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MULTIDIMENSIONAL ANALOGUES OF THE RIEMANN–HILBERT BOUNDARY VALUE PROBLEM

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Abstract. Multidimensional generalizations of the Cauchy–Riemann systems and two different types of analogues of the Riemann–Hilbert boundary value problems for these systems are considered.
Key words: elliptic systems, boundary value problems

1. Introduction

The analyticity of complex function $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$ is equivalent to the validity of system of differential equations

$$u_x - v_y = 0, \quad u_y + v_x = 0, \tag{1.1}$$

which is known under the name of the Cauchy–Riemann system. When C is a piecewise – smooth curve inside some domain of analyticity of function $f(z)$, the well–known Cauchy integral formula

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz$$

is valid for any z_0 that lies inside the curve C.

The Riemann–Hilbert problem for the Cauchy–Riemann system is the following: in a given domain $S \subset \mathbb{R}^2$ find a regular solution $\big(u(x, y), v(x, y)\big)$ of system (1.1), which satisfies boundary conditions

$$(\alpha u + \beta v)|_{\partial S} = \varphi, \ \alpha^2 + \beta^2 \neq 0, \tag{1.2}$$

where $\alpha(x, y)$, $\beta(x, y)$, $f(x, y)$ are given in S functions.

Various generalizations of this problem were investigated. For instance, a problem of finding an analytic function $f(z) = u(z)+iv(z),\ z \in S,\ z = x+iy$, which is defined in a given domain $S \subset \mathbb{C}$ and satisfies boundary conditions

$$Re[\lambda(z)f(z)]|_{\partial S} = \varphi(z),$$

where $\lambda(z),\ \varphi(z)$ are given in ∂S functions, is considered in [1, 5]. Multidimensional analogues of such problems were studied in papers [2, 4].

We consider two multidimensional structural generalizations of the Cauchy – Riemann system and formulate analogues of the Riemann–Hilbert problems for these systems. Our main goal is to demonstrate the differences in the characters of solvability of the obtained problems.

2. Multidimensional Analogues of the Cauchy–Riemann System

Let's consider $m = 2l$–dimensional system of equations

$$LU = 0, \tag{2.1}$$

where

$$U = (U_1, U_2)^T, \qquad U_1 = (u_{11}, \ldots, u_{1l}), \qquad U_2 = (u_{21}, \ldots, u_{2l}),$$
$$X = (X_1, X_2), \qquad X_1 = (x_1, \ldots, x_r), \qquad X_2 = (x_{r+1}, \ldots, x_{2r}),$$

and, by analogy with [7], L is given by a block-matrix

$$L = \begin{pmatrix} D_1 & -D_2 \\ \overline{D}_2 & \overline{D}_1 \end{pmatrix}, \tag{2.2}$$

where

$$D_1 = I\frac{\partial}{\partial x_1} + \sum_{k=1}^{r-1} M_k \frac{\partial}{\partial x_{k+1}}, \qquad \overline{D}_1 = I\frac{\partial}{\partial x_1} - \sum_{k=1}^{r-1} M_k \frac{\partial}{\partial x_{k+1}},$$
$$D_2 = I\frac{\partial}{\partial x_{r+1}} + \sum_{k=1}^{r-1} N_k \frac{\partial}{\partial x_{k+r+1}}, \qquad \overline{D}_2 = I\frac{\partial}{\partial x_{r+1}} - \sum_{k=1}^{r-1} N_k \frac{\partial}{\partial x_{k+r+1}},$$

I is $m \times m$ identity matrix and $m \times m$ real constant matrices $M_k,\ N_k$ satisfy equations

$$M_k M_j + M_j M_k = -2\delta_{kj} I, \tag{2.3}$$

$$N_k N_j + N_j N_k = -2\delta_{kj} I, \tag{2.4}$$

$$M_k N_j = N_j M_k, \quad k, j = 1, \ldots n, \tag{2.5}$$

where δ_{kj} is the Kronecker symbol.

