Mathematical Modelling and Analysis
Volume 9 Number 2, 2004, pages 169–178
 ISSN 1392-6292

MONOTONE AND CONSERVATIVE DIFFERENCE SCHEMES FOR ELLIPTIC EQUATIONS WITH MIXED DERIVATIVES[1]

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Received November 13, 2003; revised January 15, 2004

Abstract. In the paper elliptic equations with alternating-sign coefficients at mixed derivatives are considered. For such equations new difference schemes of the second order of approximation are developed. The proposed schemes are conservative and monotone. The constructed algorithms satisfy the grid maximum principle not only for coefficients of constant signs but also for alternating-sign coefficients at mixed derivatives. The *a priori* estimates of stability and convergence in the grid norm C are obtained.

Key words: monotone difference scheme, conservative difference scheme, elliptic equations, mixed derivatives, grid maximum principle

1. Introduction

For the development of difference schemes of the high order of approximation it is important to save properties of both monotonity and conservativeness because monotone schemes lead to the well-posed systems of algebraic equations. Iterative methods converge significantly better in the case of diagonally dominant matrices.

Problems of the development of difference schemes for equations with mixed derivatives were studied in papers [1, 2, 4, 11]. The conservative difference schemes for elliptic equations with mixed derivatives were considered in [5, p. 286], [6, p. 175], but these schemes do not satisfy the grid maximum principle. For elliptic and parabolic equations with mixed derivatives the monotone and conservative difference schemes were proposed in papers [7, 8, 10], but these schemes can be used only in the case of constant-sign coefficients. If coefficients at mixed derivatives changed their sign, then differential equation was rewritten in non-divergent

[1] The author thanks Prof. Oleg Iliev and Prof. Raimondas Čiegis for the statement of the problem, Prof. Piotr Matus and Dr. Mikhail Chuiko for the discussion and useful comments

form with first derivatives and monotone schemes were developed by means of the regularization principle [5, p. 183]. But after such a transformation the property of conservativeness was lost. Such situation is typical in theory of difference schemes.

In the present paper, for elliptic equations with mixed derivatives new monotone and conservative difference schemes for both constant-sign and alternating-sign coefficients are proposed. The main idea is based on using the stencil functionals with absolute values of the coefficients at mixed derivatives. For proposed difference schemes the *a priori* estimates of stability and convergence in the grid norm C are obtained. Numerical experiments confirm the theoretical results.

2. Difference scheme

In the rectangle $\bar{G} = \{0 \le x_\alpha \le l_\alpha, \quad \alpha = 1, 2\}$ with the boundary Γ we consider the Dirichlet problem for elliptic equations with mixed derivatives

$$\begin{cases} Lu - q(x)u = -f(x), & x \in G\,, \\ u = \mu(x), & x \in \Gamma, \quad x = (x_1, x_2)\,, \end{cases} \tag{2.1}$$

where

$$Lu = \sum_{\alpha,\beta=1}^{2} L_{\alpha\beta}u, \quad L_{\alpha\beta}u = \frac{\partial}{\partial x_\alpha}\left(k_{\alpha\beta}(x)\frac{\partial u}{\partial x_\beta}\right), \quad q(x) \ge c_0 > 0.$$

We suppose that the following ellipticity conditions are satisfied

$$c_1 \sum_{\alpha=1}^{2} \xi_\alpha^2 \le \sum_{\alpha,\beta=1}^{2} k_{\alpha\beta}(x)\xi_\alpha\xi_\beta \le c_2 \sum_{\alpha=1}^{2} \xi_\alpha^2, \quad x \in G, \tag{2.2}$$

where $c_1,\ c_2 > 0$ are positive constants, $\xi = (\xi_1, \xi_2)$ is an arbitrary nonzero vector.

In the rectangle $\bar{G}$ we consider the uniform grid $\bar{\omega}_h = \omega_h \cup \gamma_h$:

$$\bar{\omega}_h = \{x = (x_1^{(i_1)}, x_2^{(i_2)}):\ x_\alpha^{(i_\alpha)} = i_\alpha h_\alpha,\ h_\alpha N_\alpha = l_\alpha,\ i_\alpha = \overline{0, N_\alpha},\ \alpha = 1, 2\},$$

where ω_h is the set of inner grid nodes, γ_h is the set of boundary grid nodes.

