution related to the four states process: $A_s(x) = -\gamma_s x + W_s$, studied in [1]. Parameters are defined as:

$$W_0 = 1, \;\; W_1 = -1, \;\; W_2 = 2, \;\; W_3 = -2, \;\; \lambda_s = 0.2, \;\; \gamma_s = 0.1 \quad \forall s.$$

Results are comparable with that of the cited reference. Note the broadening of the four peaks due to the numerical viscosity of the upwind method.

6.2. Filtering of non-Markov dichotomous noise with McFadden interval PDF

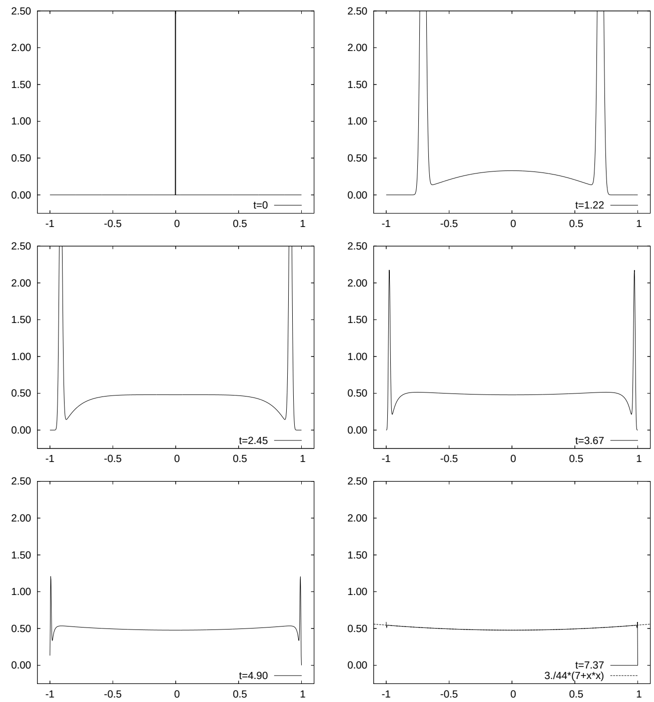

Figure 3. Six snapshots of the temporal evolution of p_k^n for filtering of dichotomous noise driven by the McFadden distribution time intervals (horizontal axis: x_k).

For this example the process is described by the Langevin equation (2.1), with functions of Eq. (2.3) and transition measure of Eq. (2.4). Intervals between switching time have the McFadden PDF: $\psi(t) = 3\,e^{-t}(1-e^{-t})^2$. The equilibrium density distribution has the form [29]:

$$p_{eq}(x) = \lim_{t\to\infty} p(x,t) = \frac{3}{44}(7+x^2), \quad |x| < 1 \tag{6.5}$$

and its integral:

$$\mathcal{F}_{eq}(x) = \lim_{t\to\infty} \mathcal{F}(x,t) = \frac{1}{44}(x^3+21x+22), \quad |x| \le 1. \tag{6.6}$$

The hazard function related to the density for switching intervals reads as:

$$\lambda(t) = \frac{3(1-e^{-t})^2}{2-e^{-t}+(1-e^{-t})^2}.$$

We note that $\lambda(0) = 0$, so that the error committed from the choice $w_0^{(i)} = 0$ (see (6.1)), is further improved. Beside it is $\lambda(t) \le 4/9$ and the convergence lemma give us more guarantee that the errors do not grow fast. Here the grid step sizes are $\Delta x = 0.002$ and $\Delta y = 8 \cdot 10^{-4}$. Integration starts with a concentrated initial density (6.2) and stops at time $T = 7.37$.

In Fig. 3 we see six snapshots of the numerical solution of the PDF (6.4). At time $t = 0$ the density of the process is concentrated to $x = 0$, then two peaks, corresponding to the two dynamical states, move towards the attraction points $x \pm 1$, and at last the stationary distribution appears.

