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THE SECOND BOUNDARY VALUE PROBLEM OF RIEMANN'S TYPE FOR BIANALYTICAL FUNCTIONS WITH DISCONTINUOUS COEFFICIENTS

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Abstract. The paper is devoted to the investigation of one of the basic boundary value problems of Riemann's type for bianalytical functions with discontinuous coefficients. In the course of work there was made out a constructive method for solution of the problem in a unit circle. There was also found out that the solution of the problem under consideration consists in consequent solutions of two Riemann's boundary value problems for analytical functions in a unit circle. Besides, the example is constructed.

Key words: bianalytical function, boundary value problem, plane with slots, index

1. Introduction

Let $L = \{t : |t| = 1\}$, $D^+ = \{z : |z| < 1\}$ and $D^- = \bar{C} \backslash \{D^+ \bigcup L\}$.

Let $G_k(t)$ $(k = 1, 2)$ – given on the contour L functions, satisfying the condition of Holder everywhere on L, except for a finite number of points, where they have simple discontinuity, and $G_k(t) \neq 0$ on the contour. Also we shall consider, that function $G_0(t)$ has derivative which satisfies the condition of Holder, except for a finite number of points, where it may have simple discontinuities. Hereinafter, following N.I. Muskhelishvili (see, for example, [2]), we shall call points of discontinuity of the functions $G_0(t)$, $G_0'(t)$ and $G_1(t)$ as knots, and remaining points of the contour L we shall name ordinary. Besides we shall rank all points of discontinuity of the function $G_0(t)$ and its derivative as knots of function $G_1(t)$.

Further we shall generally use terms and definitions accepted in [3].

Definition 1. We shall speak, that bianalytical function $F^{\pm}(z)$ in domain $D^{\pm}$ belongs to the class $A_2(D^{\pm}) \bigcap I^{(2)}(L)$, if it proceeds on the contour L together with the partial derivatives $\dfrac{\partial^{\alpha+\beta}F^{\pm}(z)}{\partial z^{\alpha}\partial\bar{z}^{\beta}}$ ($\alpha = 0, 1; \beta = 0, 1$), and so that boundary values of this function and all specified derivatives satisfy on z the condition of Holder everywhere, except for, possibly, knots, where the reversion in infinity of integrable order is possible when $\alpha + \beta < 2$.

It is required to find all piecewise bianalytical functions $F(z) = \{F^+(z), F^-(z)\}$, belonging to the class $A_2(D^{\pm}) \bigcap\ I^{(2)}(L)$, vanishing on infinity, limited near the knots of the contour and satisfying in all ordinary points of L the following boundary conditions:

$$F^+(t) = G_0(t)F^-(t) + g_0(t), \tag{1.1}$$

$$\frac{\partial F^+(t)}{\partial n_+} = -G_1(t)\frac{\partial F^-(t)}{\partial n_-} - g_1(t), \tag{1.2}$$

where $\dfrac{\partial}{\partial n_+}$ $\left(\dfrac{\partial}{\partial n_-}\right)$ – derivative on interior (exterior) normal to the contour L, $g_k(t)$ ($k = 0, 1$) – given on L functions of the class $H^{(1-k)}(L)$, and $g_0(t) = (t - c)^{\gamma_c} g^*(t)$, c – any of knots of the function $G_0(t)$, $\gamma_c > 0$ - quite defined numbers. Here, in equality (1.2), factor (-1) at $G_1(t)$ and $g_1(t)$ is entered for convenience hereinafter.

We shall name the formulated problem as *the second basic boundary value problem of Riemann's type for bianalytical functions with discontinuous coefficients in the unit circle* or in short *the problem $R_{2,2}$, and appropriate homogeneous problem ($g_0(t) \equiv g_1(t) \equiv 0$) shall be named as a problem $R^0_{2,2}$.*

Let's notice, that the problem $R_{2,2}$, stated by F. D. Gakhov as one of basic boundary value problems for bianalytical functions (see, for example, [1], p. 316) in case of continuous coefficients and smooth closed loops was explicitly investigated in the work of K. M. Rasulov (see [3]).

In the above mentioned statement the problem $R_{2,2}$ is investigated in the present work for the first time.

