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MULTIPLE SOLUTIONS OF THE FOURTH-ORDER EMDEN-FOWLER EQUATION

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Abstract. Two-point boundary value problems for the fourth-order Emden-Fowler equation are considered. If the given equation can be reduced to a quasi-linear one with a non-resonant linear part so that both equations are equivalent in some domain D, and if solution of the quasi-linear problem is located in D, then the original problem has a solution. We show that a quasi-linear problem has a solution of definite type which corresponds to the type of the linear part. If quasilinearization is possible for essentially different linear parts, then the original problem has multiple solutions.
Key words: quasi-linear equation, quasilinearization, conjugate point, i-nonresonant linear part, i-type solution

1. Introduction

Consider the nonlinear differential equation

$$x^{(4)} = f(t, x), \quad t \in I := [0, 1], \tag{1.1}$$

with the boundary conditions

$$x(0) = x'(0) = 0 = x(1) = x'(1). \tag{1.2}$$

Function $f : I \times \mathbb{R} \to \mathbb{R}$ is supposed to be continuous together with the partial derivative f_x. Then the unique solvability of the Cauchy problem

$$x(0) = x_0, \;\; x'(0) = x_1, \;\; x''(0) = x_2, \;\; x'''(0) = x_3$$

is ensured as well as the continuous dependence of solutions on initial data. Our research is motivated by the papers of R. Conti [1], L. Erbe [2], L. Jackson and K. Schrader [3], who studied oscillatory properties of solutions of two-point boundary value problems.

Consider also the quasi-linear equation

$$\big(L_4 x\big)(t) := x^{(4)} - k^4 x = F(t, x), \tag{1.3}$$

where $F, F_x, : I \times \mathbb{R} \to \mathbb{R}$ are continuous and F is bounded, that is, there exists $M \in (0, +\infty)$ such that

$$|F(t, x,)| < M \quad \forall (t, x) \in I \times \mathbb{R}.$$

If the linear part $\big(L_4 x\big)(t)$ is non-resonant with respect to the given boundary conditions (1.2), that is, the homogeneous problem $(L_4 x)(t) = 0$, (1.2) has only the trivial solution, then boundary-value problem (1.3), (1.2) is solvable. Suppose that equations (1.1) and (1.3) are equivalent in a domain

$$D(t, x) = \{(t, x);\ 0 \le t \le 1,\ |x| \le N\}.$$

If some solution $x(t)$ of the problem (1.3), (1.2) is located in this domain of equivalence $D(t, x)$, in other words, if $x(t)$ satisfies the estimate

$$|x(t)| \le N \quad \forall t \in I,$$

then it solves also the problem (1.1), (1.2). We will say for brevity that the problem (1.1), (1.2) allows a quasilinearization with respect to the domain $D(t, x)$ and the linear part is defined by $(L_4 x)(t)$.

If equation (1.1) can be reduced to another quasi-linear equation

$$\big(l_4 x\big)(t) = F_1(t,\ x),$$

which is equivalent to (1.1) in different domain $D_1(t, x)$, then the original problem (1.1), (1.2) in some cases has a solution $x_1(t) \in D_1(t, x)$. In this way we were able to obtain multiplicity results for the problem (1.1), (1.2). Similar approach was used in [8] for the second-order BVPs.

2. Quasi-Linear Problems and Types of Solutions

First we prove results for quasi-linear problems of the type (1.3), (1.2) if the following condition is satisfied for any $(t,\ x)$

$$k^4 + \frac{\partial F}{\partial x}(t,\ x) > 0. \tag{2.1}$$

In our investigation we use the oscillation theory developed by Leighton-Nehari [5] for the fourth-order linear differential equations

$$x^{(4)} - p(t)x = 0, \qquad p(t) > 0. \tag{2.2}$$

We use their definition of a conjugate point.