The maximal number of real constant $m \times m$ matrices M_k, that satisfy equations (2.3), equals

$$\rho = 8b + 2^c - 1,$$

where $m = (2a+1)2^{4b+c}$, see [3]. Matrices N_k, satisfying properties (2.4) and (2.5), could be defined as below:

$$N_k = M_{r+k} M_{2r+1}, \quad k = 1, \dots, r, \quad r \in \mathbb{N}, \quad 2r < \rho.$$

Of course, these equations are not necessary for matrices N_k.

Let's denote $x = x_1, y = x_2,\ U_1 = u, U_2 = v$, then in case of $l = r = 1$ system (2.1) coincides with the Cauchy–Riemann system (1.1) and

$$D_1 = \overline{D}_1 = \frac{\partial}{\partial x},\ D_2 = \overline{D}_2 = \frac{\partial}{\partial y}.$$

We get a four-dimensional system of type (2.1) by using matrices

$$I = \begin{pmatrix} 1\,0 \\ 0\,1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.6}$$

where $r = 2,\ l = 2$.

If we denote $u_{11} = u, u_{12} = v,\ x = x_1, y = x_2$, the Cauchy–Riemann system (1.1) can be written as the matrix equation

$$\overline{D}_1 U_1 = 0.$$

Let's denote $X = (x_1, x_2, \dots, x_n)$ (now $n = 2r$) and let's suppose that $n > 2$. By a regular solution of system (2.1) we will understand continuously differentiable function $U(X)$ which satisfies (2.1) in a given domain S and is continuous in a closure of the domain ∂S.

Statement 1. *The following equations are valid:*

$$D_1 D_2 = D_2 D_1, \qquad D_1 \overline{D}_2 = \overline{D}_2 D_1, \qquad \overline{D}_1 D_2 = D_2 \overline{D}_1,$$

$$\overline{D}_1 \overline{D}_2 = \overline{D}_2 \overline{D}_1, \qquad D_1 \overline{D}_1 = I\Delta_1, \qquad D_2 \overline{D}_2 = I\Delta_2,$$

where by Δ_k we have denoted the Laplace operator of variables $X_k,\ k = 1, 2$, respectively.

Proof. The validity of these equations is simply verified using the definitions of operators $D_k,\ \overline{D}_k,\ k = 1, 2$ and properties (2.3), (2.4) and (2.5) of matrices $M_k, N_k,\ k = 1, 2, \dots, r-1$. Let's denote

$$\overline{L} = \begin{pmatrix} \overline{D}_1 & D_2 \\ -\overline{D}_2 & D_1 \end{pmatrix},$$

then we get the equalities

$$L\overline{L} = \overline{L}L = \begin{pmatrix} I\Delta & 0 \\ 0 & I\Delta \end{pmatrix}, \qquad \Delta = \Delta_1 + \Delta_2.$$

■

Corollary 1. (2.1) is an elliptic system of equations.

Statement 2. *Function* $U = (U_1, U_2)^T$, *where*

$$U_1 = \overline{D}_1 H, \quad U_2 = -\overline{D}_2 H, \tag{2.7}$$

and $H(X) = (h_1, \ldots, h_m)$ *is arbitrary harmonic function of variables* X, *satisfies* (2.1).

Proof. Substituting $U = (\overline{D}_1 H, -\overline{D}_2 H)^T$ into system (2.1) and using equations of Statement 1, we prove that equations of (2.1) are satisfied. ■

Let's consider a boundary value problem: to find a regular solution $U(X)$ of system (2.1), which is defined in domain S and satisfies the following boundary condition:

$$U_1|_{\partial S} = F, \tag{2.8}$$

where $F(X) = (f_1, \ldots, f_l)$ is given on ∂S function. In case of

$$r = 1, \;\; D_1 = \overline{D}_1 = \frac{\partial}{\partial x_1}, \;\; D_2 = \overline{D}_2 = \frac{\partial}{\partial x_2}, \;\; \alpha = 1, \;\; \beta = 0,$$

problem (2.1), (2.8) coincides with problem (1.1), (1.2). Thus, (2.1), (2.8) can be treated as a generalized Riemann–Hilbert boundary value problem.

Let us suppose that $F(X) \equiv 0$ and consider homogeneous problem (2.1), (2.8).

