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GENERALIZED EULER-KNOPP METHOD AND CONVERGENCE ACCELERATION

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Abstract. New propositions on λ-boundedness for generalized Euler-Knopp method of summability $(\mathcal{E}, T)$, where T is a linear bounded operator from Banach space X into X, are proved. Using these results are verified a proposition on convergence acceleration by $(\mathcal{E}, T)$ and a Tauberian remainder theorem for $(\mathcal{E}, T)$.

Key words: convergence acceleration, summability methods, Tauberian remainder theorems

1. Introduction and Lemmas

Let X, Y be Banach spaces and $\mathcal{L}(X, Y)$ be a space of all linear bounded operators from Banach space X into Y. A sequence $x = (\xi_k)$ $(\xi_k \in X)$ is called $\lambda-$ bounded if

$$\exists \lim \xi_k = \xi \,\wedge\, \beta_k = \lambda_k (\xi_k - \xi) \,\wedge\, \beta_k = O(1),$$

whereas $\lambda = (\lambda_k)$ with $0 < \lambda_k \nearrow$.

Let m_X^λ be the set of all $\lambda-$bounded sequences. A sequence $x = (\xi_k)$ is called summable (see [20] and [8]) by a generalized method $\mathcal{A} = (A_{nk})$, $A_{nk} \in \mathcal{L}(X, Y)$ if $y = (\eta_n)$ with

$$\eta_n = \sum_{k=0}^{\infty} A_{nk} \xi_k \tag{1.1}$$

is convergent. Let $\mu = (\mu_k)$ with $0 < \mu_k \nearrow$. The transformation $\mathcal{A}$ is called preserving λ-boundedness (see [6] and also [1, 2, 9, 14]) if

$$\mathcal{A} m_X^\lambda \subset m_Y^\lambda.$$

The transformation $\mathcal{A}$ is called accelerating λ-boundedness if

$$\mathcal{A}m_X^\lambda \subset m_Y^\mu \tag{1.2}$$

with $\lim \mu_k/\lambda_k = \infty$. A method $\mathcal{A} = (A_{nk})$ with $A_{nk} \in \mathcal{L}(X,X)$ is called regular if $\mathcal{A}c_X \subset c_X$ and

$$\lim_n \eta_n = \lim_k \xi_k,$$

while c_X is a set of convergent sequences with $\xi_k \in X$ and η_n is defined by (1.1). We denote by I and θ the identity and zero operator on any Banach space, respectively.

Kornfeld (see [10]) proved that any regular numerical method of summability can not universally accelerate the convergence. In [6] it is proved that any regular triangular generalized method $\mathcal{A}$ satisfying the condition

$$\sum_{k=0}^{n} A_{nk} = I \tag{1.3}$$

can not accelerate the convergence. Regardless of this fact in applied mathematics linear triangular methods are used to accelerate the convergence (see [16]). Such acceleration is possible in some subsets of m_X^λ. The present article is a sequel to the inquiries [6, 16, 17, 18, 19]. Main results of convergence acceleration using nonlinear methods are presented in [4].

Let us denote by $(\mathcal{E},T)$ or shortly $\mathcal{E}$ the generalized Euler-Knopp method of summability defined (see [3, 12, 18]) by

$$E_{nk} = \begin{cases} \binom{n}{k} T^k (I-T)^{n-k}, & (k = 0,1,\dots,n), \\ \theta, \quad (k > n), \end{cases} \tag{1.4}$$

where $T \in \mathcal{L}(X,X)$, while $T \neq \theta$ and $T^0 = I$.

To prove our propositions we use the following Lemmas. It is easy to prove Lemma 1 and Corollaries 1 and 2 in the same way as the analogical assertions (see [2]) are proved in the case of number matrices.

Lemma 1. *The product of generalized Euler-Knopp methods* $(\mathcal{E},U)$ *and* $(\mathcal{E},V)$, *where* $U,V \in \mathcal{L}(X,X)$, *is Euler-Knopp method* $(\mathcal{E},UV)$.

Corollary 1. If $T \in \mathcal{L}(X,X)$ and $m \in \mathbf{N}$, then $(\mathcal{E},T)^m = (\mathcal{E},T^m)$

Corollary 2. If $T \in \mathcal{L}(X,X)$ and $T^{-1} \in \mathcal{L}(X,X)$, then the $(\mathcal{E},T^{-1})$ is the inverse of the method $(\mathcal{E},T)$.

Let $\xi_n \in X$ and $x^{(\nu)} = \left(\xi_k^{(\nu)}\right)$, while

$$\xi_n^{(0)} = \xi_n, \quad \xi_n^{(\nu+1)} = \sum_{k=0}^{n} E_{nk}\xi_k^{(\nu)} \quad (\nu \in \mathbf{N}_0), \tag{1.5}$$

E_{nk} are determined by the use of (1.4) and $x^{(\nu+1)} = \mathcal{E}x^{(\nu)}$.

