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STABILITY OF THREE-LEVEL DIFFERENCE SCHEMES WITH RESPECT TO THE RIGHT-HAND SIDE[1]

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Abstract. In this paper we investigate three-level difference schemes on non-uniform grids in time. The a priori estimates of stability with respect to the initial data and the right-hand side are obtained. New schemes of the raised order of approximation for wave equations are constructed and investigated.

Key words: Three-level difference scheme, non-uniform grid, stability

1. Introduction

The main results on the theory of the stability of operator-difference schemes have been obtained using grids uniform in time [6, 8, 9]. Necessary and sufficient conditions of stability were already obtained in the sense of the initial data and the right-hand side in finite-dimensional Hilbert spaces.

For three-level difference schemes on non-uniform grids there are a few particular results. In [1, 2] difference schemes of the first order of approximation were considered for the case $\tau_{n+1} \geq \tau_n$. In [5] a priori estimate of uniform stability with respect to initial data was received under special condition on operators and time grid. The condition on time steps leaded us to the grid satisfying the geometrical progression law $\tau_{n+1} = q\tau_n,\ q = const > 0$. In the paper [7] basic canonical forms have been first introduced for three-level difference schemes on non-uniform in time grids and important theorems concerning the stability with respect to initial data have been formulated.

In the work [4] for three-level difference schemes the a priori estimate of absolute stability of solution was obtained with respect to the initial data without assuming

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the Lipschitz–continuity of operators on a time variable. In [4] the special case of grid $\tau_{n+1} \geq \tau_n$ was discussed and the a priori estimate of stability of three-level operator-difference schemes with respect to the initial data and the right-hand side was received. However the technique introduced in [4] doesn't allow us to carry out the investigation for the inverse relations of time steps.

Investigation of stability with respect to the right-hand side of three-level difference schemes on non-uniform in time grids causes certain difficulties. In present work new a priori estimates of stability with respect to the initial data and the right-hand side are received with the use of specific technique, which consists in separate investigation of two cases $\tau_{n+1} \geq \tau_n$ and $\tau_{n+1} \leq \tau_n$ and it is represented in proofs of Theorems 1, 2. The stability of new computational methods on non-uniform grids with respect to the initial data and the right-hand side is investigated on the basis of general a priori estimates obtained for three-level operator-difference schemes. Difference schemes of the second order of local approximation are constructed and investigated on non-uniform grids in time on standard stencils for hyperbolic equations. Computational experiments for introduced schemes confirm the theoretical results received.

2. Statement of the Problem

Let us note some features of the investigation of difference schemes on non-uniform in time grids. If in the initial differential problem the coefficients are constant, approximation on a non-uniform grid leads us to operator-difference schemes dependent on grid node t_n. If we require that these operators be Lipschitz–continuous, it would lead us to an unnatural condition of the quasi-uniformity of a time grid. The second problem is connected with a reduction of the order of local approximation when we go from a uniform grid to a non-uniform one.

The main problem we solve is to build new stable three-level difference schemes of the raised order of local approximation on the non-uniform grid using the standard stencils. We consider a three-level operator-difference scheme

$$\begin{cases} Dy_{\bar{t}\hat{t}} + By_{\overset{\circ}{t}} + Ay = \varphi, \\ y_0 = u_0, \quad y_1 = u_1 \end{cases} \tag{2.1}$$

on a non-uniform in time grid

$$\hat{\bar{\omega}}_\tau = \{t_n = t_{n-1} + \tau_n,\ n \in 1, 2, \cdots, N,\ t_0 = 0,\ t_N = T\} = \hat{\omega}_\tau \cup \{0, T\}\,. \tag{2.2}$$

Here $y = y_n = y(t_n) \in H$ is the sought function; $u_0, u_1, \varphi(t_n) \in H$ are given; H is the finite-dimensional Hilbert space; D, B, A are linear operators acting in H;

$$y_{\bar{t}\hat{t}} = \frac{y_t - y_{\bar{t}}}{\tau^*}\,, \quad y_t = \frac{y_{n+1} - y_n}{\tau_{n+1}}\,, \quad y_{\bar{t}} = \frac{y_n - y_{n-1}}{\tau_n}\,, \quad y_{\overset{\circ}{t}} = \frac{y_{n+1} - y_{n-1}}{\tau_n + \tau_{n+1}}\,,$$

$$\hat{y} = y_{n+1}, \quad \check{y} = y_{n-1}, \quad \tau = \tau_n, \quad \tau_+ = \tau_{n+1}, \quad \tau^* = 0.5(\tau_n + \tau_{n+1}),$$

$$y^{(\sigma_1,\sigma_2)} = \sigma_1\hat{y} + (1 - \sigma_1 - \sigma_2)y + \sigma_2\check{y}, \quad y^{(0.5)} = 0.5(\hat{y} + \check{y}).$$