6.3. Filtering of non-Markov dichotomous noise with "gamma" interval PDF

For this example the process is described by the same Langevin equation as that the previous one, but the intervals between switching times of $\xi(t)$ of Eq. (2.1) are taken as the gamma density $\psi(t) = \mu^2 t e^{-\mu t}$ for both states [29]. Provided that $\mu = 1/2$, the equilibrium solution for the total density distribution function is:

$$p_{eq}(x) = \lim_{t\to\infty} p(x,t) = \frac{|\Gamma(1-r)|^2}{\pi^{3/2}}(1-x^2)^{-1/2}\ {}_2F_1(r, r^*; \frac{1}{2}; x^2), \quad (6.7)$$
$$r = (1+\sqrt{-1})/4, \quad |x| \le 1,$$

in which ${}_2F_1$ is a hypergeometric function. The *hazard function* (1.10) related to $\psi(t)$ results: $\lambda(t) = \dfrac{\mu^2 t}{1+\mu t}$. We see from the convergence theorem that the errors do not grow so fast, because the maximum value of $\lambda(t)$ is $\lambda_{max} = \mu/e$. We perform the numerical integration on a mesh with spatial discretization step $\Delta x = 0.004$ and temporal step $\Delta t = 0.0015$. Integration starts with distribution (6.2) and stops at time $T = 12$, where the equilibrium is supposed to be reached in good approximation.

In Fig. 4 six snapshots of the total density distribution (6.4) are plotted. The evolution behaves as in the previous example, with the exception of singularities at $x \pm 1$, for the equilibrium.

6.4. Convergence tests

Since for the above mentioned problems we know two analytical results, we can calculate the global error $\|\hat{E}\|_\infty$ for the stationary distribution functions of Eqs. (6.6) and (6.7). The analytical solution of Eq. (6.7) is evaluated by using MATLAB® with libraries for calculating the Hypergeometric function [31]. We calculate the solution with the numerical scheme until the time $T = 20$. At this time we experienced that the stationary solution is reached. This integration is repeated for some spatial step sizes Δx, with the temporal step

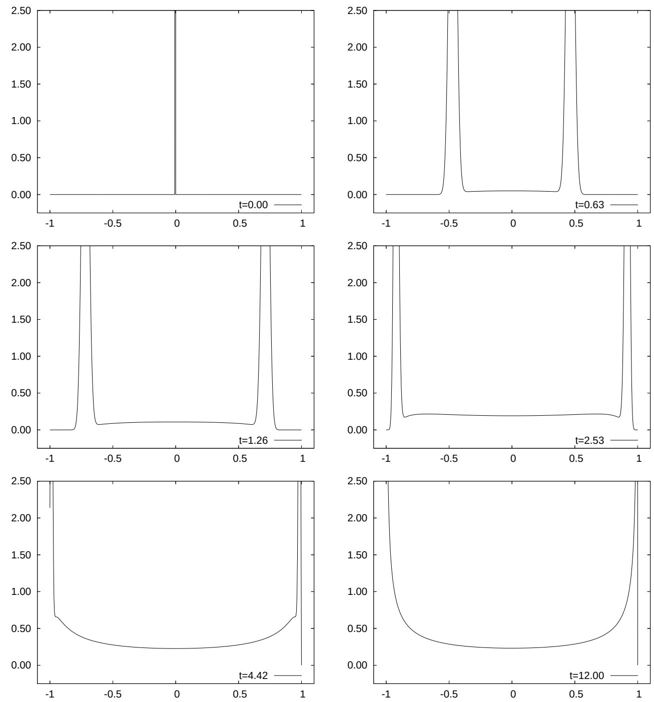

Figure 4. Six snapshots of the temporal evolution of p_k^n for filtering of dichotomous noise driven by the gamma distribution time intervals (horizontal axis: x_k).

size constraint $\alpha = \Delta y/\Delta x = 0.9\,(M+L_u \Delta x)^{-1}$, satisfying the CFL condition (5.1).