Further we will use the following notations of the theory of difference schemes [5]:

$$v^{(\pm 1\alpha)} = v(x_\alpha^{(i_\alpha)} \pm h_\alpha,\, x_{3-\alpha}^{(i_{3-\alpha})}), \quad \alpha = 1, 2,$$
$$y = y(x_1^{(i_1)}, x_2^{(i_2)}), \quad y_{\bar{x}_\alpha} = \frac{y - y^{(-1\alpha)}}{h_\alpha}, \quad y_{x_\alpha} = \frac{y^{(+1\alpha)} - y}{h_\alpha}\,.$$

On the grid $\bar{\omega}_h$ we approximate differential problem (2.1) by the difference scheme

$$\begin{cases} \Lambda y - dy = -\varphi, & x \in \omega_h\,, \\ y = \mu(x), & x \in \gamma_h\,, \end{cases} \tag{2.3}$$

where

$$\Lambda y = \sum_{\alpha,\,\beta=1}^{2} \Lambda_{\alpha\beta} y, \quad \Lambda_{\alpha\alpha} y = (a_{\alpha\alpha} y_{\bar{x}_\alpha})_{x_\alpha}, \quad \alpha = 1, 2\,,$$

$$\Lambda_{\alpha\beta} y = \frac{1}{4}\left((a^-_{\alpha\beta} y_{\bar{x}_\beta})_{x_\alpha} + (a^{-(+1\alpha)}_{\alpha\beta} y_{x_\beta})_{\bar{x}_\alpha} + (a^+_{\alpha\beta} y_{x_\beta})_{x_\alpha} + (a^{+(+1\alpha)}_{\alpha\beta} y_{\bar{x}_\beta})_{\bar{x}_\alpha} \right),$$

$$a^-_{\alpha\beta} = a_{\alpha\beta} - |a_{\alpha\beta}|, \quad a^+_{\alpha\beta} = a_{\alpha\beta} + |a_{\alpha\beta}|, \quad \alpha \neq \beta\,.$$

Here $d \geq c_0$, φ are some stencil functionals of the coefficient q and the right-hand side f respectively. The stencil functionals $a_{\alpha\beta}$ can be chosen as follows

$$a_{\alpha\beta} = k_{\alpha\beta\, i_\alpha - \frac{1}{2},\, i_\beta} = k_{\alpha\beta}(x_\alpha - 0.5h_\alpha, x_\beta)\,,$$

$$a_{\alpha\beta} = \frac{k_{\alpha\beta\, i_\alpha,\, i_\beta} + k_{\alpha\beta\, i_\alpha - 1,\, i_\beta}}{2} = \frac{k_{\alpha\beta} + k^{(-1\alpha)}_{\alpha\beta}}{2}\,,$$

$$a_{\alpha\beta} = \frac{2k_{\alpha\beta} k^{(-1\alpha)}_{\alpha\beta}}{k_{\alpha\beta} + k^{(-1\alpha)}_{\alpha\beta}}, \quad \alpha,\, \beta = 1, 2\,.$$

A difference scheme is called *conservative* (divergent), if we have algebraic sums of unknowns or functions of them only along the boundary after summation of the scheme equations over all grid nodes of the domain [3, p. 280]. If we sum up difference scheme (2.3) over grid nodes of the domain ω_h, we obtain algebraic sums of functions only along the boundary Γ. Hence, the proposed scheme is conservative.

We consider $a_{\alpha\beta} = k_{\alpha\beta}(x_\alpha - 0.5h_\alpha, x_\beta)$ and show that the grid operator Λ approximates the differential operator L with the second order. Let the coefficients $k_{\alpha\beta}(x)$ of equation (2.1), all partial derivatives up to the third order inclusively of the coefficients and up to the fourth order inclusively of the solution $u(x)$ be bounded. By using Taylor expansion of the functions $\Lambda_{\alpha\beta} u$ in the neighbourhood of the point $x \in \omega_h$, we obtain

$$\Lambda_{\alpha\alpha} u - L_{\alpha\alpha} u = O(h_1^2 + h_2^2) = O(|h|^2), \quad \alpha = 1, 2,$$

$$\Lambda_{\alpha\beta} u - L_{\alpha\beta} u = \frac{h_\beta}{4} \frac{\partial^3 u}{\partial x_\alpha \partial x_\beta^2} \left(\left| k_{\alpha\beta} + \frac{h_\alpha}{2} \frac{\partial k_{\alpha\beta}}{\partial x_\alpha} \right| - \left| k_{\alpha\beta} - \frac{h_\alpha}{2} \frac{\partial k_{\alpha\beta}}{\partial x_\alpha} \right| \right) + O(|h|^2), \quad \alpha \neq \beta\,.$$

