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THE R-FUNCTIONS METHOD IN THE BOUNDARY VALUE PROBLEM FOR A COMPLEX DOMAIN POSSESSING THE SYMMETRY

YU.S. SEMERICH
Institute for Problems in Machinery of the National Academy of Siences of Ukraine
Dm. Pozhgarskogo str., 2/10, 61046 Kharkov, Ukraine
E-mail: semerich@ipmach.kharkov.ua

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ABSTRACT

This paper considers a case of the punctual symmetry of cyclic type and application of the R-functions method in building equations of the boundaries for symmetric objects. The R-functions method combined with the Ritz method is used for solving the Laplace equation in the complex symmetric domain. The problem is described for calculating an electrostatic field in the system with crossed fields (the cylindrical magnetron of reversed construction).

Key words: The R-functions method, symmetry, the Laplace equation, cylindrical magnetron, general structure of solution

1. THE R-FUNCTIONS METHOD (RFM)

In the problems of field solving, the solution depends on the specific characteristics of fundamental physics laws on the solid forms where the field exciting, form of contact areas, forms and arrangements of field exciters.

The problem of field determination belongs to the boundary value problems for partial differential equations and requires the solution of the equation in the domain Ω

$$Au = f$$

supplemented with the boundary conditions

$$L_i u = \varphi_i \text{ at } \partial\Omega_i \ (i = 1, 2, \dots, m),$$

where $\partial\Omega_1, \partial\Omega_2, \dots, \partial\Omega_m$ are parts of the boundary $\partial\Omega$ of the domain Ω.

Let the functions u, f, φ_i and the operators A, L_i be called analytical components of the boundary problem and the domain Ω, the boundary $\partial\Omega$ and parts of boundary $\partial\Omega_i$ – geometric components.

Presenting analytic and geometric information in the boundary value statement has a lot of specific difficulties, it deals with necessity to include the geometric information into the algorithm for solving the problem. The geometric information is taken into account in the classical Fourier method or integral transformation by using a good choice of co-ordinates, in the conformal mapping method by building a convenient reflect function, in the variation methods by a good choice of coordinate sequence functions.

This paper describes the main ideas of the RFM, it presents the general approach for construction of the approximate solutions of the boundary value problems on analytical level and allows us to take into account the geometric components of the boundary value problems. The RFM is based on the R-functions theory and it is proposed by Rvachev in 1963 [4]. The essential feature of the RFM is construction of loci equations and General Structure of Solution (GSS), which exactly satisfy the boundary conditions [7].

For construction of loci equations the following systems of R-functions (R-operations) are used:

R-conjunction

$$x \wedge_\alpha y \equiv \frac{1}{1+\alpha}\left[x + y - \sqrt{x^2 + y^2 - 2\alpha xy}\right],$$

R-disjunction

$$x \vee_\alpha y \equiv \frac{1}{1+\alpha}\left[x + y + \sqrt{x^2 + y^2 - 2\alpha xy}\right], \qquad (1.1)$$

R-negation

$$\overline{x} \equiv -x,$$

where $\alpha = \alpha(x, y)$ is an arbitrary function such that $-1 < \alpha \leq 1$ [5].

Usually, the following system R_0 is used

$$x \wedge_0 y \equiv x + y - \sqrt{x^2 + y^2},$$
$$x \vee_0 y \equiv x + y + \sqrt{x^2 + y^2},$$
$$\overline{x} \equiv -x.$$

The other systems of the R-functions are described in [5].

The first two expressions of (1.1) correspond to the logical operations of the sets intersection and union, and the third expression describes the negation.

In the RFM, the complex locus is described by some logical expression between simple loci (primitives) and then the conversion to analytical geometry presentation $\omega(x, y) = 0$ or $\omega(x, y) \geq 0$ by the R-functions system is carried out. At the present time, the construction of loci equations is well studied and used by many authors [2; 3; 9].

A lot of mathematical physics problems require finding solutions in symmetric domains. In many cases taking account of symmetry results in a considerable simplification of both building complex loci equations by the RFM and finding necessary solutions. It becomes possible to conduct efficiently numerical experiments in computer modeling of the given problems. However, the method of building loci equations, possessing symmetry, using the RFM has been elaborated comparatively recently [6; 8; 10].