In Fig. 5 we show the global error $\hat{E}_k$, defined at the equilibrium, plotted for $\Delta x = \{0.1, 0.04, 0.008\}$ for the McFadden intervals. We see clearly that the maximum error decreases as Δx decreases.

In Fig. 6 we show the same test for gamma distributed intervals. Also here the error decreases, but it shows a sort of divergence near $x = \pm 1$.

In order to stress convergence, in Fig. 7, we plot the error $\|\hat{E}\|_\infty$ versus the step size Δx. We see that the McFadden's data have unitary slope, i.e. linear convergence, as we expected from Theorem (1). Instead, gamma's data are arranged with approximately $1/2$ slope. This can be explained as follows. We know [29] that the PDF of Eq. (6.7) is U-shaped having $(1-x^2)^{-1/2}\ln(1-x^2)$ infinities near $x \pm 1$. As $\delta x := 1 - x$ approaches to zero, the second spatial derivative of $\mathcal{F}_{eq}$ behaves as:

$$\partial_x p_{eq} = \partial_x^2 \mathcal{F}_{eq} \propto \delta x^{-3/2}\big(1 + \ln(2\,\delta x)\big).$$

Being $A_1(\delta x) \approx \delta x$ and by considering $\Delta x = \delta x$, we get for the error:

$$\|\hat{E}\|_\infty \approx \sqrt{\Delta x}\big(1 + \ln(2\,\Delta x)\big),$$

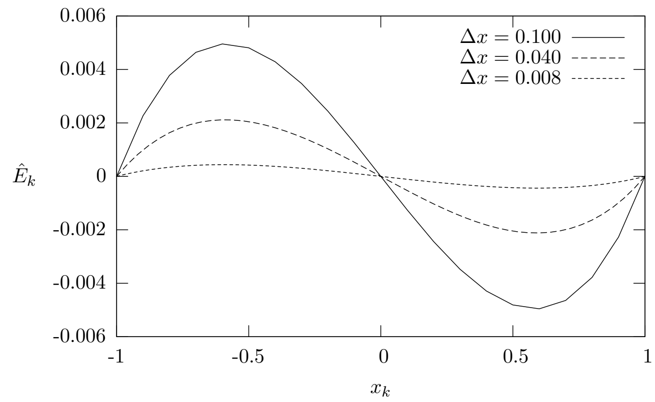

Figure 5. Global error $\hat{E}_k$ between the exact solution of Eq. (6.6) (McFadden PDF) and numerically calculated for $\Delta x = 0.1$, $\Delta x = 0.04$, $\Delta x = 0.008$.

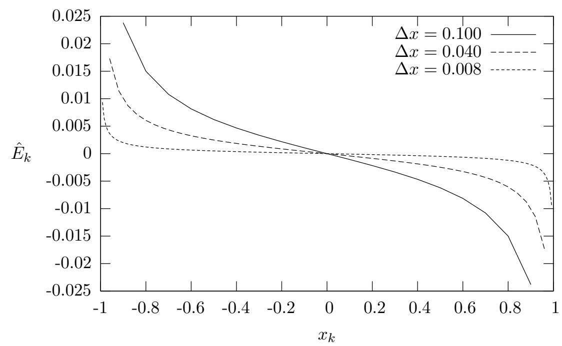

Figure 6. Global error $\hat{E}_k$ between the integral of the exact solution of Eq. (6.7) (gamma PDF) and numerically calculated for $\Delta x = 0.1$, $\Delta x = 0.04$, $\Delta x = 0.008$.

that explains what we see from the results. If we remove the endpoints from the measure of the error, we see from "gamma 90%" of Fig. 7, that, e.g. for the interval $x \in [-0.9, 0.9]$, the linear convergence order is recovered.