2. About the Solution of the Problem $R_{2,2}$

It is known (see [1, 3]), that any vanishing on infinity piecewise bianalytical function $F(z)$ with line of saltuses L is possible to represent as:

$$F(z) = \begin{cases} F^+(z) = \varphi_0^+(z) + \bar{z}\varphi_1^+(z), & z \in D^+, \\ F^-(z) = \varphi_0^-(z) + \bar{z}\varphi_1^-(z), & z \in D^-, \end{cases} \tag{2.1}$$

where $\varphi_k^{\pm}(z)$ – analytical functions in domain $D^{\pm}$ (analytical components of piecewise bianalytical function), for which the following conditions are fulfilled:

$$\Pi\{\varphi_k^-, \infty\} \geq 1 + k, \quad k = 0, 1;$$

here $\Pi\{\varphi_k^-, \infty\}$ means the order of the function $\varphi_k^-(z)$ in the point $z = \infty$.

Let's search for the solution of the problem $R_{2,2}$ as

$$F(z) = f_0(z) + (z\bar{z} - 1)f_1(z). \tag{2.2}$$

Then the functions $f_k(z)$ ($k = 0, 1$) will be connected with analytical components of the required bianalytical function $F(z)$ by the formulas:

$$\varphi_0(z) = f_0(z) - f_1(z), \qquad \varphi_1(z) = zf_1(z). \tag{2.3}$$

As known (see [1] p. 304)

$$\frac{\partial}{\partial n_\pm} = \pm i\left(t'\frac{\partial}{\partial t} - \overline{t'}\frac{\partial}{\partial \bar{t}}\right), \tag{2.4}$$

then taking into account (2.2) and the fact that the equality $\bar{t} = \dfrac{1}{t}$ is fulfilled on L, the boundary conditions (1.1) and (1.2) can be copied accordingly in the aspect:

$$f_0^+(t) = G_0(t)f_0^+(t) + g_0(t), \tag{2.5}$$

$$f_1^+(t) = G_1(t)f_1^-(t) + \frac{1}{2}\left(-t\frac{df_0^+(t)}{dt} + tG_1(t)\frac{df_0^-(t)}{dt} + g_1(t)\right). \tag{2.6}$$

The equalities (2.5) and (2.6) represent boundary conditions of usual Riemann's problems for analytical functions with discontinuous coefficients in the unit circle (see [1] or [2]).

Thus, as a matter of fact, solution of the initial problem $R_{2,2}$ is reduced to sequential solution of two auxiliary problems of Riemann (2.5) and (2.6) in classes of piecewise analytical functions with the line of saltuses L. But as in the problem $R_{2,2}$ we search the solutions, limited close to the knots of the contour and vanishing on infinity, there arises the necessity in a choice of defined classes of analytical functions at the solution of auxiliary problems (2.5) and (2.6). Therefore, at first we shall find out, in what classes it is necessary to search for solutions of boundary value problems (2.5) and (2.6).

From equalities (2.3) we can see that the function $f_0^-(z)$ on the infinity should have zero not below than the first order, and $f_1^-(z)$ – zero not below than the third order.

Let's study the behaviour of function $F(z)$ near the knots of the contour L. Let c be any of knots, then $\bar{c}c = 1$ and $|c| = 1$.

We have the following serieses of inequalities:

$$\begin{aligned}
|F(z)| &= |f_0(z) + (z\bar{z} - 1)f_1(z)| \le |f_0(z)| + |f_1(z)||z\bar{z} - 1| \\
&= |f_0(z)| + |f_1(z)||(z - c + c)(\bar{z} - \bar{c} + \bar{c}) - 1| \\
&\le |f_0(z)| + 2|f_1(z)||z - c| + |f_1(z)||z - c|^2;
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
|F(z)| &= |f_0(z) + (z\bar{z} - 1)f_1(z)| \ge |f_0(z)| - |f_1(z)||z\bar{z} - 1| \\
&= |f_0(z)| - |f_1(z)||(z - c + c)(\bar{z} - \bar{c} + \bar{c}) - 1| \\
&\ge |f_0(z)| - 2|f_1(z)||z - c| - |f_1(z)||z - c|^2.
\end{aligned} \tag{2.8}$$

Thus, for the function $F(z)$ would be limited close to the knots of the contour L, *it is necessary and enough* that the function $f_0(z)$ would be limited, and the function $f_1(z)$ supposed the evaluation:

$$|f_1(z)| \le \frac{const}{|z-c|^\alpha}, \qquad 0 \le \alpha < 1. \tag{2.9}$$

Really, if the function $f_0(z)$ is limited close to c and function $f_1(z)$ supposes the evaluation (2.9), from inequalities (2.7) it follows, that the required bianalytical function $F(z)$ will be limited in a neighbourhood of the knot c.