For any arbitrary functions $u, v \in H$ the Cauchy–Schwartz inequality and the ε-inequality hold true:

$$|(u,v)| \le \|u\|\|v\| \le \varepsilon\|u\|^2 + \frac{1}{4\varepsilon}\|v\|^2, \quad \varepsilon > 0\ . \tag{2.3}$$

For the self-adjoint and nonnegative operator A we define a semi-norm of the grid function u:

$$\|u\|_A^2 = (Au, u), \quad A = A^* \ge 0.$$

To obtain a priori estimates of stability with respect to the right-hand side we'll use stability conditions with respect to the initial data [4].

3. Auxiliary Results

Let us formulate some auxiliary results separately for cases when $\tau_{n+1} \ge \tau_n$ and $\tau_{n+1} \le \tau_n$.

Theorem 1. *Let operators of scheme* (2.1) *satisfy the following conditions*

$$D(t) = D^*(t) > 0, \quad A = A^* > 0, \tag{3.1}$$

$$R = D - \frac{\tau_n \tau_{n+1}}{4} A > 0, \tag{3.2}$$

$$B \ge \frac{\tau_{n+1} - \tau_n}{4} A, \quad \tau_{n+1} \ge \tau_n. \tag{3.3}$$

Let R, A be constant operators. Then the solution of problem (2.1) *is stable with respect to the initial data and the right-hand side and the following estimate is valid:*

$$\|y_{t,n}\|_R + \|y_n^{(0.5)}\|_A \le \sqrt{2}\left(\|y_{t,0}\|_R + \|y_0^{(0.5)}\|_A + \sum_{k=1}^{n} \tau_{k+1}\|\varphi_k\|_{R^{-1}}\right). \tag{3.4}$$

Proof. To prove the theorem we scalar multiply scheme (2.1) by $2\tau^* y_{\overset{\circ}{t}}$:

$$2\tau^*\left(D y_{\bar{t}\hat{t}}, y_{\overset{\circ}{t}}\right) + 2\tau^*\left(B y_{\overset{\circ}{t}}, y_{\overset{\circ}{t}}\right) + 2\tau^*\left(Ay, y_{\overset{\circ}{t}}\right) = 2\tau^*(\varphi, y_{\overset{\circ}{t}}).$$

Using the proof of Theorem 4.1 [4] and the following representation

$$2\tau^*(\varphi, y_{\overset{\circ}{t}}) = \tau_{n+1}(\varphi, y_t) + \tau_n(\varphi, y_{\bar{t}}),$$

we receive the energy inequality:

$$\|y_t\|_R^2 + \|y_n^{(0.5)}\|_A^2 - \left(\|y_{\bar{t}}\|_R^2 + \|y_{n-1}^{(0.5)}\|_A^2\right) + \tau^* \frac{\tau_+ - \tau}{2}\|y_{\bar{t}t}\|_R^2$$
$$+ 2\tau^*\|y_{\overset{\circ}{t}}\|^2_{B - \frac{\tau_+ - \tau}{4}A} = \tau_{n+1}(\varphi, y_t) + \tau_n(\varphi, y_{\bar{t}})\,. \tag{3.5}$$

In conditions of the theorem

$$\|y_t\|_R^2 + \|y_n^{(0.5)}\|_A^2 - (\|y_{\bar{t}}\|_R^2 + \|y_{n-1}^{(0.5)}\|_A^2) \le \tau_{n+1}(\varphi, y_t) + \tau_n(\varphi, y_{\bar{t}}).$$