Let us consider the method for construction loci equations, possessing the translation type symmetry and the punctual symmetry of cyclic type.

A case of the translation type symmetry along an axis when translated domains can be separated by some periodical systems of regions have been firstly considered in the work [5].

DEFINITION 1.1. Let $\Sigma_0 = [\sigma_0(x,y) \geq 0]$, $\sigma_0(x,y) \in C^m(\Omega)$ be some symmetry along an Y-axis domain which can be enclosed by vertical region $-a < x < a$. The domains $\Sigma_i = [\sigma_0(x - hi, y) \geq 0]$, $i = 0, \pm 1, \pm 2, \dots$ are obtained by displacement of Σ_0 along an X-axis by the value $h > 2a$, h is a step of translation. Then the boundary $\partial\Omega$ of the domain

$$\Omega = \bigcap_{i=-\infty}^{\infty} \Sigma_i$$

can be written as

$$\omega(x,y) \equiv -\sigma_0(\mu(x,a,h),y) = 0, \tag{1.2}$$

where

$$\mu(x,h) = \frac{4h}{\pi^2} \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{(2i-1)^2} \sin \frac{(2i-1)x\pi}{h}. \tag{1.3}$$

DEFINITION 1.2. If all assumptions of Definition 1.1 are fulfilled except that the domain $\Sigma_0 = [\sigma_0(x,y) \geq 0]$ has a symmetry along an Y-axis then the boundary $\partial\Omega$ of domain Ω can be written as

$$\omega(x,y) \equiv \left\{\sigma_0(\mu(x,a,h),y) \wedge_\alpha \cos\left(\frac{x\pi}{h}\right)\right\} \vee_\alpha \left\{\sigma_0(\mu(x-h,a,h),y) \wedge_\alpha \cos\left(\frac{(x-h)\pi}{h}\right)\right\}. \tag{1.4}$$

The domains possessing the translation type symmetry are showed in Fig. 1.

Let us consider a case of the punctual symmetry of cyclic type. Let $\Sigma_0 = [\sigma_0(x,y) \geq 0]$ be some domain, then $\Sigma = [\sigma(x,y) \equiv \sigma_0(x - r_0, y) \geq 0]$ is

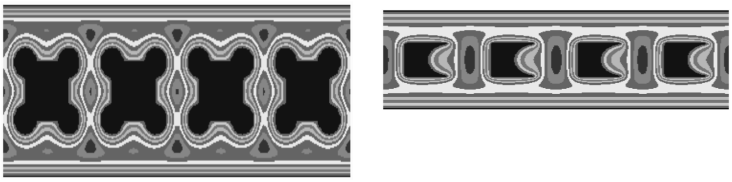

Figure 1. Domains with the translation type symmetry.

displacement on distance r_0 along an X-axis. We set the task to write locus equation $\omega(x, y) = 0$ of the domain

$$\Omega = \bigcup_{i=1}^{n} \Sigma_i$$

presented by turning the domain $\Sigma = [\sigma(x, y) \geq 0]$ on corners $\frac{2\pi k}{n}$ $(k = 0, 1, \ldots, n-1)$ around of coordinates center.

DEFINITION 1.3. Let $\sigma_0(x, y) \in C^m(\Omega)$ is a normalized function, i.e.:

$$\frac{\partial \sigma_0}{\partial \nu}|_{\Sigma_0 = 0} = 1,$$

the domain $\Sigma_0 = [\sigma_0(x, y) \geq 0]$ has symmetry along an X-axis and it can be enclosed inside the sector $-\alpha \leq \theta \leq \alpha$, $0 < \alpha < \pi/n$. Then the normalized equation $\partial\Omega = [\omega(x, y) = 0]$, $\omega \in C^m(\Omega)$ can be written as

$$\omega(x, y) \equiv \sigma_0(r \cos \mu(n\theta) - r_0, r \sin \mu(n\theta)) = 0, \tag{1.5}$$

$$r = \sqrt{x^2 + y^2}, \theta = \arctan\left(\frac{x}{y}\right),$$

where

$$\mu(n\theta) = \frac{8}{n\pi} \sum_{i=1}^{\infty} (-1)^{i+1} \frac{\sin[(2i-1)\frac{n\theta}{2}]}{(2i-1)^2}. \tag{1.6}$$

The domains possessing the punctual symmetry of cyclic type are presented in Fig. 2.