Using the inequality $\big| |a+b| - |a-b| \big| \leq 2|b|$, we have

$$|\Lambda_{\alpha\beta} u - L_{\alpha\beta} u| \leq \frac{h_1 h_2}{4} \left| \frac{\partial k_{\alpha\beta}}{\partial x_\alpha} \right| \left| \frac{\partial^3 u}{\partial x_\alpha \partial x_\beta^2} \right| + O(|h|^2) = O(|h|^2).$$

Hence,

$$\Lambda_{\alpha\beta} u - L_{\alpha\beta} u = O(|h|^2), \quad \alpha \neq \beta.$$

We suppose that the stencil functionals $d(x)$ and $\varphi(x)$ satisfy the usual conditions of approximation of the coefficient $q(x)$ and the right-hand side $f(x)$ with the second order

$$d(x) - q(x) = O(|h|^2), \quad \varphi(x) - f(x) = O(|h|^2).$$

So, difference scheme (2.3) approximates differential problem (2.1) with the second order. The stencil of difference scheme (2.3) is presented in Fig. 1.

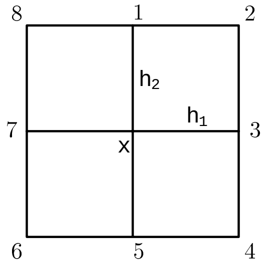

Figure 1. Stencil of difference scheme (2.3).

3. Grid maximum principle

To obtain the *a priori* estimates of stability in the grid norm C with respect to the right-hand side and the boundary conditions we will use the grid maximum principle [5, p. 258]. Therefore, we have to reduce the difference scheme to the canonical form

$$A(x)y(x) = \sum_{\xi \in S'(x)} B(x,\xi)y(\xi) + F(x), \quad x \in \bar{\omega}_h, \tag{3.1}$$

and verify the following sufficient conditions on the coefficients

$$A(x) > 0, \; B(x,\xi) \geq 0, \; D(x) = A(x) - \sum_{\xi \in S'(x)} B(x,\xi) > 0, \quad x \in \bar{\omega}_h. \tag{3.2}$$

Here $A(x)$, $B(x,\xi)$, $F(x)$ are the known grid functions, $S'(x) = S(x) \setminus \{x\}$, $S(x)$ is the stencil of the scheme.

Theorem 1. *Let us suppose that conditions* (3.2) *of the coefficients positivity are satisfied. Then for the solution of problem* (3.1) *the following a priori estimate is valid*

$$\|y\|_{\bar{C}} \leq \max\left\{ \left\| \frac{F}{D} \right\|_{C_\gamma}, \left\| \frac{F}{D} \right\|_{C} \right\}, \tag{3.3}$$

where $\|v\|_{\bar{C}} = \max_{x \in \bar{\omega}_h} |v(x)|, \quad \|v\|_C = \max_{x \in \omega_h} |v(x)|, \quad \|v\|_{C_\gamma} = \max_{x \in \gamma_h} |v(x)|$.

Let us number the nodes of the stencil of difference scheme (2.3) according to Fig. 1 and reduce the scheme to canonical form (3.1):

$$Ay = \sum_{k=1}^{8} B_k y_k + F, \quad y_k = y(x_k), \; x_k \in S'(x).$$

If $x \in \omega_h$, then values of the coefficients are defined by the following formulas

$$A = \frac{a_{11} + a_{11}^{(+11)}}{h_1^2} - \frac{|a_{12}| + |a_{12}^{(+11)}| + |a_{21}| + |a_{21}^{(+12)}|}{2h_1h_2} + \frac{a_{22} + a_{22}^{(+12)}}{h_2^2} + d,$$

$$B_1 = \frac{a_{22}^{(+12)}}{h_2^2} - \frac{|a_{12}| + a_{12} + |a_{12}^{(+11)}| - a_{12}^{(+11)} + 2|a_{21}^{(+12)}|}{4h_1h_2},$$

