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# OPTIMAL CONTROL IN MODELS WITH CONDUCTIVE-RADIATIVE HEAT TRANSFER

K. BIRGELIS

*University of Latvia*

19 Raina Blvd., Riga

E-mail: K.Birgelis@alise.lv

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**ABSTRACT**

In this paper an optimal control problem for the elliptic boundary value problem with nonlocal boundary conditions is considered. It is shown that the weak solutions of the boundary value problem depend smoothly on the control parameter and that the cost functional of the optimal control problem is Frechet differentiable with respect to the control parameter.

**Key words:** boundary value problem, elliptic equation, nonlocal boundary conditions, radiative heat transfer

## 1. INTRODUCTION

In the paper of Buikis and Fitt [1] a mathematical model was given for the process of the oil burn-out from glass fabric sheets. This model shows that during burn-out process the heat transfer via radiation is playing an essential role. In order to investigate the dependence of the inner temperature of the glass fabric from the surface temperature of the furnace's heaters, we consider a steady state heat transfer process within fabric sheet-furnace system and neglect the impact of the oil burning process.

Let the infinite cylinder $\Omega_\infty = \Sigma \times \mathbb{R} \subset \mathbb{R}^3$ represents the glass fabric sheet which is dragged through the furnace with a constant speed. For reason of simplicity we suppose that the cylinder $\Omega_\infty$ is convex and the furnace is formed from two convex heaters $\Omega_1$ and $\Omega_2$.

We assume that at an appropriate distance from the furnace the temperature inside the cylinder $\Omega_\infty$ is not affected by incoming heat flux from the furnace heaters. Thus we consider only a finite subdomain $\Omega_0$ of the cylinder $\Omega_\infty$ which lies directly nearby furnace and inner temperature of which is supposed to be highly dependent on the surface temperature of the furnace heaters.

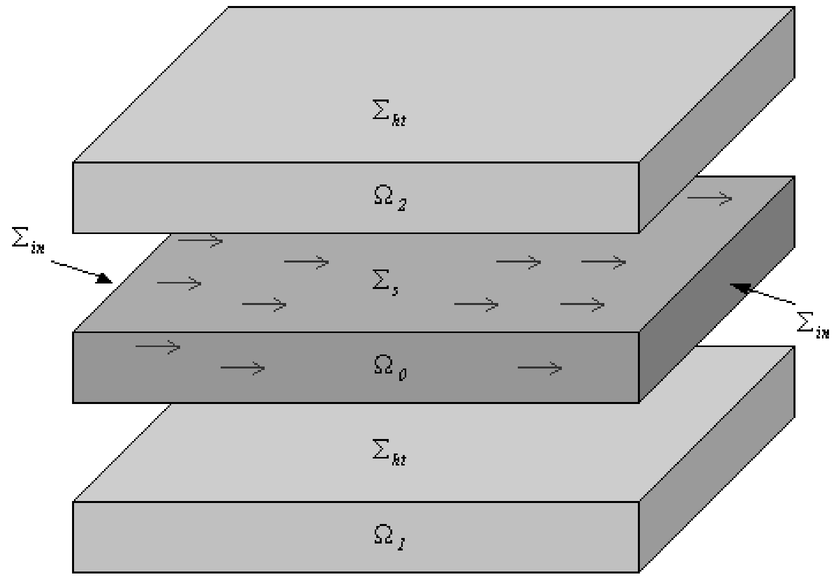


**Figure 1.** Geometry of the physical model.

We consider the steady state heat exchange when the conduction and convection occur inside the domain $\Omega_0$. The temperature $T_{in}$ is known on the contact surface $\Sigma_{in}$ between the parts $\Omega_0$ and $\Omega_\infty \setminus \Omega_0$ of the glass fabric. If $T$ is the temperature inside the domain $\Omega_0$, and $q_s$ is the heat flux through the surface $\Sigma_s$, then overall heat balance inside the domain $\Omega_0$ is determined by the following state equation:

$$\int_{\Omega_0} (k_1(\nabla T \cdot \nabla \psi) + k_2 T_{x_1} \psi)\, dv + \int_{\Sigma_s} q_s \psi \, ds = 0 \quad \forall \psi \in \dot{V}_5, \tag{1.1}$$

where $\dot{V}_5$ is an appropriate functional space.