7. Summary and Conclusions

A numerical method for approximating Liouville-Master Equation (or *forward equation*) for a class of piecewise deterministic process with memory was investigated. The method was based on a combination of *upwind*, for the hyperbolic differential operator, and a quadrature, for the non local boundary conditions. When non singular solutions are considered, the linear convergence of the time dependent numerical solution, in an uniform norm, was proved under a CFL

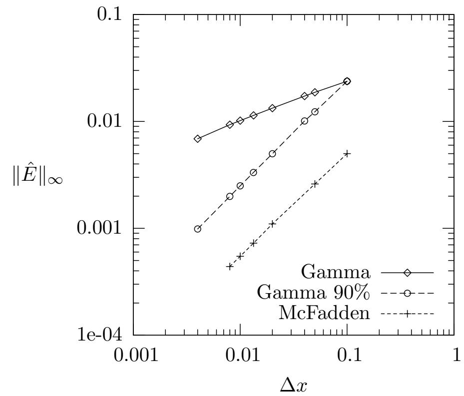

Figure 7. Convergence test for global error $\|\hat{E}\|_\infty$ vs. mesh step size Δx.

condition. The linear convergence was validated by numerical tests for the equilibrium distribution related to the filtering of non-Markov dichotomous noise, with both "gamma" and McFadden statistics of transition events, whose solutions are known in analytical form.

A. Proof of Lemma 1

Proof.

We study convergence in two parts: the first concerns the accumulation of the error along the characteristic lines of Eq. (3.6); the second concerns the error cumulated on the boundary integration of Eq. (3.7).

Let ${}^l e^i_{k,j}$ be the global error (4.1) between exact and discrete solutions. By considering the standard procedure [21, 27] that puts in relation local and global errors, we find from Eqs. (3.1) and (3.6):

$$ {}^l e^i_{k,j+1} = {}^l e^i_{k,j} - {}^l A_k \alpha \left({}^l e^i_{k+\nu,j} - {}^l e^i_{k+\nu-1,j} \right) - {}^l \lambda_j \; {}^l e^i_{k,j} \Delta y + {}^l \varepsilon^i_{kj} \Delta y, \quad \text{(A.1)} $$

where ${}^l \varepsilon^i_{kj}$ is defined in (4.6). We start our analysis from Eq. (A.1), fixed i and given j we find ${}^l e^i_{k,j+1}$. The error for ${}^l e^i_{k,0}$ is found from the boundary integral condition.

The first step is to limit the error along the characteristic line i. From Eq. (A.1) we study the case ${}^l A_k > 0$ $(\nu = 0)$:

$$ | {}^l e^i_{k,j+1} | \leq \left| 1 - {}^l A_k \alpha - {}^l \lambda_j \Delta y \right| \, | {}^l e^i_{k,j} | + {}^l A_k \alpha | \; {}^l e^i_{k-1,j} | + | {}^l \varepsilon^i_{kj} | \Delta y. $$

Let ${}^l E^i_j = \max_k | {}^l e^i_{k,j} |$ and ${}^l \mathcal{E}^i_j = \max_k | {}^l \varepsilon^i_{kj} |$ then:

$$ | {}^l e^i_{k,j+1} | \leq \left(\left| 1 - {}^l A_k \alpha - {}^l \lambda_j \Delta y \right| + {}^l A_k \alpha \right) \; {}^l E^i_j + {}^l \mathcal{E}^i_j \Delta y. $$

If we set $1 - {}^l A_k \alpha - {}^l \lambda_j \Delta y > 0$, $\forall l, j, k$, i.e. the CFL condition is satisfied, then we get:

$$| \ {}^l e^i_{k,j+1}| \leq (1 - \ {}^l\lambda_j \Delta y) \ {}^l E^i_j + \ {}^l\mathcal{E}^i_j \Delta y, \ \ \forall k, j, l.$$

Since it is valid for all k, we can write:

$$^l E^i_{j+1} \leq (1 - \ {}^l\lambda_j \Delta y) \ {}^l E^i_j + \ {}^l\mathcal{E}^i_j \Delta y, \tag{A.2}$$

that is verified for $j = 0, \dots, n$ and $i = 0, \dots, n$. Now we have to find ${}^l E^{n+1}_0$.