Definition 1. A point η is called a *conjugate point* for the point $t = 0$, if there exists a nontrivial solution $x(t)$ of equation (2.2) such that

$$x(0) = x'(0) = 0 = x(\eta) = x'(\eta).$$

For example, if the linear equation is $x^{(4)} - k^4 x = 0$, then conjugate points η satisfy the nonlinear equation

$$\cos k\eta \cosh k\eta = 1.$$

The conjugate points (or double zeros) in the oscillation theory for the fourth-order linear differential equations play the same role as the ordinary zeros in the oscillation theory for the second-order equations.

We define i-nonresonanse of the linear part and an i-type solution similarly as for the second-order quasi-linear problems [7, 8].

Definition 2. The linear part $(L_4 x)(t) := x^{(4)} - k^4 x$ is i-nonresonant with respect to the boundary conditions (1.2), if there are exactly i conjugate points in the interval $(0, 1)$ and $t = 1$ is not a conjugate point.

For example, the linear part $\big(L_4 x\big)(t) := x^{(4)} - k^4 x$ is $(n-1)$-nonresonant for any $k = \pi n$, $n \geq 1$.

Definition 3. Function $\xi(t)$ is an i-type solution of problem (1.3), (1.2), if for small enough $\alpha, \beta > 0$ the difference

$$u(t; \alpha, \beta) = x(t; \alpha, \beta) - \xi(t)$$

has exactly i double zeros (or conjugate points) in the interval $(0, 1)$ and $u(1; \alpha, \beta) \neq 0$, where $x(t; \alpha, \beta)$ is a solution of (1.3), which satisfies the initial conditions

$$x(0; \alpha, \beta) = \xi(0), \quad x'(0; \alpha, \beta) = \xi'(0),$$
$$x''(0; \alpha, \beta) = \xi''(0) + \alpha, \quad x'''(0; \alpha, \beta) = \xi'''(0) - \beta.$$

In what follows we call the solution $x(t; \alpha, \beta)$ a ***neighbouring*** solution.

Remark 1. An i-type solution $\xi(t)$ of the problem (1.3), (1.2) has the following characteristics in terms of the variational equation: if a linear equation of variations

$$y^{(4)} - k^4 y = F_x(t, \xi(t))y$$

has exactly i conjugate points in the interval $(0, 1)$ and $t = 1$ is not a conjugate point, then $\xi(t)$ is an i-type solution. However, if $t = 1$ is a conjugate point, then $\xi(t)$ may be an i-type solution, or it may be an $(i+1)$-type solution, or its type may be indefinite. The respective examples can be constructed.

The following theorem is valid (see [6, 9]).

Theorem 1. *The quasi-linear problem* (1.3), (1.2) *has an i-type solution, if the condition* (2.1) *is fulfilled and the linear part* $(L_4 x)(t) = x^{(4)} - k^4 x$ *is i-nonresonant.*

3. Green's Function

Consider the homogeneous differential equation with the linear part $(L_4x)(t)$

$$x^{(4)} - k^4 x = 0,$$

where k satisfies the non-resonance condition $\cos k \cosh k \neq 1$. We have constructed the Green function for the oscillatory fourth-order linear problem

$$\begin{cases} x^{(4)} - k^4 x = 0, \\ x(0) = x'(0) = 0 = x(1) = x'(1) \end{cases} \tag{3.1}$$

and we give the respective formula and the estimate of the Green function below.

Proposition 1. *The Green function of the problem* (3.1) *can be written in the form*

$$G_k(t,s) = \begin{cases} \frac{1}{\triangle}\Big(- u^*(t,s)\cdot v(1) - u(1)\cdot v^*(t,s) + \sum\limits_{\tau=s,\,t} \big[u(\tau)\cdot v(t+s-\tau) \\ -u(\tau-1)\cdot v(t+s-1-\tau) - u(t-\tau)\cdot v(\tau-s)\big]\Big), \ 0 \le s \le t \le 1, \\ \frac{1}{\triangle}\Big(- u^*(s,t)\cdot v(1) - u(1)\cdot v^*(s,t) + \sum\limits_{\tau=s,\,t} \big[u(\tau)\cdot v(t+s-\tau) \\ -u(\tau-1)\cdot v(t+s-1-\tau) + u(t-\tau)\cdot v(\tau-s)\big]\Big), \ 0 \le t < s \le 1, \end{cases} \tag{3.2}$$