Heat exchange between the furnace heaters and the glass fabric occurs via heat radiation emission and absorption on the surface $\Sigma_{rad} = \Sigma_s \cup \Sigma_1 \cup \Sigma_2$. If $T_{rad}$ is temperature of that surface, and $\rho = \rho(x)$ is outgoing heat radiation amount from the same surface, then both of them are tied with the integral equation:

$$\rho - (1-\epsilon)K(\rho) = \epsilon\sigma |T_{rad}|^3 T_{rad}. \tag{1.2}$$

We assume, that temperature $T_{ht}$ on the surface $\Sigma_{ht} = \Sigma_1 \cup \Sigma_2$ acts as control parameter. If we denote the temperature on the surface $\Sigma_s$ as $T_s$ then after solving the equation (1.2) we can rewrite (1.1) in the following form:

$$\int_{\Omega_0} (k_1(\nabla T \cdot \nabla \psi) + k_2 T_{x_1} \psi)\, dv + \int_{\Sigma_s} G_1(|T_s|^3 T_s)\psi\, ds$$
$$= -\int_{\Sigma_s} G_2(|T_{ht}|^3 T_{ht})\psi\, ds \quad \forall \psi \in \dot{V}_5, \quad (1.3)$$

where $G_1$, $G_2$ are some bounded linear operators.

Further, in order to minimize the temperature gradients within the $\Omega_0$ by finding optimal temperature $T_{ht}$ on the surface $\Sigma_{ht}$, we define the integral cost functional:

$$I(T) = \int_{\Omega_0} |\nabla T|^2\, dv.$$

In this paper we show that the solution of the equation (1.3) can be represented in the functional form

$$T = \Psi(T_{ht}),$$

where operator $\Psi$ is continuous and Frechet differentiable in the appropriate functional spaces. We also show continuity and Frechet differentiability of the functional $I(\Psi(T_{ht}))$.

## 2. HEAT TRANSFER EQUATION

In this section we formulate the boundary value problem, which describes heat exchange within the fabric sheet-furnace system. We also show existence of the solutions for that boundary value problem. It is important to note, that here we widely use the methodology from the paper [4], which deals with similar mathematical models of the conductive-radiative heat transfer.

Let us define geometry of the fabric sheet-furnace system:

$$\Omega_\infty := \{x \in \mathbb{R}^3 : (x_2, x_3) \in \Sigma\},$$
$$\Omega_0 := \{x \in \mathbb{R}^3 : 0 < x_1 < l, (x_2, x_3) \in \Sigma\},$$

where $l$ is a positive constant and $\Sigma \subset \mathbb{R}^2$ is a convex domain. Let $\Omega_1 \subset \mathbb{R}^3$, $\Omega_2 \subset \mathbb{R}^3$ be convex domains such that

$$\overline{\Omega}_\infty \cap \overline{\Omega}_1 = \emptyset, \quad \overline{\Omega}_\infty \cap \overline{\Omega}_2 = \emptyset, \quad \overline{\Omega}_1 \cap \overline{\Omega}_2 = \emptyset.$$

In addition, let us define:

$$\Omega_{rad} := \Omega_\infty \cup \Omega_1 \cup \Omega_2,$$
$$\Sigma_0 := \partial\Omega_0, \Sigma_1 := \partial\Omega_1, \Sigma_2 := \partial\Omega_2,$$
$$\Sigma_{ht} := \Sigma_1 \cup \Sigma_2, \Sigma_s := \Sigma_0 \backslash \partial\Omega_\infty, \Sigma_{rad} := \Sigma_{ht} \cup \Sigma_s,$$
$$\Sigma_{in} := \{x \in \Omega_\infty : x_1 = 0\} \cup \{x \in \Omega_\infty : x_1 = l\}.$$