Back, if the function $F(z)$ of the class $A_2(D^\pm)\bigcap I^{(2)}(L)$ is limited close to c, from inequalities (2.8) it follows, that the function $f_1(z)$ has to suppose the evaluation (2.9) (otherwise *all* solutions of the problem $R_{2,2}$ will not be found), so the function $f_0(z)$ has to be limited in a neighbourhood of the knot c.

Therefore, it is required to search the solution of the problem (2.5) in the class of functions, vanishing on infinity and limited near the knots, and solution of the problem (2.6) is required to search in the class of functions, having zero of the third order on infinity and infinity of the integrable order near the knots of the contour L.

Let's solve the boundary value problem of Riemann (2.5) using the method offered by F.D. Gakhov (see, for example, [1], p. 448).

Let index of the problem (2.5) be equal κ_0 in the specified class.

Then, if $\kappa_0 \ge 0$, a common solution of the problem (2.5) is set by the formula (see [1, 2]):

$$f_0(z) = X_0(z)\left(\frac{1}{2\pi i}\int_L \frac{g_0(\tau)}{X_0^+(\tau)}\frac{d\tau}{\tau - z} + P_{\kappa_0-1}(z)\right), \tag{2.10}$$

where $X_0(z)$ – canonical function of the problem (2.5), $P_{\kappa_0-1}(z)$ – the polynomial of a degree not higher then $\kappa_0 - 1$ with arbitrary complex coefficients.

In the case when $\kappa_0 < 0$, the solution of the problem (2.5) also will be expressed by the formula (2.10) with only one modification, that $P_{\kappa_0-1}(z) \equiv 0$, at observance of $|\kappa_0|$ conditions of solvability of the aspect:

$$\int_L \frac{g_0(\tau)}{X_0^+(\tau)}\tau^{k-1}d\tau = 0, \qquad k = 1,\dots,|\kappa_0|.$$

Let's define numbers γ_c specified in the statement of the problem $R_{2,2}$. Let $c_1, c_2, \dots,$ c_m be knots of the function $G_0(t)$.

Below we shall consider that

$$\begin{aligned}
\gamma_{c_k} &> \frac{1}{2\pi}(\arg G(c_k - 0) - \arg G(c_k + 0)), \quad k = 2,\dots,m\\
\gamma_{c_1} &> \frac{1}{2\pi}(argG(c_1 - 0) - argG(c_1 + 0) - 2\pi\kappa_0).
\end{aligned} \tag{2.11}$$

Further, on the found function $f_0(z)$ with the help of differentiation and in view of the formulas Sokhotzky-Plemelj (see [4], p. 333, [1, 2]), we shall find out boundary values $\dfrac{df_0^\pm(t)}{dt}$ of the function $\dfrac{df_0(z)}{dz}$.

Note 1. We shall notice, that if the knot c is not singular or c – singular, but $\ln|G_0(c-0)| - \ln|G_0(c+0)| = 0$ from the conditions (2.11), it follows that the functions $\dfrac{df_0^{\pm}(t)}{dt}$ satisfy the condition of Holder everywhere on L except for, possibly, knots, where they may have a singularity of the integrable order (knots of the first type). Otherwise functions $\dfrac{df_0^{\pm}(t)}{dt}$ will have a singularity of the first order near the knots (knots of the second type).

Further we shall solve the boundary value problem of Riemann (2.6).

Let the index of the problem (2.6) be equal κ_1 in the specified class.

As it is known (see [1, 2]), if $\kappa_1 \geq 3$, a common solution of the problem (2.6) is set by the formula:

$$f_1(z) = X_1(z) \left(\frac{1}{2\pi i} \int\limits_L \frac{Q_1(\tau)}{X_1^+(\tau)} \frac{d\tau}{\tau - z} + P_{\kappa_1 - 3}(z) \right), \tag{2.12}$$

where $X_1(z)$ – canonical function of the problem (2.6), $P_{\kappa_1-3}(z)$ – the polynomial of a degree not higher then $\kappa_1 - 3$ with arbitrary complex coefficients,

$$Q_1(t) = \frac{1}{2} \left(-t \frac{df_0^+(t)}{dt} + tG_1(t) \frac{df_0^-(t)}{dt} + g_1(t) \right).$$

If $\kappa_1 \leq 2$, the solution of the problem (2.6) also will be expressed by the formula (2.12) with only one modification that $P_{\kappa_1-3}(z) \equiv 0$, at observance of $-\kappa_1 + 2$ conditions of solvability of the aspect:

$$\int\limits_L \frac{Q_1(\tau)}{X_1^+(\tau)} \tau^{k-1} d\tau = 0, \qquad k = 1, \ldots, -\kappa_1 + 2.$$