where $\triangle = 4k^3(\cos k \cosh k - 1)$ *and* u, v *are vector-functions such that*

$$\begin{aligned} &u(\tau) = [-\sin k\tau,\, \cos k\tau], \quad v(\tau) = [\cosh k\tau,\, \sinh k\tau], \\ &u^*(t,s) = [-\sin k(s-t+1),\, \cos k(t+s-1)], \\ &v^*(t,\,s) = [\cosh\ k(t+s-1),\, \sinh k(s-t+1)], \end{aligned}$$

and $u \cdot v$ *denotes the scalar product.*

Proof. The Green function is constructed as an element of Green's matrix by reducing the linear problem (3.1) to a matrix form

$$\begin{cases} X'(t) - PX(t) = 0, \\ A_1X(0) + A_2X(1) = 0, \end{cases} \tag{3.3}$$

where $X : I \to \mathbb{R}^4$, $P, A_i \in \mathbb{R}^{4\times 4}$ $(i = 1,2)$. The Green matrix can be obtained by formula

$$\mathbb{G}(t,s) = \begin{cases} Y(t)\big(A_1Y(0) + A_2Y(1)\big)^{-1}A_1Y(0)Y^{-1}(s), \quad 0 \le s \le t \le 1, \\ -Y(t)\big(A_1Y(0) + A_2Y(1)\big)^{-1}A_2Y(1)Y^{-1}(s), \quad 0 \le t < s \le 1, \end{cases}$$

where $Y(t)$ $(Y : I \to \mathbb{R}^{4\times 4})$ is a fundamental matrix of system in (3.3) [4]. ■

Proposition 2. *Function $G_k(t, s)$ can be estimated by*

$$\big|G_k(t,s)\big| \le \Gamma_k := \frac{(5+\sqrt{2})\sqrt{\cosh 2k} + \sinh k + 1}{4k^3\big|\cos k \cosh k - 1\big|}. \tag{3.4}$$

Proof. The proof follows from a property of the scalar product $|u\cdot v| \le |u|\,|v|$ taking into consideration that

$$|u(\tau)| \le 1, \quad |v(\tau)| \le \sqrt{\cosh 2k},$$
$$|u^*(t,s)| \le \sqrt{2}, \quad |v^*(t,s)| \le \sqrt{\cosh 2k}.$$

■

We can improve this estimate for some numbers k. For instance, if $k = \pi n$, $(n = 1, 2, \ldots)$ Green's function $G_k(t, s)$ can be simplified. We express hyperbolic sine and cosine in terms of the exponential functions and obtain the following estimates

$$\big|G_k(t,s)\big| \le \frac{(1+\sqrt{2})e^k}{k^3(e^k+1)} =: \Gamma_1(k), \quad k = (2n-1)\pi, \tag{3.5}$$

$$\big|G_k(t,s)\big| \le \frac{(1+\sqrt{2})e^k}{k^3(e^k-1)} =: \Gamma_2(k), \quad k = 2n\pi. \tag{3.6}$$

4. The Emden-Fowler Equation

We apply the obtained estimates (3.5), (3.6) and Theorem 1 to the Emden-Fowler type equation

$$x^{(4)} = \lambda^2 |x|^p \operatorname{sign} x \tag{4.1}$$

with the boundary conditions (1.2), where $\lambda \neq 0$, $p > 0$, $p \neq 1$.