Let us also introduce notations for the physical characteristics:

- $T = T(x)$ - temperature in the domain $\Omega_0$;
- $T_{rad} = T_{rad}(x)$ - temperature on the surface $\Sigma_{rad}$;
- $\rho = \rho(x)$ - outgoing heat radiation amount from the surface $\Sigma_{rad}$;
- $q_{rad} = q_{rad}(x)$ - radiative heat flux through the surface $\Sigma_{rad}$;
- $q_s = q_s(x)$ - radiative heat flux through the surface $\Sigma_s$;
- $\sigma$ - Stephan-Boltzmann constant;
- $0 \leq \epsilon = \epsilon(x) \leq 1$ - emissivity of the surface $\Sigma_{rad}$;
- $T_s = T_s(x)$ - temperature on the surface $\Sigma_s$;
- $T_{ht} = T_{ht}(x)$ - temperature on the surface $\Sigma_{ht}$;
- $T_{in} = T_{in}(x)$ - temperature on the surface $\Sigma_{in}$;
- $k_1 > 0$ - thermal conductivity of the glass fabric;
- $k_2$ - normalized pulling velocity of the glass fabric sheet.

In the points $(x, y) \in \Sigma_{rad} \times \Sigma_{rad}$ we define functions

$$\theta(x,y) := \begin{cases} 1, & \text{if } \{z \in \mathbb{R}^3 : z = \lambda x + (1-\lambda)y,\ 0 < \lambda < 1\} \cap \Omega_{rad} = \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

$$k(x,y) := \frac{\cos\left(\nu(x), (y-x)\right) \cos\left(\nu(y), (x-y)\right)}{\pi |x-y|^2} \theta(x,y),$$

where $\nu(x)$, $\nu(y)$ are outward normal vectors with respect to the surface $\Sigma_{rad}$.

The function $k(x, y)$ is defined almost everywhere on the set $\Sigma_{rad} \times \Sigma_{rad}$. This statement follows from the fact, that domains $\Omega_0$, $\Omega_1$, $\Omega_2$ are convex and therefore outward normal $\nu$ is defined almost everywhere on the boundaries of these domains.

The following result can be proved:

**Lemma 2.1.** *The mapping*

$$\rho \mapsto \int_{\Sigma_{rad}} k(x,y)\rho(y)\, ds(y) \tag{2.1}$$

*defines the integral operator* $K \in \mathfrak{L}(L_1(\Sigma_{rad}), L_\infty(\Sigma_{rad}))$. *In addition,*

$$\|K\|_{\mathfrak{L}(L_p(\Sigma_{rad}), L_p(\Sigma_{rad}))} < 1, \quad 1 \leq p \leq \infty.$$

*Notation 1.* Here and in the following text by $\mathfrak{L}(X, Y)$ we denote the standard space of linear bounded operators from $X$ into $Y$.

We assume that radiative heat exchange within the fabric sheet-furnace system occurs via heat radiation emission and absorption on the surface $\Sigma_{rad}$. Therefore the physical characteristics $T_{rad} \in L_4(\Sigma_{rad})$, $\rho \in L_1(\Sigma_{rad})$ are connected by the integral equation

$$(I-(1-\epsilon)K)(\rho) = \epsilon\sigma|T_{rad}|^3 T_{rad}, \tag{2.2}$$

as well the characteristics $q_{rad} \in L_1(\Sigma_{rad})$, $\rho \in L_1(\Sigma_{rad})$ are tied by the following equation

$$q_{rad} = (I-K)(\rho).$$

The following results are valid.