Let us introduce a notation

$$G = \|y_t\|_R^2 + \|y_n^{(0.5)}\|_A^2. \tag{3.6}$$

Then taking into account that $\tau_{n+1} \ge \tau_n$ and using the Cauchy–Schwartz inequality (2.3), we have

$$\begin{aligned} \tau G_{\bar{t}} &\le \tau_+(\varphi, y_t) + \tau(\varphi, y_{\bar{t}}) \le \tau_+\|\varphi\|_{R^{-1}}\|y_t\|_R + \tau\|\varphi\|_{R^{-1}}\|y_{\bar{t}}\|_R \\ &\le \tau_+\|\varphi\|_{R^{-1}}(\|y_t\|_R + \|y_{\bar{t}}\|_R) \le \tau_+\|\varphi\|_{R^{-1}}(G^{1/2} + \check{G}^{1/2}). \end{aligned} \tag{3.7}$$

Using the following identity [3]:

$$\left(G^{1/2}\right)_{\bar{t}} = \frac{G^{1/2} - \check{G}^{1/2}}{\tau_n} = \frac{G - \check{G}}{\tau_n(G^{1/2} + \check{G}^{1/2})} = \frac{G_{\bar{t}}}{G^{1/2} + \check{G}^{1/2}} \tag{3.8}$$

from (3.7) we get the estimate

$$\tau_n \left(G^{1/2}\right)_{\bar{t}} \le \tau_{n+1}\|\varphi\|_{R^{-1}},$$

or

$$\left(\|y_{t,n}\|_R^2 + \|y_n^{(0.5)}\|_A^2\right)^{1/2} \le \left(\|y_{\bar{t},n}\|_R^2 + \|y_{n-1}^{(0.5)}\|_A^2\right)^{1/2} + \tau_{n+1}\|\varphi_n\|_{R^{-1}}.$$

And then we have

$$\left(\|y_{t,n}\|_R^2 + \|y_n^{(0.5)}\|_A^2\right)^{1/2} \le \left(\|y_{t,0}\|_R^2 + \|y_0^{(0.5)}\|_A^2\right)^{1/2} + \sum_{k=1}^{n} \tau_{k+1}\|\varphi_k\|_{R^{-1}}. \tag{3.9}$$

Now using the evident relations:

$$|a| + |b| \le \sqrt{2(a^2 + b^2)}, \quad \sqrt{a^2 + b^2} \le |a| + |b|, \tag{3.10}$$

the statement of the theorem follows from inequality (3.9). ■

In order to formulate the theorem about stability for the inverse relations of time steps we rewrite the three–level operator–difference scheme (2.1) in the following form:

$$\begin{cases} D\dfrac{y_{n+1} - 2y_n + y_{n-1}}{\tau_n\tau_{n+1}} + \left(B - \dfrac{\tau_{n+1} - \tau_n}{\tau_{n+1}\tau_n}D\right)\dfrac{y_{n+1} - y_{n-1}}{2\tau^*} + Ay_n = \varphi_n, \\ y_0 = u_0, \quad y_1 = u_1\,. \end{cases} \tag{3.11}$$

Theorem 2. *Let operators $D(t), B(t), A$ satisfy the following conditions:*

$$D(t) = D^*(t) > 0, \quad B(t) > 0, \quad A = A^* > 0, \tag{3.12}$$

$$R = D - \frac{\tau_n \tau_{n+1}}{4} A > 0, \tag{3.13}$$

and operators A, R are constant. Let

$$\tau_{n+1} \le \tau_n \ . \tag{3.14}$$

Then operator-difference scheme (2.1) *is stable with respect to the initial data and the right-hand side and the a priori estimate is true:*

$$\|y_{t,n}\|_R + \|y_n^{(0.5)}\|_A \le \frac{\tau_1 \sqrt{2}}{\tau_{n+1}} \left\{ \|y_t(0)\|_R + \|y_0^{(0.5)}\|_A + \sum_{k=1}^{n} \tau_k \|\varphi_k\|_{R^{-1}} \right\} \ . \tag{3.15}$$