$$B_2 = \frac{|a_{12}^{(+11)}| + a_{12}^{(+11)} + |a_{21}^{(+12)}| + a_{21}^{(+12)}}{4h_1h_2} \geq 0,$$

$$B_3 = \frac{a_{11}^{(+11)}}{h_1^2} - \frac{2|a_{12}^{(+11)}| + |a_{21}| + a_{21} + |a_{21}^{(+12)}| - a_{21}^{(+12)}}{4h_1h_2},$$

$$B_4 = \frac{|a_{12}^{(+11)}| - a_{12}^{(+11)} + |a_{21}| - a_{21}}{4h_1h_2} \geq 0,$$

$$B_5 = \frac{a_{22}}{h_2^2} - \frac{|a_{12}| - a_{12} + |a_{12}^{(+11)}| + a_{12}^{(+11)} + 2|a_{21}|}{4h_1h_2},$$

$$B_6 = \frac{|a_{12}| + a_{12} + |a_{21}| + a_{21}}{4h_1h_2} \geq 0,$$

$$B_7 = \frac{a_{11}}{h_1^2} - \frac{2|a_{12}| + |a_{21}| - a_{21} + |a_{21}^{(+12)}| + a_{21}^{(+12)}}{4h_1h_2},$$

$$B_8 = \frac{|a_{12}| - a_{12} + |a_{21}^{(+12)}| - a_{21}^{(+12)}}{4h_1h_2} \geq 0,$$

$$D = d \geq c_0 > 0, \qquad F = \varphi\,.$$

For $x \in \gamma_h$, the coefficients of the canonical form are given by:

$$A = 1, \quad B = 0, \quad D = 1, \quad F = \mu\,.$$

Further we will assume that the following condition is satisfied

$$\max\{k_1,\ k_2\} \leq \frac{h_1}{h_2} \leq \min\{k_3,\ k_4\}, \tag{3.4}$$

where

$$k_1 = \frac{|a_{12}| - a_{12} + |a_{12}^{(+11)}| + a_{12}^{(+11)} + 2|a_{21}|}{4a_{22}}\,,$$

$$k_2 = \frac{|a_{12}| + a_{12} + |a_{12}^{(+11)}| - a_{12}^{(+11)} + 2|a_{21}^{(+12)}|}{4a_{22}^{(+12)}}\,,$$

$$k_3 = \frac{4a_{11}}{2|a_{12}| + |a_{21}| - a_{21} + |a_{21}^{(+12)}| + a_{21}^{(+12)}}\,,$$

$$k_4 = \frac{4a_{11}^{(+11)}}{2|a_{12}^{(+11)}| + |a_{21}| + a_{21} + |a_{21}^{(+12)}| - a_{21}^{(+12)}}\,.$$

Lemma 1. *Let coefficients of differential equation* (2.1) *satisfy the following inequality*

$$k_{\alpha\alpha} \geq |k_{\alpha\beta}^{(\pm 1\alpha,\pm 1\beta)}|, \quad \alpha,\beta = 1,2. \tag{3.5}$$

If we choose $h_1 = h_2 = h$, then condition (3.4) *is always satisfied.*

Proof. Let condition (3.5) be satisfied and $h_1 = h_2$. In order to prove that in this case condition (3.4) is always satisfied, we have to show that

$$k_1 \leq 1, \quad k_2 \leq 1, \quad k_3 \geq 1, \quad k_4 \geq 1\,.$$

First we prove that $k_1 \leq 1$, i.e.,

$$|a_{12}| - a_{12} + |a_{12}^{(+11)}| + a_{12}^{(+11)} + 2|a_{21}| \leq 4a_{22}. \tag{3.6}$$

Let $a_{12} = 0.5(k_{12}^{(-11)} + k_{12}),\ a_{21} = 0.5(k_{21}^{(-12)} + k_{21}),\ a_{22} = 0.5(k_{22}^{(-12)} + k_{22})$. In this case formula (3.6) can be rewritten in the form

$$\big|k_{12}^{(-11)}+k_{12}\big|+\big|k_{12}+k_{12}^{(+11)}\big|+2\big|k_{21}^{(-12)}+k_{21}\big|-k_{12}^{(-11)}+k_{12}^{(+11)} \leq 4\big(k_{22}^{(-12)}+k_{22}\big). \tag{3.7}$$