Statement 3. *Homogeneous problem* (2.1), (2.8) *has infinitely many linearly independent solutions.*

Proof. By substitution of U into system (2.1) we find that functions $U = (U_1, U_2)^T$, where

$$U_1 = \overline{D}_1 H, \quad U_2 = -\overline{D}_2 H,$$

satisfy equations (2.1) and the boundary condition $U_1|_{\partial S} = 0$ when $H(X) = \Phi_1(X_1)\Phi_2(X_2)$, where $\Phi_1(X_1)$ and $\Phi_2(X_1)$ are arbitrary functions satisfying

$$\overline{D}_1 \Phi_1(X_1) = \Delta_2 \Phi_2(X_1) = 0.$$

■

Remark 1. It is clear that nonhomogeneous problem (2.1), (2.8) has solution not for arbitrary boundary value function $F(X)$.

3. The Other Type of Multidimensional Analogues of the Cauchy–Riemann Equations

In this section we consider system (1.1) and represent it in the following form:

$$\left(I\frac{\partial}{\partial x}+M_1\frac{\partial}{\partial y}\right)(u,v)^T = 0,$$

where I and M_1 are given by (2.6). As $M_1^2=-I$, this system is equivalent to the matrix equation

$$DU=O, \tag{3.1}$$

where O is 2×2 zero matrix and differential matrix operator D is defined by

$$D=I\frac{\partial}{\partial x}+M_1\frac{\partial}{\partial y},\quad U(x,y)=Iu(x,y)+M_1v(x,y).$$

If we denote $z = Ix + M_1y$, then $U(x,y) = Iu(x,y) + M_1v(x,y)$ can be treated as a function of a complex variable $z = Ix + M_1y$ with the real I and the imaginary M_1 units. Let's denote the conjugate complex number by $\overline{z} = Ix - M_1y$ and the conjugate differential operator by $\overline{D}=I\frac{\partial}{\partial x}-M_1\frac{\partial}{\partial y}$. Then

$$D\overline{D}=\overline{D}D=I\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)=I\Delta.$$

If we define $D_1, D_2, \overline{D}_1$ and $\overline{D}_2$ as below:

$$\begin{aligned}
D_1 &= I\frac{\partial}{\partial x_1}+M_1\frac{\partial}{\partial x_2}, & \overline{D}_1 &= I\frac{\partial}{\partial x_1}-M_1\frac{\partial}{\partial x_2},\\
D_2 &= I\frac{\partial}{\partial x_3}+N_1\frac{\partial}{\partial x_4}, & \overline{D}_2 &= I\frac{\partial}{\partial x_3}-N_1\frac{\partial}{\partial x_4},
\end{aligned}$$

where $N_1 = M_1$, we will find equations (2.3), (2.4), (2.5) being satisfied and will see that elliptic differential operator L given by (2.2) can be represented as:

$$L = I\frac{\partial}{\partial x_1}+M_1\frac{\partial}{\partial x_2}+M_2\frac{\partial}{\partial x_3}+M_3\frac{\partial}{\partial x_4}.$$

Here I is 4×4 identity,

$$M_1=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix},\quad M_2=\begin{pmatrix}0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&-1&0&0\end{pmatrix},\quad M_3=\begin{pmatrix}0&0&0&1\\0&0&-1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix}.$$

Matrices M_1, M_2, M_3 satisfy (2.3) and, additionally, equations

$$M_1M_2=M_3,\quad M_1M_3=-M_2,\quad M_2M_3=M_1. \tag{3.2}$$

Let us denote L by D, then equation

$$D(Iu_1 + M_1u_2 + M_2u_3 + M_3u_4) = O \tag{3.3}$$

where O is 4×4 zero matrix, in case of $u_3 \equiv u_4 \equiv 0$ is equivalent to equation (3.1) and, consequently, to the Cauchy–Riemann system (1.1). Due to (2.3) and (3.2), matrices I, M_1, M_2, M_3 can play a role of matrix representation of quaternion basis. When considering $z = (x_1, x_2, x_3, x_4)$ as a quaternion variable

$$\mathbf{z} = x_1 I + x_2 M_1 + x_2 M_2 + x_3 M_3,$$

equation (3.3) coincides with the Fueter conditions, see [8]. If u_1, u_2, u_3, u_4 are complex functions, system (3.3) is equivalent to the homogeneous Maxwell's equations, see [6].