Proof. Let us scalarly multiply scheme (3.11) by $\tau_n \tau_{n+1}(y_{n+1} - y_{n-1})$ and use representations for scalar products given in Theorem 4.1 [4]:

$$\begin{aligned} &\big(D((y_{n+1} - y_n) - (y_n - y_{n-1})), (y_{n+1} - y_n) + (y_n - y_{n-1})\big) \\ &+ \left(\left(\frac{\tau_n \tau_{n+1}}{2\tau^*} B - \frac{\tau_{n+1} - \tau_n}{2\tau^*} D \right) (y_{n+1} - y_{n-1}), y_{n+1} - y_{n-1} \right) \\ &+ \tau_n \tau_{n+1} (A y_n, y_{n+1} - y_{n-1}) = \tau_n \tau_{n+1} (\varphi, y_{n+1} - y_{n-1}). \end{aligned}$$

Taking into consideration conditions of the theorem (3.12), (3.13) and the following identity

$$\tau_n \tau_{n+1} (\varphi, y_{n+1} - y_{n-1}) = \tau_n \tau_{n+1} (\varphi, \tau_{n+1} y_t) + \tau_n \tau_{n+1} (\varphi, \tau_n y_{\bar{t}}),$$

we get the energy inequality

$$\begin{aligned} &\tau_{n+1}^2 \|y_t\|_R^2 + \tau_n \tau_{n+1} \|y_n^{(0.5)}\|_A^2 - (\tau_n^2 \|y_{\bar{t}}\|_R^2 + \tau_n \tau_{n+1} \|y_{n-1}^{(0.5)}\|_A^2) \\ &\qquad \le \tau_n \tau_{n+1} (\varphi, \tau_{n+1} y_t) + \tau_n \tau_{n+1} (\varphi, \tau_n y_{\bar{t}}). \end{aligned} \tag{3.16}$$

Since $\tau_{n+1}^2 \le \tau_n \tau_{n+1} \le \tau_n^2$ (see (3.14)), then expression (3.16) takes the following form:

$$\tau_{n+1}^2 (\|y_t\|_R^2 + \|y_n^{(0.5)}\|_A^2) - \tau_n^2 (\|y_{\bar{t}}\|_R^2 + \|y_{n-1}^{(0.5)}\|_A^2) \le \tau_n^2 ((\varphi, \tau_{n+1} y_t) + (\varphi, \tau_n y_{\bar{t}})).$$

Using notation (3.6) and the Cauchy–Schwartz inequality, we rewrite the last relation:

$$\begin{aligned} \tau_n (\tau_{n+1}^2 G_n)_{\bar{t}} &\le \tau_n^2 (\tau_{n+1} (\varphi, y_t) + \tau_n (\varphi, y_{\bar{t}})) \\ &\le \tau_n^2 \|\varphi\|_{R^{-1}} (\tau_{n+1} G_n^{1/2} + \tau_n G_{n-1}^{1/2}). \end{aligned} \tag{3.17}$$

Let us note that the following identity similar to (3.8) is valid:

$$\begin{aligned} \left(\tau_{n+1} G_n^{1/2} \right)_{\bar{t}} = \frac{\tau_{n+1} G_n^{1/2} - \tau_n G_{n-1}^{1/2}}{\tau_n} &= \frac{\tau_{n+1}^2 G_n - \tau_n^2 G_{n-1}}{\tau_n (\tau_{n+1} G_n^{1/2} + \tau_n G_{n-1}^{1/2})} \\ &= \frac{(\tau_{n+1}^2 G_n)_{\bar{t}}}{\tau_{n+1} G_n^{1/2} + \tau_n G_{n-1}^{1/2}}. \end{aligned} \tag{3.18}$$

Taking into account identity (3.18) we receive from (3.17) the following estimate:

$$\tau_n(\tau_{n+1}^2 G_n)_{\bar{t}} \leq \tau_n^2 \|\varphi\|_{R^{-1}}$$

or

$$\tau_{n+1}(\|y_{t,n}\|_R^2 + \|y_n^{(0.5)}\|_A^2)^{1/2} \leq \tau_n(\|y_{t,n-1}\|_R^2 + \|y_{n-1}^{(0.5)}\|_A^2)^{1/2} + \tau_n^2\|\varphi\|_{R^{-1}}. \tag{3.19}$$

Recursively we get

$$\tau_{n+1}(\|y_{t,n}\|_R^2 + \|y_n^{(0.5)}\|_A^2)^{1/2} \leq \tau_1 \left\{ (\|y_t(0)\|_R^2 + \|y_0^{(0.5)}\|_A^2)^{1/2} + \sum_{k=1}^{n} \tau_k \|\varphi_k\|_{R^{-1}} \right\}. \tag{3.20}$$