As condition (3.5) is valid, then $k_{22} \geq |k_{21}|,\ k_{22}^{(-12)} \geq |k_{21}^{(-12)}|$ and we have

$$|k_{21}^{(-12)} + k_{21}| \leq |k_{21}^{(-12)}| + |k_{21}| \leq k_{22}^{(-12)} + k_{22}\,.$$

Thus instead of (3.7) we have to prove that

$$|k_{12}^{(-11)} + k_{12}| + |k_{12} + k_{12}^{(+11)}| - k_{12}^{(-11)} + k_{12}^{(+11)} \leq 2(k_{22}^{(-12)} + k_{22}). \tag{3.8}$$

1. Let assume that $k_{12}^{(-11)} + k_{12} \geq 0$, $k_{12} + k_{12}^{(+11)} \geq 0$. Then inequality (3.8) can be rewritten in the form:

$$k_{12} + k_{12}^{(+11)} \leq k_{22}^{(-12)} + k_{22}.$$

It is easy to see that this inequality is valid under condition (3.5).

2. Let assume that $k_{12}^{(-11)} + k_{12} \geq 0$, $k_{12} + k_{12}^{(+11)} \leq 0$. In this case from (3.8) we obtain: $k_{22}^{(-12)} + k_{22} \geq 0$. From ellipticity condition (2.2) for $\xi = (0,1)$ we have $0 < c_1 \leq k_{22} \leq c_2$. Hence, the required inequality holds true.

3. Let assume that $k_{12}^{(-11)} + k_{12} \leq 0$, $k_{12} + k_{12}^{(+11)} \geq 0$, then formula (3.8) has the form:

$$-k_{12}^{(-11)} + k_{12}^{(+11)} \leq k_{22}^{(-12)} + k_{22}.$$

This inequality is valid under condition (3.5).

4. Let assume that $k_{12}^{(-11)} + k_{12} \leq 0$, $k_{12} + k_{12}^{(+11)} \leq 0$. In this case we rewrite inequality (3.8) in the form:

$$-k_{12} - k_{12}^{(-11)} \leq k_{22}^{(-12)} + k_{22}.$$

This inequality is true under condition (3.5).

Hence, $k_1 \leq 1$ if condition (3.5) is satisfied. Analogously we prove that $k_2 \leq 1$, $k_3 \geq 1$, $k_4 \geq 1$. ■

Theorem 2. *Let us suppose, that for all $x \in \omega_h$ condition* (3.4) *is satisfied. Then difference scheme* (2.3) *is stable with respect to the right-hand side and the boundary conditions, and for its solution the following a priori estimate is valid*

$$\|y\|_{\bar{C}} \leq \max\{\|\mu\|_{C_\gamma},\ c_0^{-1}\,\|\varphi\|_C\}. \tag{3.9}$$

Proof. It is easy to see that the coefficients $B_{2k} \geq 0,\ k = \overline{1,4}$ without any limitations. The coefficients $B_{2k-1} \geq 0,\ k = \overline{1,4}$ under condition (3.4):

$$\begin{aligned}
B_1 &= \frac{1}{h_1 h_2}\big(a_{22}^{(+12)}\frac{h_1}{h_2} - \frac{|a_{12}| + a_{12} + |a_{12}^{(+11)}| - a_{12}^{(+11)} + 2|a_{21}^{(+12)}|}{4}\big)\\
&\geq \frac{1}{h_1 h_2}\big(a_{22}^{(+12)}k_2 - \frac{|a_{12}| + a_{12} + |a_{12}^{(+11)}| - a_{12}^{(+11)} + 2|a_{21}^{(+12)}|}{4}\big) = 0,\\
B_3 &= \frac{1}{h_1^2}\big(a_{11}^{(+11)} - \frac{h_1}{h_2}\frac{2|a_{12}^{(+11)}| + |a_{21}| + a_{21} + |a_{21}^{(+12)}| - a_{21}^{(+12)}}{4}\big)\\
&\geq \frac{1}{h_1^2}\big(a_{11}^{(+11)} - k_4\frac{2|a_{12}^{(+11)}| + |a_{21}| + a_{21} + |a_{21}^{(+12)}| - a_{21}^{(+12)}}{4}\big) = 0,\\
B_5 &= \frac{1}{h_1 h_2}\big(a_{22}\frac{h_1}{h_2} - \frac{|a_{12}| - a_{12} + |a_{12}^{(+11)}| + a_{12}^{(+11)} + 2|a_{21}|}{4}\big)\\
&\geq \frac{1}{h_1 h_2}\big(a_{22}k_1 - \frac{|a_{12}| - a_{12} + |a_{12}^{(+11)}| + a_{12}^{(+11)} + 2|a_{21}|}{4}\big) = 0,\\
B_7 &= \frac{1}{h_1^2}\big(a_{11} - \frac{h_1}{h_2}\frac{2|a_{12}| + |a_{21}| - a_{21} + |a_{21}^{(+12)}| + a_{21}^{(+12)}}{4}\big)\\
&\geq \frac{1}{h_1^2}\big(a_{11} - k_3\frac{2|a_{12}| + |a_{21}| - a_{21} + |a_{21}^{(+12)}| + a_{21}^{(+12)}}{4}\big) = 0.
\end{aligned}$$