**Lemma 2.2.** *For known parameter $T_{rad} \in L_4(\Sigma_{rad})$ the equation (2.2) has exactly one solution $\rho \in L_1(\Sigma_{rad})$. Therefore, if $T_{rad} \in L_4(\Sigma_{rad})$, then we have the following representations:*

$$\rho = (I-(1-\epsilon)K)^{-1}(\epsilon\sigma|T_{rad}|^3 T_{rad}),$$
$$q_{rad} = (I-K)(I-(1-\epsilon)K)^{-1}(\epsilon\sigma|T_{rad}|^3 T_{rad}).$$

**Lemma 2.3.** *If $T_s \in L_4(\Sigma_s)$, $T_{ht} \in L_4(\Sigma_{ht})$, then we have the following representation:*

$$\begin{aligned} q_s &= G_1(|T_s|^3 T_s) + G_2(|T_{ht}|^3 T_{ht}) \\ &= E_1(|T_s|^3 T_s) - F_1(|T_s|^3 T_s) + G_2(|T_{ht}|^3 T_{ht}), \end{aligned} \tag{2.3}$$

*where $G_1 \in \mathfrak{L}(L_1(\Sigma_s), L_1(\Sigma_s))$, $G_2 \in \mathfrak{L}(L_1(\Sigma_{ht}), L_\infty(\Sigma_s))$,*
*$E_1 \in \mathfrak{L}(L_1(\Sigma_s), L_1(\Sigma_s))$, $F_1 \in \mathfrak{L}(L_1(\Sigma_s), L_\infty(\Sigma_s))$.*

*In addition, the following properties hold:*

- *$G_1 \in \mathfrak{L}(L_p(\Sigma_s), L_p(\Sigma_s))$, $E_1 \in \mathfrak{L}(L_p(\Sigma_s), L_p(\Sigma_s))$ and $\|I-G_1\|_{\mathfrak{L}(L_p(\Sigma_s),L_p(\Sigma_s))} < 1$ for all constants $1 \le p \le \infty$;*
- *if $u \in L_1(\Sigma_s)$ and $u \ge 0$, then $(I-G_1)(u) \ge 0$, $E_1(u) \ge 0$ and $F_1(u) \ge 0$;*
- *if $u \in L_1(\Sigma_{ht})$ and $u \ge 0$, then $G_2(u) \ge 0$;*
- *there exists a constant $0 \le c < 1$ such that $(I-G_1)(1) \le c$;*
- $$\int_{\Sigma_s} G_1(u)v\,ds = \int_{\Sigma_s} uG_1(v)\,ds$$

  *for all elements $u \in L_{p_1}(\Sigma_s)$, $v \in L_{p_2}(\Sigma_s)$, where constants $1 \le p_1, p_2 \le \infty$ and $1/p_1 + 1/p_2 = 1$.*

In order to formulate the state equation, which describes the heat balance within the domain $\Omega_0$, we introduce specific functional spaces

$$V_p := W_2^1(\Omega_0) \cap L_p(\Sigma_s),$$
$$\dot{V}_p := \{u \in V_p : u\,|_{\Sigma_{in}} = 0\}, \quad 1 \le p \le \infty.$$

The following results are valid.

**Lemma 2.4.** *The integral*

$$I_1(u, v, \psi) = \int_{\Omega_0} (k_1(\nabla(u+v) \cdot \nabla\psi) + k_2(u+v)_{x_1}\psi)\, dv + \int_{\Sigma_s} G_1(|u+v|^3(u+v))\psi\, ds$$

*is defined for all elements $v \in V_5$, $u \in \dot{V}_5$, $\psi \in \dot{V}_5$. The mapping*

$$(u, v) \mapsto I_1(u, v, \cdot)$$

*defines an operator $A_1 : \dot{V}_5 \times V_5 \mapsto \dot{V}_5^*$, which has the following properties:*