Analogically as in the case of number matrices (see [15]) it is possible to prove the next Lemma.

Lemma 2. *If* $(\mathcal{E}, T)$ *is a generalized Euler-Knopp method of summability defined by* (1.4), *and sequence* $x^{(\nu)} = \left(\xi_k^{(\nu)}\right)$ *is defined by* (1.5), *then*

$$\Delta\xi_n^{(\nu+1)} = \sum_{k=0}^{n} \frac{k}{n} \binom{n}{k} T^k (I-T)^{n-k} \Delta\xi_k^{(\nu)}, \tag{1.6}$$

while

$$\Delta\xi_n^{(\nu)} = \begin{cases} \xi_n^{(\nu)} - \xi_{n-1}^{(\nu)} & (n \in \boldsymbol{N}), \\ \xi_n^{(\nu)} & (n = 0). \end{cases}$$

Lemma 3. (see [12]). *Method* $(\mathcal{E}, T)$ *is regular if and only if*

$$\|T\| + \|I - T\| \leq 1, \quad \|I - T\| < 1. \tag{1.7}$$

Remark 1. As

$$1 = \|I\| = \|T + (I - T)\| \leq \|T\| + \|(I - T)\|,$$

then the first inequality in (1.7) implies

$$\|T\| + \|I - T\| = 1.$$

Remark 2. If $T = cI$ with $0 < c \leq 1$, then the method $(\mathcal{E}, T)$ is regular.

Lemma 4. (see [6]). *Let us have* $\mathcal{A} = (A_{nk})$, $A_{nk} \in \mathcal{L}(X, Y)$, *and* $e_X(\varsigma) := (\varsigma, \varsigma, \varsigma, \ldots)$ *with* $\varsigma \in X$. *If*

$$\exists \lim_n A_{nk} = A_k \quad (k \in \boldsymbol{N}_0) \tag{1.8}$$

in norm, then the conditions

$$\mathcal{A}e_X(\varsigma) \in m_Y^{\mu} \quad (\varsigma \in X), \quad \sum_k \lambda_k^{-1}\|A_k\| < \infty, \tag{1.9}$$

$$\mu_n \sum_k \lambda_k^{-1}\|A_{nk} - A_k\| = O(1) \tag{1.10}$$

are sufficient for the inclusion (1.2).

2. Convergence Preservation and Convergence Acceleration

Proposition 1. (see also [18]). *If* X *is a Banach space and* $(\mathcal{E}, T)$ *is determined by* (1.4) , *then the conditions* (1.7) *and*

$$\mu_n \|I - T\|^n \sum_{k=0}^{n} \binom{n}{k} \frac{1}{\lambda_k} \left(\frac{\|T\|}{\|I - T\|} \right)^k = O(1) \tag{2.1}$$

are sufficient for the inclusion

$$(\mathcal{E}, T)\, m_X^{\lambda} \subset m_X^{\mu}. \tag{2.2}$$

Proof. Let us verify the conditions of the Lemma 4 by fixing $\mathcal{A} = \mathcal{E}$. By Lemma 4 the conditions (1.7) are sufficient for the regularity of the method $\mathcal{E}$. As $\mathcal{E}$ is regular, then (see [12]) $A_k = \theta \ (k \in \mathbf{N}_0)$. The second condition (1.9) follows from $A_k = \theta \ (k \in \mathbf{N}_0)$. Using (1.1) and (1.4) we get for $\varsigma \in X$ that

$$\eta_n = \sum_{k=0}^{n} \binom{n}{k} T^k (I-T)^{n-k} \zeta = (T + (I-T))^n \varsigma = I\varsigma = \varsigma \quad (n \in \mathbf{N}_0).$$

So we have

$$\eta = \lim_n \eta_n = \lim_n \varsigma = \varsigma,$$
$$\mu_n (\eta_n - \eta) = \mu_n (\varsigma - \varsigma) = 0, \quad \mathcal{E}e_X(\varsigma) \in m_X^{\mu} \quad (\varsigma \in X).$$

That means the first condition (1.9) is satisfied. As $A_k = \theta \ (k \in \mathbf{N}_0)$, by the second condition (1.7) we get

$$\|A_{nk} - A_k\| = \|A_{nk}\| = \left\| \binom{n}{k} T^k (I-T)^{n-k} \right\|$$
$$\leq \binom{n}{k} \|T\|^k \|(I-T)\|^{n-k} \overset{n\to\infty}{\to} 0.$$