Now using relations (3.10) from the last inequality one can easy get a priori estimate (3.15). ■

Remark 1. Let us note that stability estimates (3.4), (3.15) are received without using the Gronuoll lemma, which is usually applied for investigation of stability with respect to the right-hand side, and the estimates don't include the constant e^{cT}, which becomes large with the growth of T. If the series

$$\sum_{k=1}^{n} \tau_{k+1}\|\varphi_k\|_{R^{-1}}, \quad \sum_{k=1}^{n} \tau_k \|\varphi_k\|_{R^{-1}}$$

converge when $n \to \infty$, then estimates (3.4), (3.15) express the global stability of three-level difference scheme (2.1).

4. Stability with Respect to the Right-Hand Side on Arbitrary Grids

Let us combine the results obtained and formulate general theorem about uniform stability of three-level operator-difference schemes. We assume an arbitrary time grid, where principle of mesh refinement changes k times [4]:

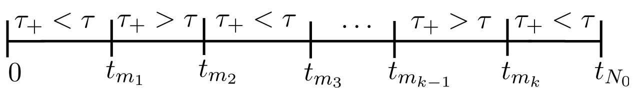

Figure 1. The non-uniform time grid.

Theorem 3. *We assume that operators of difference scheme* (2.1) $D(t)$, $B(t)$, A *satisfy the following conditions:*

$$D(t) = D^*(t) > 0, \quad A = A^* > 0, \tag{4.1}$$

$$R = D - \frac{\tau_n \tau_{n+1}}{4} A > 0, \quad B \geq \max\left\{\frac{\tau_{n+1} - \tau_n}{4} A,\ 0\right\}, \tag{4.2}$$

and A, R *are constant operators. We also assume that time steps are interrelated as*

$$\begin{gathered} \frac{\tau_{m_j}}{\tau_{m_{j+1}}} \leq c_{m_{j+1}} \leq c_0, \quad j = 0, 1, \ldots, k, \\ \tau_{m_0} = \tau_1, \quad \tau_{m_{k+1}} = \tau_{N_0}, \end{gathered} \tag{4.3}$$

where k *is the finite number of changes of mesh refinement principle.*

Then the solution of problem (2.1) *is stable with respect to the initial data and the right-hand side, and for arbitrary* τ_n *the following a priori estimate holds true (an absolute stability):*

$$\begin{aligned} \left(\|y_{t,n}\|_R^2 + \|y_n^{(0.5)}\|_A^2\right)^{1/2} &\leq c_0^k \left(\|y_{t,0}\|_R^2 + \|y_0^{(0.5)}\|_A^2\right)^{1/2} \\ &+ \sum_{s=1}^{N_0-1} \max\{\tau_s, \tau_{s+1}\} \|\varphi_s\|_{R^{-1}}. \end{aligned} \tag{4.4}$$

Proof. Let $t \in [t_{m_k}, t_{N_0}]$ and time steps become finer to the end of the interval. Then according to Theorem 2 (see (3.20)) the following estimate is valid:

$$\begin{gathered} \|y_{n+1}\|_1 \leq \frac{\tau_{m_k}}{\tau_{N_0}} \left\{\|y_{m_k}\|_1 + \sum_{s=m_k}^{N_0-1} \tau_s \|\varphi_s\|_{R^{-1}}\right\}, \\ n = m_k, m_k + 1, \ldots, N_0 - 1, \end{gathered} \tag{4.5}$$

where

$$\|y_{n+1}\|_1 = \left(\|y_{t,n}\|_R^2 + \|y_n^{(0.5)}\|_A^2\right)^{1/2}.$$

Let in the moment t_{m_k} the principle of mesh refinement changes, i. e., the time steps become related as $\tau_+ > \tau$. Then according to the Theorem 1 (see (3.9))

$$\begin{gathered} \|y_{n+1}\|_1 \leq \|y_{m_{k-1}}\|_1 + \sum_{s=m_{k-1}}^{m_k-1} \tau_{s+1} \|\varphi_s\|_{R^{-1}}, \\ n = m_{k-1}, m_{k-1} + 1, \cdots, m_k - 1. \end{gathered}$$