- *it maps the set $\dot{V}_\infty \times V_\infty$ in the space $V_2^*$;*
- *the mapping $A_1(\cdot, v)$ is coercive for all elements $v \in V_5$;*
- *the mapping $A_1(\cdot, v)$ can be represented as sum of a radial monotone operator $B_v : \dot{V}_5 \mapsto \dot{V}_5^*$ and a weakly continuous operator $C_v : \dot{V}_5 \mapsto \dot{V}_5^*$ for all elements $v \in V_5$.*

**Lemma 2.5.** *The integral*

$$I_2(z, \psi) = -\int_{\Sigma_s} G_2(|z|^3 z)\psi\, ds$$

*is defined for all elements $\psi \in \dot{V}_2$, $z \in L_\infty(\Sigma_{ht})$. The mapping*

$$z \mapsto I_2(z, \cdot)$$

*defines an operator $A_2 : L_\infty(\Sigma_{ht}) \mapsto \dot{V}_2^*$, which has Frechet derivative $A_2'[z]$ for all $z \in L_\infty(\Sigma_{ht})$.*

The physical characteristics $T \in V_5$, $q_s \in L_{5/4}(\Sigma_s)$ are tied by the following heat balance equation

$$\int_{\Omega_0} (k_1(\nabla T \cdot \nabla\psi) + k_2 T_{x_1}\psi)\, dv + \int_{\Sigma_s} q_s\psi\, ds = 0, \quad \forall\psi \in \dot{V}_5. \tag{2.4}$$

Further we assume that there exists an element $w_0 \in V_5$ such that it's trace on the surface $\Sigma_{in}$ is equal to $T_{in}$. We also assume that the temperature $T \in V_5$ can be represented as the sum

$$T = w + w_0,$$

where $w \in \dot{V}_5$.

Then from the equation (2.4) we get the final equation

$$A_1(w, w_0) = A_2(T_{ht}), \tag{2.5}$$

which ties the temperature $T_{ht} \in L_\infty(\Sigma_{ht})$ with the parameters $w \in \dot{V}_5$, $w_0 \in V_5$.

**Lemma 2.6.** *For known parameters $T_{ht} \in L_\infty(\Sigma_{ht})$, $w_0 \in V_\infty$ the equation (2.5) has at least one solution $w \in \dot{V}_\infty$. In addition, the following estimate is valid:*

$$\|w\|_{\dot{V}_\infty} \le c(w_0, T_{ht})\|w\|_{\dot{V}_5}.$$

*Proof.* Properties of the operators $A_1$, $A_2$ ensure existence of the element $w \in \dot{V}_5$, which is a solution of the equation (2.5) ([2]). Boundedness of the solution $w \in \dot{V}_5$ can be proved by using Mozer's iteration method. Here it is possible to adopt the already existing proof for the case, when boundedness of the solutions for an elliptic Dirichlet problem is proved([3]). ■

## 3. LINEARIZED EQUATION

In this section we study the linearized equation of the equation (2.5).

The following results can be proved:

**Lemma 3.1.** *The integral*

$$I_3(u, \psi) = \int_{\Omega_0} (k_1(\nabla u \cdot \nabla \psi) + k_2 u_{x_1} \psi)\, dv$$

*is defined for all elements $u \in \dot{V}_2$, $\psi \in \dot{V}_2$. The mapping*

$$u \mapsto I_3(u, \cdot)$$

*defines an operator $A_3 \in \mathfrak{L}(\dot{V}_2, \dot{V}_2^*)$, where also the inverse operator $A_3^{-1} \in \mathfrak{L}(\dot{V}_2^*, \dot{V}_2)$ exists.*

**Lemma 3.2.** *The integral*

$$I_4(u, v, \psi) = \int_{\Sigma_s} G_1(|v|u)\psi\, ds$$

*is defined for all elements* $u \in \dot{V}_2$, $v \in L_2(\Sigma_s)$, $\psi \in \dot{V}_2$. *The mapping*

$$(u, v) \mapsto I_4(u, v, \cdot)$$

*defines an operator* $A_4 : \dot{V}_2 \times L_2(\Sigma_s) \mapsto \dot{V}_2^*$, *where the operator* $A_1(\cdot, v) \in \mathfrak{L}(\dot{V}_2, \dot{V}_2^*)$ *is compact for all elements* $v \in L_2(\Sigma_s)$.