So the condition (1.8) is satisfied. The condition (1.10) follows from the condition (2.1). The conditions of Lemma 4 are satisfied and from (1.2) we get (2.2). This completes the proof. ■

Corollary 3. The conditions (1.7) and

$$\lambda_n \|I-T\|^n \sum_{k=0}^{n} \binom{n}{k} \frac{1}{\lambda_k} \left(\frac{\|T\|}{\|I-T\|} \right)^k = O(1) \tag{2.3}$$

are sufficient for

$$(\mathcal{E}, T)\, m_X^{\lambda} \subset m_X^{\lambda}. \tag{2.4}$$

Corollary 4. If

$$\|T\| = \|I-T\| = \frac{1}{2}, \tag{2.5}$$

then

$$\frac{\lambda_n}{2^n} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{\lambda_k} = O(1) \tag{2.6}$$

implies (2.4).

As any regular triangular generalized method $\mathcal{A}$, satisfying the condition (1.3) can not accelerate the convergence (see [6]), then the following assertion is valid.

Corollary 5. If the conditions (1.7) are satisfied, then the generalized Euler-Knopp method $(\mathcal{E}, T)$ can not accelerate the convergence.

Lemma 5. *The conditions*

$$\|T\| = \tau,\ \|I - T\| = 1 - \tau,\ 0 < \tau < 1 \tag{2.7}$$

and

$$\lambda_n = O(1)\,(n+1)^{O(1)} \tag{2.8}$$

imply (2.4).

Proof. It follows from (2.7) that conditions (1.7) are satisfied and the condition (2.3) takes a form

$$\sum_{k=0}^{n} \binom{n}{k} \frac{\lambda_n}{\lambda_k} \tau^k\,(1-\tau)^{n-k} = O(1). \tag{2.9}$$

The conditions $0 < \tau < 1$ and (2.8) imply (2.9) (see [15]). It follows from Corollary 3 that (2.4) is valid. ■

3. Tauberian Remainder Theorems

In [13] the first Tauberian theorems for the generalized methods of summability are proved. In [7] Tauberian theorems for semigroups are studied. In [5] statistical Tauberian theorems in metric spaces are examined. In [11] Tauberian conditions, under which statistical convergence follows from statistical summability, are studied. In [15] several Tauberian remainder theorems for Euler-Knopp methods in the case of number matrices and $X = \mathbf{R}$ are proved.

Proposition 2. *If the conditions*

$$0 < \varphi_n \uparrow, \quad \varphi_n/n \downarrow \ \ (n \in \boldsymbol{N}_0), \tag{3.1}$$

$$\lambda_n\,\varphi_n \left\|\Delta\xi_n^{(\nu)}\right\| = O(1) \tag{3.2}$$

and (2.3) *are satisfied, then*

$$\lambda_n\,\varphi_n \left\|\Delta\xi_n^{(\nu+1)}\right\| = O(1), \tag{3.3}$$

whereas $\xi_n^{(\nu+1)}$ *is defined by* (1.5).

Proof. Using (1.6) we get

$$\left\|\Delta\xi_n^{(\nu+1)}\right\| \leq \sum_{k=0}^{n} \frac{k}{n} \binom{n}{k}\ \|T\|^k\,\|I-T\|^{n-k} \left\|\Delta\xi_k^{(\nu)}\right\|. \tag{3.4}$$

Applying (3.4), (2.3) and (3.2) we obtain

$$\lambda_n \varphi_n \left\| \Delta \xi_n^{(\nu+1)} \right\| \le \sum_{k=0}^{n} \frac{\varphi_n}{n} \frac{k}{\varphi_k} \binom{n}{k} \frac{\lambda_n}{\lambda_k} \|T\|^k \|I - T\|^{n-k}$$
$$= O(1) \sum_{k=0}^{n} \binom{n}{k} \frac{\lambda_n}{\lambda_k} \|T\|^k \|I - T\|^{n-k} = O(1).$$

So the conditions (2.3) and (3.1)–(3.2) imply (3.3). ■

Corollary 6. The conditions (2.3), (3.1) and

$$\lambda_n \, \varphi_n \, \|\Delta \xi_n\| = O(1) \tag{3.5}$$

imply (3.2), whereas $\xi_n^{(\nu)}$ $(\nu \in \mathbf{N})$ is defined by (1.5).

Proposition 3. *Let* $\nu \in \boldsymbol{N}_0$. *If* $\lambda = (\lambda_k)$, $x^{(\nu)} = \left(\xi_k^{(\nu)}\right)$ *and* T *are satisfying the conditions* (2.8),

$$\lambda_n \sqrt{n+1} \left\| \Delta \xi_n^{(\nu)} \right\| = O(1), \tag{3.6}$$

$$(\mathcal{E}, T) \, x^{(\nu)} \in m_X^{\lambda} \tag{3.7}$$

and (2.5), *then* $x^{(\nu)} \in m_X^{\lambda}$.