Theorem 2. *If there exists some k in the form $k = \pi i$, $(i = 1, 2, \ldots)$, which satisfies one of the following inequalities*

$$k\frac{(1+\sqrt{2})e^k}{(e^k+1)} < \beta \frac{p^{\frac{p}{p-1}}}{|p-1|}, \quad \textit{for} \quad k = (2n-1)\pi, \tag{4.2}$$

$$k\frac{(1+\sqrt{2})e^k}{(e^k-1)} < \beta \frac{p^{\frac{p}{p-1}}}{|p-1|}, \quad \textit{for} \quad k = 2n\pi, \tag{4.3}$$

where β is a positive root of the equation $\beta^p = \beta + (p-1)\cdot p^{\frac{p}{1-p}}$, then there exists an $(i-1)$-type solution of problem (4.1), (1.2).

Proof. Let us consider instead of equation (4.1) the equivalent one

$$x^{(4)} - k^4 x = \lambda^2 |x|^p \operatorname{sign} x - k^4 x,$$

where k satisfies $\cos k \cosh k \neq 1$. Denote

$$f_k(x) := \lambda^2 |x|^p \operatorname{sign} x - k^4 x.$$

We can calculate the value of function $f_k(x)$ at the extremum point x_{extr}. Set

$$M_k = |f_k(x_{extr})| = \lambda^{\frac{2}{1-p}} \left(\frac{k^4}{p}\right)^{\frac{p}{p-1}} |p-1|. \tag{4.4}$$

Choose $N_k > 0$ such that

$$|x(t)| \leq N_k \;\Rightarrow\; |f_k(x)| \leq M_k, \quad \forall t \in I$$

(similar type arguments were used in [8] for the second-order problems). Computations give that

$$N_k = \left(\frac{k^4}{\lambda^2}\right)^{\frac{1}{p-1}} \beta, \tag{4.5}$$

where β is a positive root of the equation

$$\beta^p = \beta + (p-1)p^{\frac{p}{1-p}}.$$

Next we consider the quasi-linear equation

$$x^{(4)} - k^4 x = \varphi(x) \cdot \{\lambda^2 \cdot |x|^p \operatorname{sign} x - k^4 x\} =: F_k(x), \tag{4.6}$$

where $F_k(x) := \varphi(x) f_k(x)$. Function $\varphi(x)$ satisfies

$$\varphi(x) = \begin{cases} 1, & |x| \leq N_k, \\ 0, & |x| \geq N_k + \varepsilon_1 \end{cases}$$

and $0 < \varphi(x) < 1$ for remaining values of x. Then

$$\max_{t \in I, x \in \mathbb{R}} \big|F_k(x)\big| \leq M_k + \varepsilon_2,$$

moreover ε_1 and ε_2 can be made arbitrarily small. So it can be assumed that function $F_k(x)$ is smooth and bounded by M_k.

Quasi-linear problem (4.6), (1.2) can be written in the integral form

$$x(t) = \int_0^1 G_k(t,s) F_k(x(s))\, ds,$$

where $G_k(t,s)$ is the Green function given by (3.2). Then

$$\big|x(t)\big| \leq \Gamma_k M_k,$$

here Γ_k is an estimate of Green's function in (3.4). If moreover the inequality

$$\Gamma_k M_k < N_k \tag{4.7}$$

holds, then equations (4.1) and (4.6) are equivalent in the domain

$$\Omega_k = \{(t, x) : \ 0 \le t \le 1, \ |x| < N_k\}.$$

In other words if inequality (4.7) holds, then the original problem (4.1), (1.2) allows for quasilinearization with respect to the domain Ω_k and the linear part $(L_4 x)(t) = x^{(4)} - k^4 x$.

Notice that in the domain of equivalence Ω_k condition (2.1) is fulfilled (i.e. $k^4 + \dfrac{dF_k}{dx} > 0$). So it follows from Theorem 1 that if the linear part $(L_4 x)(t) = x^{(4)} - k^4 x$ is i-nonresonant, then the quasi-linear problem (4.6), (1.2) has an i-type solution, if moreover the inequality (4.7) holds, then the original problem (4.1), (1.2) also has an i-type solution.