Next, let us define the linearized equation

$$\int_{\Omega_0} (k_1(\nabla u \cdot \nabla\psi) + k_2 u_{x_1}\psi)\, dv + \int_{\Sigma_s} G_1(|v|u)\psi\, ds = f(\psi) \quad \forall\psi \in \dot{V}_2, \quad (3.1)$$

where $v \in L_2(\Sigma_s)$, $f \in \dot{V}_2^*$ are fixed parameters, but $u \in \dot{V}_2$ is the unknown parameter.

If we take into account definitions of the operators $A_3$, $A_4$, then the equation (3.1) can be rewritten in the equivalent form

$$A_3(u) + A_4(u, v) = f, \quad (3.2)$$

where $v \in L_2(\Sigma_s)$, $f \in \dot{V}_2^*$, but $u \in \dot{V}_2$.

**Theorem 3.1.** *For every parameter* $v \in L_2(\Sigma_s)$ *there exists an operator* $(A_3(\cdot) + A_4(\cdot, v))^{-1} \in \mathfrak{L}(\dot{V}_2^*, \dot{V}_2)$.

*Proof* We fix an element $v \in L_2(\Sigma_s)$. Let us prove the fact, that the equation

$$A_3(u) + A_4(u, v) = 0 \quad (3.3)$$

can have only a trivial solution $u \in \dot{V}_2$.

Let us assume the opposite, i.e. that equation (3.3) has a nontrivial solution $u \in \dot{V}_2$. Let us also fix the family of sample functions

$$\{u_\tau = \max\{\min\{\frac{u}{\tau}, 1\}, -1\} : 0 < \tau \leq 1\}.$$

Properties of the operators $A_3(\cdot)$, $A_4(\cdot, v)$ ensure that for a sufficiently small constant $0 < \delta \leq 1$ we have the estimate

$$\int_{\Omega_0} (k_1(\nabla u \cdot \nabla u_\tau) + k_2 u_{x_1} u_\tau)\, dv + \int_{\Sigma_s} G_1(|v|u) u_\tau\, ds > 0 \quad \forall\tau \in (0, \delta],$$

which contradicts with the fact, that the element $u \in \dot{V}_2$ is a solution of the equation (3.3).

Next, the fact that the equation (3.3) can have only trivial solution and properties of the operators $A_3(\cdot)$, $A_4(\cdot, v)$ ensure that the operator $(A_3(\cdot) + A_4(\cdot, v))^{-1} \in \mathfrak{L}(\dot{V}_2^*, \dot{V}_2)$ exists.

■

## 4. CONTINUOUS DEPENDENCE

In this section we prove that the solutions of the equation (2.5) depend continuously on the control parameter $T_{ht} \in L_\infty(\Sigma_s)$. As a consequence we get the uniqueness of the solutions of (2.5) and continuity of the cost functional with respect to the control parameter $T_{ht} \in L_\infty(\Sigma_s)$.

**Lemma 4.1.** *Let $\{T^n_{ht}\}_{n\in\mathbb{N}} \subset L_\infty(\Sigma_s)$ be a sequence which converges to an element $T^0_{ht} \in L_\infty(\Sigma_s)$ in the $L_\infty(\Sigma_s)$-norm. Let $\{u^n\}_{n\in\mathbb{N}} \subset \dot{V}_\infty$ be the corresponding sequence of the solutions of the equation (2.5) and let $u^0 \in V_\infty$ be the solution of (2.5) that corresponds to the element $T^0_{ht}$. Then sequence $\{u^n\}_{n\in\mathbb{N}}$ converges to the element $u^0$ in the $\dot{V}_2$-norm.*