Statement 4. *Equation* (3.3) *is equivalent to the system of equations*

$$DU = \mathsf{0},$$

where $U(z) = (u_1, u_2, u_3, u_4)^T$, $z = (x_1, x_2, x_3, x_4)$, 0 *is the* 4*-dimensional zero vector.*

Proof. Due to (2.3), matrices I, M_1, M_2, M_3 are linearly independent, thus the validity of a statement is implied by the following matrix

$$(I\frac{\partial}{\partial x_1} + M_1\frac{\partial}{\partial x_2} + M_2\frac{\partial}{\partial x_3} + M_3\frac{\partial}{\partial x_4})(u_1 I + u_2 M_1 + u_3 M_2 + u_4 M_3)$$
$$= w_1 I + w_2 M_1 + w_3 M_2 + w_4 M_3$$

and vector

$$D(u_1, -u_2, -u_3, -u_4)^T = (w_1, -w_2, -w_3, -w_4)^T$$

equations. ∎

Let us denote by

$$\overline{D} = I\frac{\partial}{\partial x_1} - M_1\frac{\partial}{\partial x_2} - M_2\frac{\partial}{\partial x_3} - M_3\frac{\partial}{\partial x_4}$$

the conjugate operator to D. Then $D\overline{D} = \overline{D}D = I\Delta$, where the Laplace operator of variables x_1, x_2, x_3, x_4 is denoted by Δ. Variable $\mathbf{z}$ in a matrix form

$$\mathbf{z} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ -x_2 & x_1 & -x_4 & x_3 \\ -x_3 & x_4 & x_1 & -x_2 \\ -x_4 & -x_3 & x_2 & x_1 \end{pmatrix}$$

represents the 4-dimensional Hadamard matrix of the Williamson type. This type matrices appear in statistics, engineering, optical communications and other areas. Considering the following sample of the 8-dimensional Hadamard matrix of the Williamson type

$$\mathrm{z} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ -x_2 & x_1 & -x_4 & x_3 & -x_6 & x_5 & x_8 & -x_7 \\ -x_3 & x_4 & x_1 & -x_2 & -x_7 & -x_8 & x_5 & x_6 \\ -x_4 & -x_3 & x_2 & x_1 & -x_8 & x_7 & -x_6 & x_5 \\ -x_5 & x_6 & x_7 & x_8 & x_1 & -x_2 & -x_3 & -x_4 \\ -x_6 & -x_5 & x_8 & -x_7 & x_2 & x_1 & x_4 & -x_3 \\ -x_7 & -x_8 & -x_5 & x_6 & x_3 & -x_4 & x_1 & x_2 \\ -x_8 & x_7 & -x_6 & -x_5 & x_4 & x_3 & -x_2 & x_1 \end{pmatrix},$$

we can represent it as a sum of matrices:

$$\mathrm{z} = Ix_1 + \sum_{k=1}^{7} M_k x_{k+1},$$

where I is 8×8 identity and matrices $M_k,\ k = 1, 2, \ldots, 7$ can be found from a previous expression of z. They satisfy (2.3) and together with I they can be used as a basis of a right-hand matrix representation of octonions. Matrix equation

$$\mathrm{DU} = \Big(I\frac{\partial}{\partial x_1} + \sum_{k=1}^{7} M_k \frac{\partial}{\partial x_{k+1}}\Big)\Big(Iu_1 + \sum_{k=1}^{7} M_k u_{k+1}\Big) = O, \tag{3.4}$$

where O is a 8×8 zero matrix, is equivalent to (3.3) when $u_k \equiv 0,\ k = 5, \ldots, 8$ and operator D has properties analogous to properties of D.

Remark 2. The analogue of a Statement 4 with respect to equation (3.4) is not valid.