DEFINITION 1.4. If the domain $\Sigma_0 = [\sigma_0(x, y) \geq 0]$ does not have a symmetry

Figure 2. Domains with the punctual symmetry of cyclic type.

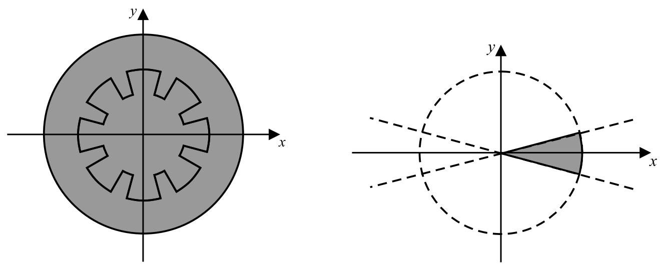

Figure 3. a) Domain Ω. b) Domain Ω_1.

along an Y-axis then we use

$$\omega(x,y) \equiv \left\{\sigma_0(r\cos\mu(n\theta) - r_0, r\sin\mu(n\theta)) \wedge_\alpha \cos\left(\frac{n\theta}{2}\right)\right\}$$
$$\vee_\alpha \left\{\sigma_0\left(r\cos\mu\left(n\left(\theta - \frac{2\pi}{n}\right)\right) - r_0, r\sin\mu\left(n\left(\theta - \frac{2\pi}{n}\right)\right)\right)\right.$$
$$\left.\wedge_\alpha \cos\left(\frac{n}{2}\left(\theta - \frac{2\pi}{n}\right)\right)\right\}. \tag{1.7}$$

Construction of Loci Equation

Let us write locus equation of the domain Ω, shown in Fig.3a. As it can be seen from Fig.3a, the domain Ω has the punctual symmetry of cyclic type. Let us choose the domain Ω_1, which is shown in Fig.3b, as a translation element.

For construction of Boolean set for the geometric domain in Fig. 3b, we choose the primitives Σ_i, represented by inequalities $\sigma_i(x,y) \geq 0$ as halfspaces

$$\Sigma_I = \left[\sigma_I \equiv \left(\frac{ax+by}{\sqrt{a^2+b^2}}\right) \geq 0\right],$$

$$\Sigma_{II} = \left[\sigma_{II} \equiv \left(\frac{cx+dy}{\sqrt{c^2+d^2}}\right) \geq 0\right],$$

or as circle of radius r_1

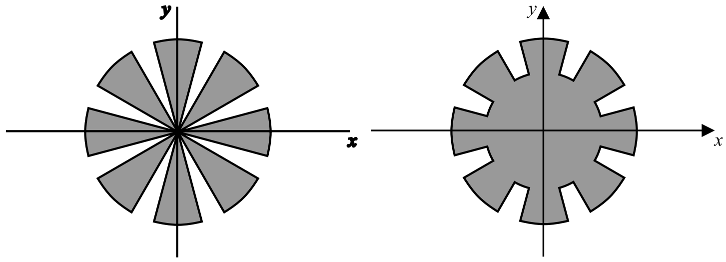

Figure 4. a) Domain Ω_2. b) Domain Ω_3.

$$\Sigma_{III} = \left[\sigma_{III} \equiv \frac{1}{2r_1}\left(r_1^2 - x^2 - y^2\right) \geq 0\right],$$

where $a = \frac{1}{4}$, $b = 1$, $c = \frac{1}{4}$, $d = -1$, $r_1 = 1.8$.

The domain shown in Fig. 3b is defined by the Boolean set expression

$$\Sigma_I \cap \Sigma_{II} \cap \Sigma_{III}.$$

Replacing symbols Σ_i by inequalities σ_i and the Boolean operation $\cap$ by the corresponding R-functions $\wedge_0$ we immediately obtain a single implicit function

$$\omega_1(x, y) = \sigma_I \wedge_0 \sigma_{II} \wedge_0 \sigma_{III},$$

which is positive inside the domain, negative outside, and takes zero value on the boundary of domain.