The case $\nu = 1$ gives the same result:

$$| \ {}^l e^i_{k,j+1}| \leq |1 + \ {}^l A_k \alpha - \ {}^l\lambda_j \Delta y|| \ {}^l e^i_{k,j}| - \ {}^l A_k \alpha| \ {}^l e^i_{k+1,j}| + | \ {}^l\mathcal{E}^i_j|\Delta y$$

and for the maximum error on x:

$$| \ {}^l e^i_{k,j+1}| \leq \left(\left|1 - | \ {}^l A_k|\alpha - \ {}^l\lambda_j \Delta y\right| + | \ {}^l A_k|\alpha\right) \ {}^l E^i_j + \ {}^l\mathcal{E}^i_j \Delta y,$$

Let $1 - | \ {}^l A_k|\alpha - \ {}^l\lambda_j \Delta y > 0$, we get Eq. (A.2), so that it is valid independently by the sign of $A_l(x)$. By iterating this expression we find:

$$^l E^i_{m+1} \leq \Pi^m_{j=0}(1 - \ {}^l\lambda_j \Delta y) \ {}^l E^i_0 + (m+1)\Delta y \ {}^l\mathcal{E}^i, \tag{A.3}$$

where ${}^l\mathcal{E}^i = \max_j \ {}^l\mathcal{E}^i_j$. When the norm over all states is considered:

$$\|E^i_m\|_1 \leq \|E^i_0\|_1 (1 - L_u \Delta y)^m + m\Delta y \|\mathcal{E}^i\|_1.$$

Being ${}^l\lambda_j \geq 0$ we have good chances to get an upper bound to the error for increasing time.

Now we study the second step, i.e. the error ${}^l e^i_{k,0}$ (or ${}^l E^i_0$) along the boundary condition. Here $j = 0$ then $n = i$.

From Eq. (3.3) as calculated with quadrature on the exact solution $\phi_l(x, y, \xi)$, we can write:

$$\phi_s(x_k, 0, t_i) = \sum_{l=1}^{S} q_{sl} \Big(\sum_{j=0}^{i} \Delta y \, w^{(i)}_j \ {}^l\lambda(y_j) \phi_l(x_k, y_j, t_i - y_j) + \ {}^l R^i_k \Big).$$

Subtracting side by side this equation and (3.7):

$$^s e^i_{k,0} = \sum_{l=1}^{S} q_{sl} \Big(\sum_{j=0}^{i} \Delta y \, w^{(i)}_j \ {}^l\lambda_j \ {}^l e^{i-j}_{kj} + \ {}^l R^i_k \Big)$$

and by applying the triangle inequality we get:

$$| \ {}^s e^i_{k,0}| \leq \sum_{l=1}^{S} q_{sl} \Big(\sum_{j=1}^{i} \Delta y \, w^{(i)}_j \ {}^l\lambda_j | \ {}^l e^{i-j}_{kj}| + | \ {}^l R^i_k| \Big).$$

Let ${}^l R^i = \max_k | \ {}^l R^i_k|$ and being $q_{sl} \geq 0$, $w^{(i)}_j \geq 0$, we get:

$$^s E^i_0 \leq \sum_{l=1}^{S} q_{sl} \big(\sum_{j=0}^{i} \Delta y \, w^{(i)}_j \ {}^l\lambda_j \ {}^l E^{i-j}_j + \ {}^l R^i \big). \tag{A.4}$$