Let $\mathsf{M}_k,\ k = 1, \ldots, n-1$ be any $m \times m$ real constant matrices, satisfying (2.3), I is $m \times m$ identity, and let's consider matrix equation

$$\mathsf{DU} = \Big(\mathsf{I}\frac{\partial}{\partial x_1} + \sum_{k=1}^{n-1} \mathsf{M}_k \frac{\partial}{\partial x_{k+1}}\Big)\Big(\mathsf{I}\mathsf{u}_1 + \sum_{k=1}^{n-1} \mathsf{M}_k \mathsf{u}_{k+1}\Big) = \mathsf{O}, \tag{3.5}$$

where O is $m\times m$ zero matrix. Operator L of equation (2.1) is also an operator of type of D:

$$\mathsf{D} = \Big(\mathsf{I}\frac{\partial}{\partial x_1} + \sum_{k=1}^{n-1} \mathsf{M}_k \frac{\partial}{\partial x_{k+1}}\Big),$$

where $m = 2l,\ n = 2r$, and for $k \neq r+1, k = 1, 2, \ldots, n-1$

$$\mathsf{M}_{r+1} = \begin{pmatrix} O & -I \\ I & O \end{pmatrix}, \ \mathsf{M}_k = \begin{pmatrix} M_k & -N_k \\ -N_k & -M_k \end{pmatrix},$$

O and I are $l \times l$ zero and identity matrices correspondingly. Let us denote by

$$Z = \mathsf{I}x_1 + \sum_{k=1}^{n-1} \mathsf{M}_k x_{k+1}, \quad \overline{Z} = \mathsf{I}x_1 - \sum_{k=1}^{n-1} \mathsf{M}_k x_{k+1},$$

$$Re(Z) = x_1, \quad Im(Z) = \sum_{k=1}^{n-1} M_k x_{k+1}, \ \{Z\} = \mathsf{Z},$$

where $x_k \in \mathbb{R}, \ k = 1, \dots, n$ and

$$\mathsf{U} = \mathsf{I}\mathsf{u}_1 + \sum_{k=1}^{n-1} \mathsf{M}_k \mathsf{u}_{k+1}, \qquad \overline{\mathsf{U}} = \mathsf{I}\mathsf{u}_1 - \sum_{k=1}^{n-1} \mathsf{M}_k \mathsf{u}_{k+1},$$

$$\mathsf{D} = \mathsf{I}\frac{\partial}{\partial x_1} + \sum_{k=1}^{n-1} \mathsf{M}_k \frac{\partial}{\partial x_{k+1}}, \qquad \overline{\mathsf{D}} = \mathsf{I}\frac{\partial}{\partial x_1} - \sum_{k=1}^{n-1} \mathsf{M}_k \frac{\partial}{\partial x_{k+1}}.$$

Let's consider the following problem: find twice continuously differentiable in a given domain $\mathsf{S} \subset \mathsf{Z}$ solution $\mathsf{U}(\mathsf{z})$ of equation (15), satisfying boundary value condition

$$Re\mathsf{U}\Big|_{\partial\mathsf{S}} = \mathsf{f}(\mathsf{z}), \tag{3.6}$$

where $\mathsf{f}(\mathsf{z})$ is given in $\partial\mathsf{S}$ function. This boundary value problem in case of $m = n = 1$ is equivalent to the problem (1.1), (1.2), where $\alpha \equiv 1, \ \beta \equiv 0$, consequently, problem (3.5), (3.6) is a multidimensional analogue of Riemann–Hilbert problem for the Cauchy–Riemann equations.

Let S be any star-shaped domain of type

$$\mathsf{S} = \{Z \in \mathsf{Z}, \ \forall t \in [0, 1) \Rightarrow (tZ) \in \mathsf{Z}\}.$$

The given below statement follows ideas of quaternion analysis.