Note 2. Generally speaking, absolute term $Q_1(t)$ of the problem (2.6) satisfies the condition of Holder everywhere on L except for, possibly, knots $c_1, c_2, \ldots, c_m$, where it may have singularity of the first order (knots of the second type), and remaining knots, where it may have an integrable singularity. And, if the knot of the second type of the problem (2.5) is the singular knot of the problem (2.6), then the problem $R_{2,2}$ will be *insoluble* in the class $A_2(D^{\pm}) \bigcap I^{(2)}(L)$.

Further on the found functions $f_0(z)$ and $f_1(z)$, using the formulas (2.3), we restore analytical components of the required piecewise bianalytical function, and then the piecewise bianalytical function $F(z)$ itself under the formula (2.1).

Thus, the following basic outcome is fair.

Theorem 1. *Let* $L = \{t : |t| = 1\}$, $D^+ = \{z : |z| < 1\}$ *and* $D^- = \bar{C} \backslash \{D^+ \bigcup L\}$. *Then the solution of the problem* $R_{2,2}$ *is reduced to the sequential solution of two scalar boundary value problems of Riemann* (2.5) *and* (2.6) *with discontinuity coefficients in classes of analytical functions in the unit circle, and that the solution of the problem* (2.5) *is searched in the class of functions vanishing on infinity and limited in the knots of the contour; and the solution of the problem* (2.6) *is searched in the class of functions, having on infinity zero of the third order and infinity of the integrable order in the knots of the contour* L. *The problem* $R_{2,2}$ *is solvable if and only if* *the problems* (2.5) *and* (2.6) *in the specified classes of functions are simultaneously solvable and knots of the second type are not singular for the coefficient* $G_1(t)$ *of the problem* (2.6)*.*

Example 1. Let $L = \{t : |t| = 1\}$, $D^+ = \{z : |z| < 1\}$ and $D^- = \bar{C}\backslash\{D^+ \bigcup L\}$. It is required to find all piecewise bianalytical functions $F(z) = \{F^+(z), F^-(z)\}$ belonging to the class $A_2(D^\pm) \bigcap I^{(2)}(L)$ vanishing on infinity, limited near the knots of the contour and satisfying in all ordinary points L the following boundary conditions:

$$F^+(t) = G_0(t)F^-(t) + (t-1)^{\frac{3}{2}}(t+1)^{\frac{3}{2}}, \tag{2.13}$$

$$\frac{\partial F^+(t)}{\partial n_+} = -t^4\frac{\partial F^-(t)}{\partial n_-} - (3t^2(t^2-1)^{\frac{1}{2}} + 2t^2), \tag{2.14}$$

Here

$$G_0(t) = \begin{cases} 1, & t \in L_1 = \{t : t = e^{is}, 0 \leq s \leq \pi\}, \\ -1, & t \in L_2 = \{t : t = e^{is}, \pi \leq s \leq 2\pi\}, \end{cases}$$

$G_1(t) = t^4$, $g_0(t) = (t-1)^{\frac{3}{2}}(t+1)^{\frac{3}{2}}$ and $g_1(t) = (3t^2(t^2-1)^{\frac{1}{2}} + t^2)$.

Using equalities (2.2) – (2.4), the boundary condition (2.13) will take the following aspect:

$$f_0^+(t) = G_0(t)f_0^-(t) + (t-1)^{\frac{3}{2}}(t+1)^{\frac{3}{2}}, \tag{2.15}$$

Knots of the problem (2.15) are the points $t = 1$ and $t = -1$, in which function $G_0(t)$ has simple discontinuity.

Let's calculate the index of the problem (2.15). Let's choose as the initial point $t = 1$. We have,

$$G_0(1+0) = 1 = e^{i0}, \quad \theta_1 = 0,$$

the change of argument of the function $G_0(t)$ on the arc L_1 will be equal

$$\Delta\theta_1 = [argG_0(t)]_{L_1} = 0.$$

Therefore

$$G_0(-1-0) = 1 = e^{i0}.$$

Let $G_0(-1+0) = -1 = e^{i\theta_2}$. Let's choose a value θ_2 so that the inequality is fulfilled

$$0 \leq 0 - \theta_2 < 2\pi,$$

That is $\theta_2 = -\pi$

The change of argument of the function $G_0(t)$ on the arc L_2 will be equal to zero. So $G_0(1-0) = -1 = e^{-i\pi}$.