Let ${}^l\underline{\lambda} = \min_j \ {}^l\lambda_j$, from Eq. (A.3) is:

$$ {}^lE^i_{j+1} \le (1 - \ {}^l\underline{\lambda}\Delta y)^{j+1} \ {}^lE^i_0 + (j+1)\Delta y \ {}^l\mathcal{E}^i $$

and inserting it in (A.4), we find:

$$ {}^sE^i_0 \le \sum_{l=1}^{S} q_{sl}\Big(\Delta y \sum_{j=0}^{i} w^{(i)}_j \ {}^l\lambda_j[(1 - \ {}^l\underline{\lambda}\Delta y)^j \ {}^lE^{i-j}_0 + j\Delta y \ {}^l\mathcal{E}^{i-j}] + \ {}^lR^i\Big). $$

This inequality puts in relation the error on the boundary condition ($j = 0$) with those at early times. Let $\bar{w}^{(i)} = \max_{j\ge1} w^{(i)}_j$ and ${}^l\bar{\lambda} = \max_{j\ge1} \ {}^l\lambda_j$, then the following inequality

$$ {}^sE^i_0 \le \sum_{l=1}^{S} q_{sl}\Big(\Delta y \, \bar{w}^{(i)} \ {}^l\bar{\lambda} \sum_{j=1}^{i}[(1 - \ {}^l\underline{\lambda}\Delta y)^j \ {}^lE^{i-j}_0 + \ j\Delta y \ {}^l\mathcal{E}^{i-j}] + \Delta y \, w^{(i)}_0 \ {}^l\lambda_0 \ {}^lE^i_0 + \ {}^lR^i\Big) $$

is valid for $i \ge 1$. We have introduced the sum starting from $j = 0$, so that, in order to find ${}^sE^i_0$, we have to solve a system of equations. However for convergence we do not need to solve it at all. By moving ${}^lE^i_0$ to the left hand side and summing over the states, we find:

$$ \sum_s \ {}^sE^i_0 - \sum_{s,l} q_{sl}\Delta y \, w^{(i)}_0 \ {}^l\lambda_0 \ {}^lE^i_0 \le \sum_s \sum_{l=1}^{S} q_{sl}\big(\Delta y \, \bar{w}^{(i)} \ {}^l\bar{\lambda} \sum_{j=1}^{i}[(1 - \ {}^l\underline{\lambda}\Delta y)^j \ {}^lE^{i-j}_0 + j\Delta y \ {}^l\mathcal{E}^{i-j}] + \ {}^lR^i\big). $$

By using the fundamental property of stochastic matrix:

$$ \sum_l (1 - \bar{w}_0 L_0 \Delta y) \ {}^lE^i_0 \le \sum_{j=1}^{i} \Delta y \, \bar{w}^{(i)} \sum_l \ {}^l\bar{\lambda}[(1 - \ {}^l\underline{\lambda}\Delta y)^j \ {}^lE^{i-j}_0 + j\Delta y \ {}^l\mathcal{E}^{i-j}] + \sum_l \ {}^lR^i, $$

where $\bar{w}_0 = \max_i w^{(i)}_0$, $L_0 = \max_l \lambda_0$. Let $a := 1 - \bar{w}_0 L_0 \Delta y > 0$, we get:

$$ \sum_l \ {}^lE^i_0 \le a^{-1}\Delta y \sum_{j=1}^{i} \bar{w}^{(i)} \sum_l \ {}^l\bar{\lambda}[(1 - \ {}^l\underline{\lambda}\Delta y)^j \ {}^lE^{i-j}_0 + j\Delta y \ {}^l\mathcal{E}^{i-j}] + \sum_l \ {}^lR^i, $$

where $0 < a < 1$. Let $L_u = \max_l \ {}^l\bar{\lambda}$, $L_d = \min_l \ {}^l\underline{\lambda}$, $b = (1 - L_d\Delta y)$ provided that from CFL it is surely $L_d\Delta y \le 1$, we have:

$$ \|E^i_0\|_1 \le a^{-1}\Delta y \, \bar{w}^{(i)} \ L_u \sum_{j=1}^{i}[b^j\|E^{i-j}_0\|_1 + j\Delta y\|\mathcal{E}^{i-j}\|_1] + \|R^i\|_1 $$

and with the summation index change $m = i - j$, it becomes:

$$b^{-i}\|E_0^i\|_1 \le K\Delta y\Big(\Delta y\frac{i(i+1)}{2}\|\bar{\mathcal{E}}\|_1 + \sum_{m=0}^{i-1} b^{-m}\|E_0^m\|_1\Big) + \|R^i\|_1 b^{-i},$$

where $\|\bar{\mathcal{E}}\|_1 = \max_i \|\mathcal{E}^i\|_1$, and $K = \max_i(a^{-1}\bar{w}^{(i)}\ L_u)$. From the discrete Gronwall lemma (cf. in Th. 1.5.4 and Corol. 1.5.1 of [7]), we obtain:

$$b^{-i}\|E_0^i\|_1 \le \Big(K\Delta y^2\frac{i(i+1)}{2}\|\bar{\mathcal{E}}\|_1 + \|R^i\|_1 b^{-i}\Big)(1+K\Delta y)^i.$$

Let $\|\bar{R}\|_1 = \max_i \|R^i\|_1$, we find:

$$\|E_0^i\|_1 \le \|\bar{R}\|_1(1+K\Delta y)^i + \|\bar{\mathcal{E}}\|_1 K\Delta y^2\frac{(i+1)i}{2}[b(1+K\Delta y)]^i$$

and finally:

$$\|E_0^i\|_1 \le \|\bar{R}\|_1 \exp\left(Kt_i\right) + \|\bar{\mathcal{E}}\|_1 K\frac{t_{i+1}^2}{2}\exp\left(\left(K-L_d\right)t_i\right).$$

■

References

[1] M. Annunziato. A finite difference method for piecewise deterministic Markov processes. http://arxiv.org/abs/math.NA/0606588.

[2] M. Annunziato. Non-gaussian equilibrium distribution arising from the Langevin equation. *Phys. Rev. E*, **65**, 21113–1–6, 2002.

[3] M. Annunziato, P. Grigolini and J. Riccardi. Fluctuation-dissapation process without a time scale. *Phys. Rev. E*, **61**, 4801–4808, 2000.

[4] M. Annunziato, P. Grigolini and B. J. West. Canonical and noncanonical equilibrium distribution. *Phys. Rev. E*, **64**, 011107–1–13, 2001.

[5] I. Bena, C. Van den Broeck, R. Kawai and K. Lindenberg. Drift by dichotomous Markov noise. *Phys. Rev. E*, **68**, 041111–1–12, 2003.

[6] M. Bologna, P. Grigolini and B. J. West. Strange kinetics: conflict between density and trajectory description. *Chem. Phys.*, **284**, 115–128, 2002.

[7] H. Brunner and P.J. Van der Houwen. *The numerical solution of Volterra equation.* North-Holland, 1986.

[8] A. Budini and M. O Cáceres. Functional characterization of generalized Langevin equations. *J. Phys. A: Math. Gen.*, **37**, 5959–5981, 2004.

[9] M. O Cáceres. Computing a non-Maxwellian velocity distribution from first principles. *Phys. Rev E*, **67**, 016102–016104, 2003.

[10] O. L.V. Costa and F. Dufour. On the Poisson equation for piecewise-deterministic Markov processes. *SIAM J.Control Optim.*, **42**, 985–1001, 2003.

[11] D.R. Cox. The analysis of non-Markovian stochastic processes by the inclusion of supplementary variables. *Proc. Camb. Phil. Soc.*, **51**, 433–441, 1955.

[12] M.H.A. Davis. Piecewise-deterministic Markov processes: A general class of non-diffusion stochastic models. *J. of the Royal Stat. Soc. Series B*, **46**, 353–388, 1984.