Coefficient $A > 0$, if the following condition is true

$$\max\left\{\frac{|a_{21}|}{2a_{22}},\ \frac{|a_{21}^{(+12)}|}{2a_{22}^{(+12)}}\right\} \leq \frac{h_1}{h_2} \leq \min\left\{\frac{2a_{11}}{|a_{12}|},\ \frac{2a_{11}^{(+11)}}{|a_{12}^{(+11)}|}\right\}.$$

This statement follows from the following inequalities

$$\begin{aligned}
A = {}& \frac{1}{h_1^2}\Big(a_{11} - \frac{h_1}{h_2}\frac{|a_{12}|}{2}\Big) + \frac{1}{h_1^2}\Big(a_{11}^{(+11)} - \frac{h_1}{h_2}\frac{|a_{12}^{(+11)}|}{2}\Big) + \frac{1}{h_1 h_2}\Big(a_{22}\frac{h_1}{h_2} - \frac{|a_{21}|}{2}\Big)\\
&+ \frac{1}{h_1 h_2}\Big(a_{22}^{(+12)}\frac{h_1}{h_2} - \frac{|a_{21}^{(+12)}|}{2}\Big) + d \geq \frac{1}{h_1^2}\Big(a_{11} - \frac{2a_{11}}{|a_{12}|}\frac{|a_{12}|}{2}\Big)\\
&+ \frac{1}{h_1^2}\Big(a_{11}^{(+11)} - \frac{2a_{11}^{(+11)}}{a_{12}^{(+11)}}\frac{|a_{12}^{(+11)}|}{2}\Big) + \frac{1}{h_1 h_2}\Big(a_{22}\frac{|a_{21}|}{2a_{22}} - \frac{|a_{21}|}{2}\Big)\\
&+ \frac{1}{h_1 h_2}\Big(a_{22}^{(+12)}\frac{|a_{21}^{(+12)}|}{2a_{22}^{(+12)}} - \frac{|a_{21}^{(+12)}|}{2}\Big) + d = d > 0.
\end{aligned}$$

Note, that the above condition is weaker than condition (3.4), i.e., it holds true if condition (3.4) is valid. We verify directly that for any grid node $x \in \omega_h$ the coefficient $D > 0$:

$$D = A - \sum_{k=1}^{8} B_k = d(x) \geq c_0 > 0.$$

For $x \in \gamma_h$, the coefficients of the canonical form are given by: $A = 1 > 0, \quad B = 0, \quad D = 1 > 0$. Now, all the conditions of Theorem 1 are satisfied. *A priori* estimate (3.3) provides the required inequality (3.9). ■

4. Convergence

Let us consider now the problem of convergence of the proposed difference scheme. Substituting $y = z + u$ into equations (2.3) we get the following problem for the error of the discrete solution

$$\begin{cases} \Lambda z - dz = -\psi, \quad x \in \omega_h \,, \\ z = 0, \quad x \in \gamma_h \,, \end{cases} \tag{4.1}$$

where $\psi = \Lambda u - du + \varphi$ denotes the error of approximation of difference scheme (2.3) corresponding to the exact solution of differential problem (2.1). It was shown above that the proposed difference scheme approximates the given differential problem with the second order, thus

$$\|\psi\|_C = M(h_1^2 + h_2^2),$$

where $M > 0$ is a positive constant which does not depend on the grid steps h_1, h_2.