Substituting the last inequality to (4.5), we get the estimate

$$\begin{gathered} \|y_{n+1}\|_1 \leq \frac{\tau_{m_k}}{\tau_{N_0}} \left(\|y_{m_{k-1}}\|_1 + \sum_{s=m_{k-1}}^{N_0-1} \max\{\tau_s, \tau_{s+1}\} \|\varphi_s\|_{R^{-1}}\right), \\ n = m_{k-1}, \ldots, N_0 - 1. \end{gathered} \tag{4.6}$$

Recursively continuing (4.6) and taking into account steps interrelations (4.3), we receive the required estimate of stability. ■

5. Examples

5.1. Wave equation

In the domain $\bar{Q} = \bar{\Omega} \cup [0, T]$, $\bar{\Omega} = \{0 \le x \le l\}$ it is necessary to find the solution of the first boundary value problem for the one dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad 0 < x < l, \quad t > 0, \tag{5.1}$$

$$u(0,t) = u(l,t) = 0, \quad u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = \bar{u}_0(x). \tag{5.2}$$

On the uniform space grid $\bar{\omega}_h = \{x_i = ih, \; i = \overline{0,N}, \; hN = l\}$ and the non-uniform time grid (2.2) we replace the problem (5.1), (5.2) by the class of difference schemes with weights

$$y_{\bar{t}\hat{t}} + Ay^{(\sigma_1,\sigma_2)} = \varphi, \quad y_0 = u_0, \quad y_1 = u_1, \tag{5.3}$$

where

$$y^{(\sigma_1,\sigma_2)} = \sigma_1 \hat{y} + (1 - \sigma_1 - \sigma_2) y + \sigma_2 \check{y},$$
$$(Av)_i = -v_{\bar{x}x,i} = (v_{i+1} - 2v_i + v_{i-1})/h^2, \quad i = \overline{1, N-1}, \quad v_0 = v_N = 0,$$
$$\varphi(t) = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_{N-1}(t))^T, \quad \varphi_i(t) = f(x_i, t);$$

the operator $A : H \to H$, where the linear space $H = \Omega_h$ consists of a set of vectors $v = (v_1(t), v_2(t), \ldots, v_{N-1}(t))^T$; a scalar product and a norm in H are assigned as usual

$$(y, v) = \sum_{i=1}^{N-1} h y_i v_i, \quad \|y\| = \sqrt{(y,y)}.$$

A reduction of scheme (5.3) to the canonical form was done and the validity of all requirements of Theorem 3 were proved in [4]. Therefore the following theorem is true.

Theorem 4. *Assume that*

$$\sigma_1 \ge \sigma_2 + \max\left\{ \frac{\tau_n - \tau_{n+1}}{2(\tau_{n+1} + \tau_n)} \;, \; 0 \right\}, \quad \sigma_1 + \sigma_2 = \frac{1}{2}. \tag{5.4}$$

Then, the difference scheme with weights (5.3) *is stable in the sense of the initial data and the right-hand side and the following a priori estimate holds true*

$$\left(\|y_{t,n}\|^2 + \|y_n^{(0.5)}\|_A^2 \right)^{1/2} \le c_0^k \left(\|y_{t,0}\|^2 + \|y_0^{(0.5)}\|_A^2 \right)^{1/2} + \sum_{s=1}^{N_0-1} \max\{\tau_s, \tau_{s+1}\} \|\varphi_s\|.$$

The parameters σ_j are defined by taking into account the second order accuracy approximation condition:

$$\sigma_1 \tau_+ - \sigma_2 \tau = \frac{\tau_+ - \tau}{3}$$

and the stability requirement (5.4). Thus we get, that (see, [4]):

$$\sigma_1 = \frac{2\tau_+ + \tau}{6(\tau_+ + \tau)}, \quad \sigma_2 = \frac{\tau_+ + 2\tau}{6(\tau_+ + \tau)}. \tag{5.5}$$

5.2. Numerical experiment

The accuracy of the proposed scheme (5.3), (5.5) was examined in numerous tests. The obtained results were compared with results obtained by using classical schemes for numerical solution of differential problem (5.1), (5.2) on uniform and non-uniform grids in time. Numerical experiment was undertaken in the domain $\bar{\Omega} = [0, 1],\ t \in [0, 1]$. The exact solution is given as

$$u(x,t) = \sin\left(\frac{\pi x}{l}\right)\sin\left(\frac{\pi t}{l}\right) + t^2(x^2 - lx), \quad l = 1\,.$$