[13] M.H.A. Davis. *Markov Models and Optimizations.* Chapman & Hall/CRC, London, 1993.

[14] R. Filliger and M. O. Hongler. Supersymmetry in random two-velocity processes. *Physica A*, **332**, 141–150, 2004.

[15] C. W. Gardiner. *Handbook of Stochastic Methods.* Springer-Verlag, Berlin Heidelberg New York, 1985. 2nd Ed.

[16] T. Hillen and H. G. Othmer. The diffusion limit of transport equations derived from velocity-jump processes. *SIAM J. Appl. Math.*, **61**, 751–775, 2000.

[17] W. Horsthemke. Spatial instabilities in reaction random walks with direction-independent kinetics. *Phys. Rev E*, **60**, 2651–2663, 1999.

[18] E. Jakeman and E. Renshaw. Correlated random-walk model for scattering. *J. Opt. Soc. Am. A*, **4**, 1206–1212, 1987.

[19] E. Jakeman and K. D. Ridley. The statistics of a filtered telegraph signal. *J. Phys. A*, **32**, 8803–8821, 1999.

[20] K. Kitahara and et al. Phase diagrams of noise induced transitions. *Prog. Theor. Phys.*, **64**, 1233–1247, 1980.

[21] Randall J. LeVeque. *Numerical Methods for conservation Laws.* Birkhäuser, 1992.

[22] R. Mankin, A. Ainsaar, A. Haljas and E. Reiter. Trichotomous-noise-induced catastrophic shifts in symbiotic ecosystems. *Phys. Rev. E*, **65**, 051108–1–9, 2002.

[23] J. Masoliver and J. M. Porrà. Harmonic oscillators driven by colored noise: Crossovers, resonances, and spectra. *Phys. Rev. E*, **48**, 4309–4319, 1993.

[24] G.N. Milstein and Yu.M. Repin. Action of Markov process on a system of differential equations. *Differential Equations*, **5**, 1010–1019, 1972. Translated from: Differentsial Uravneniya **5**, 1969 pp. 1371–1384.

[25] H. Mori. Statistical-mechanical theory of transport in fluids. *Phys. Rev.*, **112**, 1829–1842, 1958.

[26] A. Morita. Free Brownian motion of a particle driven by a dichotomous random force. *Phys. Rev. A*, **41**, 754–760, 1990.

[27] K. W. Morton and D. F. Mayers. *Numerical Solution of Partial Differential Equations.* Cambridge University Press, 1994.

[28] H. G. Othmer, S. R. Dunbar and W. Alt. Models of dispersal in biological systems. *J. Math. Biol.*, **26**, 263–298, 1988.

[29] R. F. Pawula. Approximating distributions from moments. *Phys.Rev.A*, **36**, 4996–5007, 1987.

[30] R. F. Pawula and O. Rice. On filtered binary processes. *IEEE Trans. Inf. Th.*, **IT-32**, 63–72, 1986.

[31] W. F. Perger, A. Bhalla and M. Nardin. A numerical evaluator for the generalized hypergeometric 1 series. *Comp. Phys. Commun.*, **77**, 249–254, 1993. (MATLAB® translation by B. E. Barrowes, see MATLAB® Central File Exchange). This routine uses a gamma function for complex arguments, see P. Godfrey, `http://my.fit.edu/~gabdo/gamma.m`.

[32] C. Van den Broeck and P. Hänggi. Activation rates for nonlinear stochastic flows driven by non-Gaussian noise. *Phys. Rev. A*, **30**, 2730–2736, 1984.

[33] W. M. Wonham. *J. Electron. Control*, **6**, 376–383, 1959.

[34] R. Zwanzig. *Lectures in Theoretical Physics (Boulder)*, volume 3 ed. W.E. Broton (New York: Interscience), 1961.