Statement 5. *Let $\mathsf{v}(Z)$ be any real-valued twice continuously differentiable harmonic in given domain S function. Then equation (3.5) has solution $\mathsf{U}(Z)$, which satisfies condition*

$$Re\,\mathsf{U}(Z) = \mathsf{v}(Z).$$

Proof. Function

$$\mathsf{U}(Z) = \mathsf{I}\mathsf{v}(Z) + Im \int_0^1 \tau^{n-2}(\overline{\mathsf{D}}\mathsf{v})(\tau Z)Z\, d\tau,$$

satisfies (3.5). Really,

$$Re \int_0^1 \tau^{n-2}(\overline{\mathsf{D}}\mathsf{v})(\tau Z)Z\, d\tau = Re \int_0^1 \tau^{n-2}\big(\mathsf{I}\frac{\partial \mathsf{v}}{\partial x_1}(\tau Z) - \sum_{k=1}^{n-1} \mathsf{M}_k \frac{\partial \mathsf{v}}{\partial x_{k+1}}(\tau Z)\big)(\mathsf{I}x_1 + \sum_{k=1}^{n-1} \mathsf{M}_k x_{k+1})\, d\tau$$

$$= Re \int_0^1 \tau^{n-2} \mathsf{I}(x_1 \frac{\partial \mathsf{v}}{\partial x}(\tau Z) + \sum_{k=1}^{n-1} x_{k+1} \frac{\partial \mathsf{v}}{\partial x_{k+1}}(\tau Z))\, d\tau$$

$$= \int_0^1 \tau^{n-2} \frac{d\mathsf{v}}{d\tau}(\tau Z)\, d\tau = \mathsf{v}(Z) - (n-2) \int_0^1 \tau^{n-1} \mathsf{v}(\tau Z)\, d\tau.$$

Therefore

$$\mathsf{U}(Z) = \int_0^1 \tau^{n-2} (\overline{\mathsf{D}}\mathsf{v})(\tau Z) Z\, d\tau + \mathsf{I}(n-2) \int_0^1 \tau^{n-1} \mathsf{v}(\tau Z)\, d\tau.$$

Since v and $\overline{\mathsf{D}}\mathsf{v}$ have continuous derivatives in S, derivatives of integrals could be replaced with the integrals of derivatives of the integrands. Applying properties (2.3) of matrices M_k, we have:

$$\mathsf{DU}(Z) = \int_0^1 \tau^{n-2} \mathsf{D}(\overline{\mathsf{D}}\mathsf{v}(\tau Z)) Z\, d\tau + \int_0^1 \tau^{n-2} \big((\overline{\mathsf{D}}\mathsf{v})(\tau Z)$$
$$+ (\overline{\mathsf{D}}\mathsf{v})(\tau Z) \sum_{k=1}^{n-1} {\mathsf{M}_k}^2 \big)\, d\tau + (n-2) \int_0^1 \tau^{n-2} (\mathsf{Du})(\tau Z)\, d\tau.$$

Since $\mathsf{v}(Z)$ is a harmonic function in a given domain S, we have:

$$\mathsf{D}((\overline{\mathsf{D}}\mathsf{v})(\tau Z)) = \mathsf{I}\tau(\Delta \mathsf{v})(\tau Z) = \mathsf{O}.$$

This and the equations

$$(\overline{\mathsf{D}}\mathsf{v})(\tau Z) + (\overline{\mathsf{D}}\mathsf{v})(\tau Z) \sum_{k=1}^{n-1} {\mathsf{M}_k}^2 = -(n-2)(\overline{\mathsf{D}}\mathsf{v})(\tau Z)$$

imply that $\mathsf{DU}(Z) = \mathsf{O}$. ■

Corollary 2. Problem (3.5), (3.6) in a star-shaped domain S has a solution when scalar Dirichlet problem for Laplace equation with respect to variables $x_1, x_2, \ldots, x_n$ has a solution with the same initial data.

Remark 3. Homogeneous boundary value problem (3.5), (3.6) has infinitely many linearly independent solutions. Really, any function $\mathsf{U}(Z) = \overline{\mathsf{D}}\tilde{\mathsf{U}}(Z)$, where

$$\tilde{\mathsf{U}}(Z) = \sum_{k=1}^{n-1} \mathsf{M}_k \mathsf{u}_{k+1},$$

is arbitrary harmonic function of variables Z, satisfies (3.5) and boundary conditions $Re\mathsf{U}\Big|_{\partial \mathsf{S}} = 0$.

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