*Proof* Let us define sequences:

$$\{\delta T^n_{ht} := T^n_{ht} - T^0_{ht}\}_{n\in\mathbb{N}},$$
$$\{\delta u^n := u^n - u^0\}_{n\in\mathbb{N}}.$$

The following equality

$$A_1(u^n, w_0) - A_1(u^0, w_0) = A_2(T^n_{ht}) - A_2(T^0_{ht}) \tag{4.1}$$

holds for every index $n \in \mathbb{N}$.

It is possible to prove that the equality (4.1) can be rewritten as

$$A_3(\delta u^n) + A_4(\delta u^n, a^n) = A_2(b^n \delta T^n_{ht}), \tag{4.2}$$

where

$$a^n = \int_0^1 4|u^0 + T_0 + \tau\delta u^n|^3\, d\tau,$$
$$b^n = \int_0^1 4|T^0_{ht} + \tau\delta T^n_{ht}|^3\, d\tau$$

and $n \in \mathbb{N}$.

Next, we prove that $\|\delta u^n\|_{L_2(\Sigma_s)} \to 0$ as $n \to \infty$. Let us assume the opposite. Then without a loss of generality we get that for every index $n \in \mathbb{N}$ there exists a constant $\delta > 0$ such that

$$\|\delta u^n\|_{L_2(\Sigma_s)} \geq \delta. \tag{4.3}$$

Properties of the operator $A_1$ guarantee boundedness of the sequence $\{\|\delta u^n\|_{\dot{V}_2}\}_{n\in\mathbb{N}}$. Therefore we get that there exists an element $\delta u^* \in \dot{V}_2$ such that $\delta u^n \rightharpoonup \delta u^*$ as $n \to \infty$.

Since embedding of the space $\dot{V}_2$ in the space $L_2(\Sigma_s)$ is compact then

$$\|\delta u^n - \delta u^*\|_{L_2(\Sigma_s)} \to 0$$

as $n \to \infty$. If we take into account the estimate (4.3) then we get

$$\|\delta u^*\|_{L_2(\Sigma_s)} \geq \delta.$$

Without a loss of generality we can prove that

$$A_3(\delta u^n) + A_4(\delta u^n, a^n) - A_2(b^n \delta T^n_{ht}) \rightharpoonup A_3(\delta u^*) + A_4(\delta u^*, a^*) \tag{4.4}$$

as $n \to \infty$ and where

$$a^* = \int_0^1 4|u^0 + T_0 + \tau\delta u^*|^3 \, d\tau.$$

If we combine the equality (4.2) with (4.4), then we get that the equation (3.3) has a nontrivial solution $u^*$. But Theorem 3.1 states that such fact it is impossible, therefore we must assume that $\|\delta u^n\|_{L_2(\Sigma_s)} \to 0$ as $n \to \infty$.

Next, we prove that $\|\delta u^n\|_{\dot{V}_2} \to 0$ as $n \to \infty$. The fact, that $\|\delta u^n\|_{L_2(\Sigma_s)} \to 0$ as $n \to \infty$ ensures that

$$\|A_4(\delta u^n, a^n) - A_2(b^n \delta T^n_{ht})\|_{\dot{V}_2} \underset{n\to\infty}{\longrightarrow} 0. \tag{4.5}$$

But then it follows from the equality (4.2) that

$$\|A_3(\delta u^n)\|_{\dot{V}_2} \underset{n\to\infty}{\longrightarrow} 0. \tag{4.6}$$

Since the operator $A_4$ is invertable, then from (4.6) it follows that $\|\delta u^n\|_{\dot{V}_2} \to 0$ as $n \to \infty$.