Let us consider values k of the form $k = \pi i, i \ge 1$. For such k the linear part $(L_4 x)(t) = x^{(4)} - k^4 x$ is $(i-1)$-nonresonant and the Green function $G_k(t, s)$ satisfies either the estimate $\Gamma_1(k)$ (see (3.5)) or $\Gamma_2(k)$ (see (3.6)). It follows from (4.4), (4.5), (3.5), (3.6) that the inequality (4.7) reduces respectively either to (4.2) or (4.3). The proof is complete. ■

Corollary 1. If there exist $k = \pi i$, $i = 1, 2, \ldots, m$, which satisfies inequalities (4.2), (4.3), then there exist at least m solutions of different types of problem (4.1), (1.2).

5. Example

In Table 1 given in Appendix the results of calculations are provided. They show that certain k given in the form $k = \pi n$, $n = 1, 2 \ldots$ satify inequalities (4.2) and (4.3). For instance, if $p = \frac{8}{7}$, then there exist three values of k ($k = \pi$, $k = 2\pi$, $k = 3\pi$), which satisfy the inequalities above, that means that there exist at least three solutions of different types.

We have computed different solutions for the problem

$$\begin{cases} x^{(4)} = 810|x|^{\frac{8}{7}} \operatorname{sign} x, \\ x(0) = x'(0) = 0 = x(1) = x'(1). \end{cases} \tag{5.1}$$

The solid line in Figure 1 indicates a trivial solution of problem (5.1) and dashed line presents the corresponding neighbouring solution (see Definition 3). Their difference has no double zeros (the conjugate points) in the interval (0, 1), so a trivial solution is a 0-type solution.

Figure 2*a* illustrates the second solution of problem (5.1) (solid line). It is a 1-type solution, because the difference between neighboring solution (dashed line) and this solution has exactly one double zero (conjugate point) in some point of the open interval (0, 1) (see Figure 2*b*). The initial data of the 1-type solution is given by

$$x''(0) = 1.1, \quad x'''(0) = -5.03461937.$$

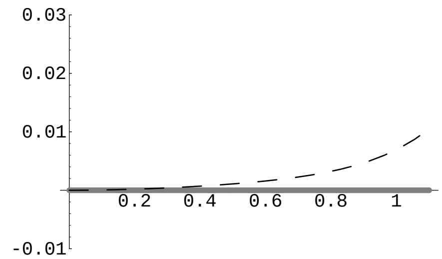
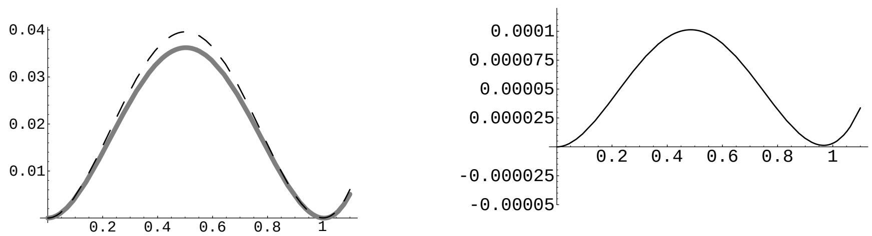
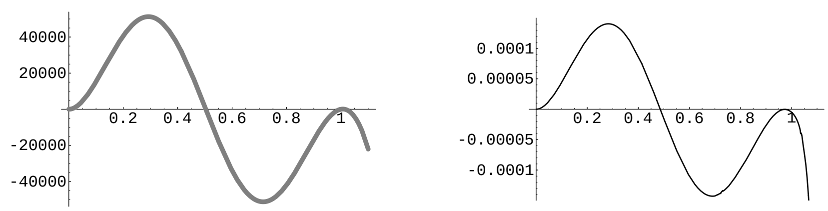


Figure 1. 0-type solution of the problem (5.1).



a b

Figure 2. a) 1-type solution of the problem (5.1), b) the difference between neighboring solution and 1-type solution



a b

Figure 3. a - 2-type solution of the problem (5.1), b - difference between neighboring solution and 2-type solution