Proof. Let $m = [n/2]$. If

$$\rho_1(n) = \lambda_n \sum_{k=0}^{m} \binom{2n}{k} T^k \, (I - T)^{2n-k} \left(\xi_k^{(\nu)} - \xi_n^{(\nu)}\right),$$

$$\rho_2(n) = \lambda_n \sum_{k=m+1}^{2n-m-1} \binom{2n}{k} T^k \, (I - T)^{2n-k} \left(\xi_k^{(\nu)} - \xi_n^{(\nu)}\right),$$

$$\rho_3(n) = \lambda_n \sum_{k=2n-m}^{2n} \binom{2n}{k} T^k \, (I - T)^{2n-k} \left(\xi_k^{(\nu)} - \xi_n^{(\nu)}\right)$$

and $\rho(n) = \rho_1(n) + \rho_2(n) + \rho_3(n)$ is equal to

$$\rho(n) = \lambda_n \left(\xi_{2n}^{(\nu+1)} - \xi_n^{(\nu)}\right). \tag{3.8}$$

If $0 \le k \le m$, then

$$\xi_k^{(\nu)} - \xi_n^{(\nu)} = - \sum_{i=k+1}^{n} \Delta \xi_i^{(\nu)}$$

and (3.6) implies

$$\left\| \xi_k^{(\nu)} - \xi_n^{(\nu)} \right\| = O(1). \tag{3.9}$$

Therefore using (2.5), (3.9) and Stirling's formula, we get

$$\|\rho_1(n)\| \leq \lambda_n \sum_{k=0}^{m} \binom{2n}{k} \|T\|^k \|I-T\|^{2n-k} \left\|\xi_k^{(\nu)} - \xi_n^{(\nu)}\right\|$$
$$= O\left(n\lambda_n 2^{-2n}\right) \sum_{k=0}^{m} \binom{2n}{k} = O\left(n^2 \lambda_n 2^{-2n}\right) \binom{2n}{m} = O\left(1\right).$$

If $m+1 \leq k \leq 2n-m-1$, then (3.6) and (2.8) imply

$$\left\|\xi_k^{(\nu)} - \xi_n^{(\nu)}\right\| = O(1) \frac{|n-k|}{\lambda_n \sqrt{n+1}}. \tag{3.10}$$

As

$$\sum_{k=0}^{m} \binom{2n}{k} |n-k| = n\binom{2n}{k},$$

then using (2.5), (3.10) and Stirling's formula we get

$$\|\rho_2(n)\| \leq \lambda_n \sum_{k=m+1}^{2n-m-1} \binom{2n}{k} \|T\|^k \|I-T\|^{2n-k} \left\|\xi_k^{(\nu)} - \xi_n^{(\nu)}\right\|$$
$$= \lambda_n 2^{-2n} \sum_{k=m+1}^{2n-m-1} \binom{2n}{k} O(1) \frac{|n-k|}{\lambda_n \sqrt{n+1}}$$
$$= O\left(2^{-2n}\right) \frac{1}{\sqrt{n+1}} \sum_{k=m+1}^{2n-m-1} \binom{2n}{k} |n-k| = O(1).$$

If $2n-m \leq k \leq 2n$, then (3.6) implies (3.9). Using (2.8), (2.5) , (3.9) and Stirling's formula we get

$$\|\rho_3(n)\| = O(1).$$

Therefore using (3.8) we obtain

$$\|\rho(n)\| = O(1). \tag{3.11}$$

As

$$\lambda_n \left(\xi_n^{(\nu)} - \xi\right) = \lambda_n \left(\xi_n^{(\nu)} - \xi_{2n}^{(\nu+1)}\right) + \lambda_n \left(\xi_{2n}^{(\nu+1)} - \xi\right),$$

while

$$\lim_{n\to\infty} \xi_{2n}^{(\nu+1)} = \xi,$$

then using the conditions (3.11), (3.7) and (2.8) we finish the proof. ■

Corollary 7. If $\nu \in \mathbf{N}_0$, $\lambda = (\lambda_k)$, $x = (\xi_k)$ and T are satisfying the conditions (2.8), (2.5) and

$$\lambda_n \sqrt{n+1} \, \|\Delta\xi_n\| = O(1), \quad (\mathcal{E}, T^n) \, x \in m_X^\lambda, \tag{3.12}$$

then $x \in m_X^\lambda$.

Proof. Using Corollary 6 and Proposition 3 we get step by step the assertion of Corollary 7. ■

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