■

As direct consequences of the Lemma 4.1 we formulate the following theorems:

**Theorem 4.1.** *For known parameters $T_{ht} \in L_\infty(\Sigma_{ht})$, $w_0 \in V_\infty$ the equation (2.5) has exactly one solution $w \in \dot{V}_\infty$. The mapping*

$$T_{ht} \mapsto w$$

*defines an operator $\Phi : L_\infty(\Sigma_s) \mapsto \dot{V}_2$, which is continuous.*

**Theorem 4.2.** *The cost functional*

$$I(T_{ht}) = \int_{\Omega_0} |\nabla(\Phi(T_{ht}) + w_0)|^2 \, dv \tag{4.7}$$

*is continuous with respect to $T_{ht} \in L_\infty(\Sigma_s)$.*

## 5. FRECHET DIFFERENTIABILITY

In this section we prove that solutions of the equation (2.5) depend smoothly on the control parameter $T_{ht} \in L_\infty(\Sigma_s)$. We also get Frechet differentiability of the cost functional with respect to the control parameter $T_{ht} \in L_\infty(\Sigma_s)$.

**Theorem 5.1.** *The operator $\Phi$ has Frechet derivative $\Phi'[T_{ht}]$ for all $T_{ht} \in L_\infty(\Sigma_s)$.*

*Proof* Let us fix an element $T_{ht}^0 \in L_\infty(\Sigma_s)$. It is possible to prove that there exists a constant $\delta > 0$ such that for every element $T_{ht} \in L_\infty(\Sigma_s)$ with $\|T_{ht} - T_{ht}^0\|_{L_\infty(\Sigma_s)} < \delta$ the following equality holds:

$$A_1(\Phi(T_{ht}), w_0) = A_1(\Phi(T_{ht}^0), w_0) + A_3(\Phi(T_{ht}) - \Phi(T_{ht}^0)) \\ + A_4(\Phi(T_{ht}) - \Phi(T_{ht}^0), 4|\Phi(T_{ht}^0) + w_0|^3) + o(\|\Phi(T_{ht}) - \Phi(T_{ht}^0)\|_{\dot{V}_2}). \quad (5.1)$$

At the same time

$$A_2(T_{ht}) = A_2(T_{ht}^0) + A_2'[T_{ht}^0](T_{ht} - T_{ht}^0) + o(\|T_{ht} - T_{ht}^0\|_{L_\infty(\Sigma_s)}) \quad (5.2)$$

and

$$\|\Phi(T_{ht}) - \Phi(T_{ht}^0)\|_{\dot{V}_2} \leq c(T_{ht}^0)\|T_{ht} - T_{ht}^0\|_{L_\infty(\Sigma_s)}. \quad (5.3)$$

If we take into account (5.1), (5.2) and the estimate (5.3) then we get

$$\frac{\|\Phi(T_{ht}) - \Phi(T_{ht}^0) - (A_3 + A_4)^{-1} A_2'[T_{ht}^0](T_{ht} - T_{ht}^0)\|_{\dot{V}_2}}{\|T_{ht} - T_{ht}^0\|_{L_\infty(\Sigma_s)}} \to 0$$

as $\|T_{ht} - T_{ht}^0\|_{L_\infty(\Sigma_s)} \to 0$.
■

**Theorem 5.2.** *The cost functional*

$$I(T_{ht}) = \int_{\Omega_0} |\nabla(\Phi(T_{ht}) + w_0)|^2 \, dv \quad (5.4)$$

*has Frechet derivative for all $T_{ht} \in L_\infty(\Sigma_s)$.*

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## Optimalus valdymas modeliuose su laidžiu-radioaktyviu šilumos pernešimu

K. Birgelis

Darbe nagrinėjamas nelokalaus kraštinio uždavinio optimalaus valdymo uždavinys. Parodyta, kad silpnasis kraštinio uždavinio sprendinys tolydžiai priklauso nuo valdomojo parametro, taigi, optimalaus valdymo tikslo funkcija yra diferencijuojama Freše prasme pagal valdomus parametrus.