Let's define the whole number κ_0, satisfying the following condition:

$$0 \leq -\pi - 2\pi\kappa_0 < 2\pi.$$

Thus, the index of the problem (2.15) will be equal to -1.

The common solution of the problem will look like:

$$f_0^+(z) = (z^2 - 1)^{\frac{3}{2}},$$
$$f_0^-(z) \equiv 0,$$

at observance of one condition of solvability:

$$\int_L \tau(\tau^2 - 1)d\tau = 0,$$

which is obviously fulfilled.

The boundary values of the derivative of the solution of the problem (2.15) will be set by the formulas:

$$\frac{df_0^+(t)}{dt} = 3t(t^2 - 1)^{\frac{1}{2}}, \tag{2.16}$$

$$\frac{df_0^-(t)}{dt} \equiv 0. \tag{2.17}$$

With the account (2.16) – (2.17), the boundary condition (2.14) will take the aspect:

$$f_1^+(t) = t^4 f_1^-(t) + t^2. \tag{2.18}$$

Generally speaking, the coefficient of the problem (2.18) is the continuous function on L, but following the statement of the problem (2.18) the knots will be represented by the points $t = 1$ and $t = -1$.

Let's calculate the index of the problem (2.18). Let's choose as the initial point $t = 1$. We have,

$$G_1(1 + 0) = 1 = e^{i0}, \quad \theta_1 = 0,$$

The change of argument of function $G_1(t)$ on the arc L_1 will be equal

$$\Delta\theta_1 = [argG_1(t)]_{L_1} = 4\pi.$$

Therefore

$$G_1(-1 - 0) = 1 = e^{i4\pi}.$$

Let $G_1(-1 + 0) = -1 = e^{i\theta_2}$. Let's choose a value θ_2 so that the inequality is fulfilled

$$-2\pi < 4\pi - \theta_2 \leq 0,$$

That is $\theta_2 = 4\pi$

The change of argument of function $G_1(t)$ on the arc L_2 will be equal 4π. So $G_1(1 - 0) = 1 = e^{i8\pi}$.

Let's define whole number κ_1, satisfying the following condition:

$$-2\pi < 8\pi - 2\pi\kappa_1 \leq 0.$$

Thus, the index of the problem (2.18) will be equal to 4.

Hence, the common solution of the problem (2.18) will look like:

$$f_1^+(z) = z^2 + a_1 z + a_0,$$
$$f_1^-(z) = \frac{a_1}{z^3} + \frac{a_0}{z^4},$$

where a_0 and a_1 – arbitrary complex coefficients.

On the found functions $f_0(z)$ and $f_1(z)$, using (2.3), we restore analytical components of the required piecewise bianalytical function $F(z)$:

$$\varphi_0^+(z) = (z^2-1)^{\frac{3}{2}} - z^2 + a_1 z + a_0, \tag{2.19}$$
$$\varphi_0^-(z) = -\frac{a_1}{z^3} - \frac{a_0}{z^4}, \tag{2.20}$$
$$\varphi_1^+(z) = z^3 + a_1 z^2 + a_0 z, \tag{2.21}$$
$$\varphi_1^-(z) = \frac{a_1}{z^2} + \frac{a_0}{z^3}. \tag{2.22}$$

Thus, common solution of the problem (2.13) – (2.14) is represented by the formula:

$$F(z) = \begin{cases} F^+(z) = \varphi_0^+(z) + \bar{z}\varphi_1^+(z), & z \in D^+, \\ F^-(z) = \varphi_0^-(z) + \bar{z}\varphi_1^-(z), & z \in D^-, \end{cases}$$

where the functions $\varphi_0^{\pm}(z)$ and $\varphi_1^{\pm}(z)$ are defined by the formulas (2.19) – (2.22).

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Antrasis kraštinis uždavinys Rimano tipo bianalizinėms funkcijoms su trūkiais koeficientais

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Darbe sprendžiamas antrasis kraštinis uždavinys Rimano tipo bianalizinėms funkcijoms su trūkiais koeficientais. Parodoma, kad sprendžiamas uždavinys suvedamas į sprendimą dviejų Rimano uždavinių analizinėms funkcijoms su trūkiais koeficientais. Randamos analizinių funkcijų klasės, kuriose gali būti sprendinys. Pateikiamas pavyzdys, iliustruojantis nagrinėjamo uždavinio sprendimo algoritmą.