Figure 3*a* illustrates a 2-type solution of problem (5.1). It is difficult to show the graph of the respective neighboring solution, because two lines almost coincide. Nevertheless, the difference between neighboring solution and this solution is presented in Figure 3 *b* and it has one simple zero and one double zero in the open interval (0, 1). The initial data of the 2-type solution is essentially different from previous one:

$$x''(0) = 4099959.008, \quad x'''(0) = -31634999.21.$$

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Appendix

Table 1. Numbers β and values of k of the form $\pi i\,(i = 1,\, 2\,\ldots)$ for certain values of p.

p	β	k_i
$p = \frac{4}{5}$	$\beta \approx 1.3632$	$k_1 = \pi;\ k_2 = 2\pi$
$p = \frac{5}{6}$	$\beta \approx 1.3553$	$k_1 = \pi;\ k_2 = 2\pi$
$p = \frac{6}{7}$	$\beta \approx 1.3499$	$k_1 = \pi;\ k_2 = 2\pi;\ k_3 = 3\pi$
$p = \frac{7}{8}$	$\beta \approx 1.3461$	$k_1 = \pi;\ k_2 = 2\pi;\ k_3 = 3\pi$
$p = \frac{8}{9}$	$\beta \approx 1.3431$	$k_1 = \pi;\ k_2 = 2\pi;\ k_3 = 3\pi;\ k_4 = 4\pi$
$p = \frac{9}{10}$	$\beta \approx 1.3407$	$k_1 = \pi;\ k_2 = 2\pi;\ k_3 = 3\pi;\ k_4 = 4\pi$
$p = \frac{10}{11}$	$\beta \approx 1.3388$	$k_1 = \pi;\ k_2 = 2\pi;\ k_3 = 3\pi;\ k_4 = 4\pi;\ k_5 = 5\pi$
$p = \frac{11}{12}$	$\beta \approx 1.3373$	$k_1 = \pi;\ k_2 = 2\pi;\ k_3 = 3\pi;\ k_4 = 4\pi;\ k_5 = 5\pi$
$p = \frac{12}{13}$	$\beta \approx 1.3359$	$k_1 = \pi;\ k_2 = 2\pi;\ k_3 = 3\pi;\ k_4 = 4\pi;\ k_5 = 5\pi$
...	...	...
$p = \frac{13}{12}$	$\beta \approx 1.3065$	$k_1 = \pi;\ k_2 = 2\pi;\ k_3 = 3\pi;\ k_4 = 4\pi;\ k_5 = 5\pi$
$p = \frac{12}{11}$	$\beta \approx 1.3053$	$k_1 = \pi;\ k_2 = 2\pi;\ k_3 = 3\pi;\ k_4 = 4\pi;\ k_5 = 5\pi$
$p = \frac{11}{10}$	$\beta \approx 1.3038$	$k_1 = \pi;\ k_2 = 2\pi;\ k_3 = 3\pi;\ k_4 = 4\pi$
$p = \frac{10}{9}$	$\beta \approx 1.3019$	$k_1 = \pi;\ k_2 = 2\pi;\ k_3 = 3\pi;\ k_4 = 4\pi$
$p = \frac{9}{8}$	$\beta \approx 1.2998$	$k_1 = \pi;\ k_2 = 2\pi;\ k_3 = 3\pi$
$p = \frac{8}{7}$	$\beta \approx 1.2969$	$k_1 = \pi;\ k_2 = 2\pi;\ k_3 = 3\pi$
$p = \frac{7}{6}$	$\beta \approx 1.2933$	$k_1 = \pi;\ k_2 = 2\pi;\ k_3 = 3\pi$
$p = \frac{6}{5}$	$\beta \approx 1.2884$	$k_1 = \pi;\ k_2 = 2\pi$
$p = \frac{5}{4}$	$\beta \approx 1.2813$	$k_1 = \pi;\ k_2 = 2\pi$