In order to check the second order of approximation and convergence of difference scheme (5.3), (5.5), the initial boundary-value problem was solved on the sequence of grids: $\omega_h \times \omega_\tau,\ \omega_{h/2} \times \omega_{\tau/2},\ \omega_{h/4} \times \omega_{\tau/4},\ \ldots$. The non-uniform time grid ω_τ was built using the random-number generator. The remaining grids were obtained by dividing each time interval into two equal parts. Thus the number of points where $\tau_j \neq \tau_{j+1}$ remains constant for all experiments. In this sense the applied time grids become close to the uniform grid.

The absolute error of solution y is given by

$$z^N = \max_{(x,t)\in\omega_{h\tau}} |y(x,t) - u(x,t)|, \quad \omega_{h\tau} = \omega_h \times \omega_\tau.$$

Since in real problems the exact solution is usually unknown, let us introduce the aposteriori error estimate of the solution $y_{h\tau}$, which can be obtained using the Runge estimator:

$$D^N = \frac{1}{3} \max_{(x,t)\in\omega_{2(h\tau)}} |y_{h\tau}(x,t) - y_{2(h\tau)}(x,t)|.$$

The second aposteriori estimate

$$p^N = \log_2(D^{N/2}/D^N)$$

gives the convergence order of the approximation $y_{h\tau}$. The results of the experiments for the time level t=1 are presented in Tab. 1.

Table 1. Convergence analysis for a test problem.

$Nx \times Nt$	20×20	40×40	80×80	160×160	320×320	640×640
z_N	0.015670	0.004022	0.001017	0.000255	0.000064	0.000016
D^N	0.014482	0.003883	0.001001	0.000254	0.000064	0.000016
p^N		1.89914	1.95513	1.97902	1.98868	2.0

References

[1] A. Bokov. Energetic inequalities for three–level difference schemes on grids changing in time. *Godishn. Vissh. Uchebni Zabed. Prilozh. Mat.*, **12**(2), 87 – 96, 1976.

[2] E. Diakonov and A. Bokov. Energetic inequalities for difference schemes on grids changing in time. *Dokl. Bolg. AN.*, **28**(2), 157 – 160, 1975.

[3] B. Jovanovich, S. Lemeshevsky and P. Matus. On the stability of differential–operator equations and operator–difference schemes as $t \to \infty$. *Comput. Meth. Appl. Math.*, **2**(2), 153 – 170, 2002.

[4] P. Matus and E. Zyuzina. Three–level difference schemes on nonuniform in time grids. *Comput. Meth. Appl. Math.*, **1**(3), 265 – 284, 2001.

[5] V. Mazhukin, P. Matus and I. Mozolevski. About stability of three–level difference schemes on non-uniform time grids. *Dokl. NAN Belarusi*, **44**(6), 23 – 25, 2000.

[6] A. Samarskii. *The Theory of Difference Schemes*. Marcel Dekker Inc., New York-Basel, 2001.

[7] A. Samarskii, P. Vabishchevich, E. Makarevich and P. Matus. Stability of three–level difference schemes on non-uniform in time grids. *Dokl. Russ. Acad. Nauk*, **376**(6), 738 – 741, 2001.

[8] A. Samarskii, P. Vabishchevich and P. Matus. *Difference Schemes with Operator Factors*. Kluwer Academic Publishers, Boston/Dordrecht/London, 2002.

[9] A. A. Samarskii and A. V. Goolin. *Stability of Difference Schemes*. Nauka, Moskow, 1973. (in Russian)

Trisluoksnių baigtinių skirtumų schemų stabilumas dešiniosios pusės atžvilgiu

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Straipsnyje nagrinėjamos trisluoksnės baigtinių skirtumų schemos su netolygiu laikinu žingsniu. Gauti aprioriniai stabilumo įverčiai pradinių duomenų ir dešiniosios pusės atžvilgiu. Pasiūlytos naujos aukštesnės aproksimacijos eilės baigtinių skirtumų schemos vienmatei bangos lygčiai. Pateikti skaitinio eksperimento rezultatai.