Using Theorem 2 for the solution of problem (4.1), it can be verified that the following theorem takes place.

Theorem 3. *Let us suppose that for all* $x \in \omega_h$, *condition* (3.4) *is satisfied. Then the solution of difference scheme* (2.3) *converges to the exact solution of differential problem* (2.1), *and the following a priori estimate*

$$\|y - u\|_C \leq \frac{M}{c_0} \left(h_1^2 + h_2^2 \right)$$

is valid.

Remark 1. Results above can be easily extended to p–dimensional ($p \geq 2$) elliptic equations with mixed derivatives.

Remark 2. The proposed approach can be also applied for the development of the conservative monotone difference schemes for multidimensional parabolic equations with mixed derivatives.

5. Numerical results

To solve problem (2.1) by means of difference scheme (2.3) we use the modified strongly implicit method [9]. Therefore, we reduce difference scheme (2.3) to the system of algebraic equations

$$[A]y = C\,.$$

Here A is a nine-diagonal matrix. Then we consider matrix $[A+P]$, which is the product of the lower triangular matrix $[L]$ and the upper triangular matrix $[U]$, and develop the iterative process

$$[A+P]y^{n+1} = C + [P]y^n\,.$$

Since $[A+P] = [L][U]$ we obtain the following numerical algorithm

$$[L][U]y^{n+1} = C + [P]y^n\,.$$

Matrices $[L]$, $[U]$ and $[P]$ are defined in [9].

Numerical experiments were carried out in domain $\bar{G} = [0,1]\times[0,1]$. We choose the coefficients: $k_{11} = 1,\quad k_{12} = k_{21} = \cos(\pi(x_1+x_2)),\quad k_{22} = 1,\quad q = 1$. It is easy to see that $k_{\alpha\beta}$ satisfy ellipticity condition (2.2). The exact solution is given as $u = \sin(4\pi x_1)\sin(4\pi x_2)$. By substituting the exact solution into (2.1), we obtain the boundary conditions and the right-hand side f.

Table 1. The convergence order of difference scheme (2.3).

$N \times N$	32×32	64×64	128×128	256×256	512×512
z^N	0.043264	0.010734	0.002687	0.000654	0.000167
D^N	0.049900	0.010923	0.002690	0.000672	0.000164
p^N	2.19	2.02	2.00	2.03	2.04

The results of the numerical experiments are presented in Tab. 1, where

$$z^N = \max_{x\in\omega_h} |y_h(x) - u(x)|$$

is the global error of the discrete solution. Since the exact solution is usually unknown, we have computed the solution on the grids ω_h, $\omega_{h/2}$, $\omega_{h/4}$, etc. Then the *aposteriori* error estimate of the solution y_h can be obtained by using the Runge estimator:

$$D^N = \frac{1}{3}\max_{x\in\omega_{2h}} |y_h(x) - y_{2h}(x)|\,.$$

Here we take the difference between the values of the solution on the grid with $N/2$ nodes and the solution at the same point on the grid with N nodes.

The second aposteriori estimator $p^N = \log_2(D^{N/2}/D^N)$ estimates the convergence order of the approximation y_h.

6. Conclusions

In this paper new difference scheme for elliptic equations with mixed derivatives and alternating coefficients is presented. The proposed scheme is conservative, has the second order of approximation and satisfies the grid maximum principle. For the developed numerical algorithms the a priori estimates of stability and convergence in the uniform norm are obtained.

The proposed approach to the construction of monotone conservative difference schemes can be also applied to the development of monotone and conservative numerical algorithms for multidimensional parabolic equations with mixed derivatives.

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Monotoniškos ir konservatyvios baigtinių skirtumų schemos eliptinio tipo lygtims su mišriomis išvestinėmis

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Straipsnyje nagrinėjamos eliptinio tipo lygtys su mišriomis išvestinėmis. Šioms diferencialinėms lygtims pasiūlytos naujos antros eilės baigtinių skirtumų schemos, kurios yra monotoniškos ir konservatyvios. Sukonstruoti algoritmai tenkina skaitinį maksimumo principą, kai koeficientai prie mišriųjų išvestinių gali būti bet kokio ženklo. Gauti aprioriniai įverčiai maksimumo normoje. Įrodyta baigtinių skirtumų schemų stabilumas ir konvergavimas.