Expression for domain Ω_2, which is shown in Fig.4a, can be written with the help of (1.5) in the form

$$\omega_2(x, y) \equiv \omega_1(r \cos \mu(n\theta), r \sin \mu(n\theta)) = 0.$$

Expression for domain Ω_3 from Fig. 4b can be written in the form

$$\omega_3(x, y) \equiv \omega_2 \vee_0 \omega_4 = 0,$$

where

$$\omega_4(x, y) \equiv \frac{1}{2r_2}(r_2^2 - x^2 - y^2) = 0$$

defines a circle of radius $r_2 = 1.3$.

Finally, the locus equation shown in Fig. 3a can be written in the form

$$\omega(x, y) \equiv \overline{\omega_4} \wedge_0 \omega_5 = 0,$$

where

$$\omega_5(x,y) \equiv \frac{1}{2r_3}(r_3^2 - x^2 - y^2) = 0$$

defines a circle of radius $r_3 = 3$.

Thus, we derived the real function which defines the boundary equation for the complex domain Ω. Fig. 5 shows a surface plot of the function $\omega(x,y)$ which is positive inside the domain and is zero on the boundary. Equation $\omega = 0$ defines the geometry of the domain implicitly, such function $\omega(x,y)$ is called *implicit* function for the specified geometric domain.

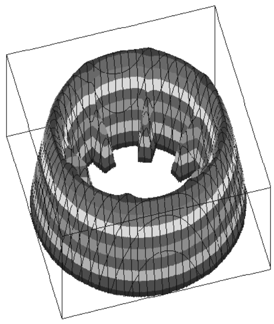

Figure 5. Surface plot of the function ω.

2. BOUNDARY VALUE PROBLEM

Let us consider the example illustrating the application of the RFM for the solution of boundary value problems. We want to find the solution of electrostatic problem in the system with crossed fields (the cylindrical magnetron of reversed construction). In this case, the distribution of potential is described by the Laplace equation

$$-\triangle u(x,y) = 0, \tag{2.1}$$

for $(x,y) \in \Omega$ (Fig. 3a) and it satisfies the Dirichlet boundary conditions

$$u\,|_{\partial\Omega_1} = U_a, \quad u\,|_{\partial\Omega_2} = 0, \tag{2.2}$$

where $\partial\Omega_1$ is the interior boundary of the domain Ω, $\partial\Omega_2$ is the exterior one.

As it is shown in [5], the solution of this problem can be find in the form

$$u = u_0 + u_1 = \omega\Phi + \varphi,$$

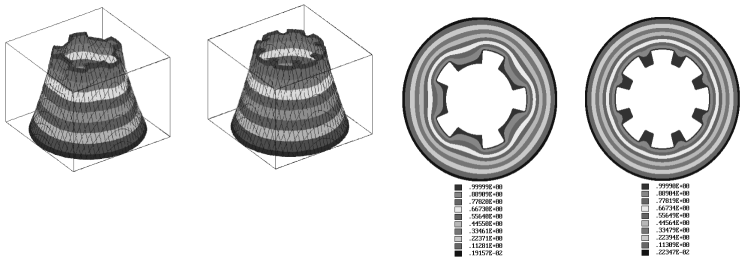

Figure 6. The numerical solution for various cases of the anode block of resonator.

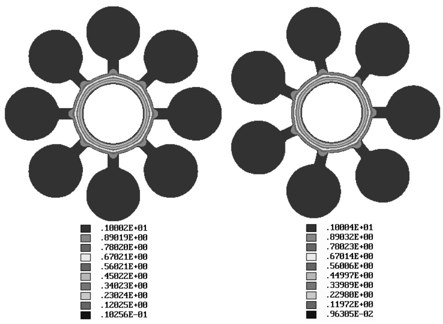

Figure 7. The numerical solution for various number of resonator cells.

where $\omega(x, y) = 0$ is the boundary equation of the domain Ω, Φ is the unknown component, $\varphi(x, y)$ is the elementary function, constructed by the following expression [5]

$$\varphi = \frac{U_a \omega_2}{\omega_1 + \omega_2}.$$

The unknown component $\Phi(x, y)$ is approximated in the form

$$\Phi(x, y) = \sum_{i=1}^{n} c_i \psi_i(x, y),$$

where $\psi_i(x, y)$ is some complete system of functions, for example, algebraic polynomials, Chebyshev's polynomials, trigonometric functions, splines, or atomaric functions [5]. The $\{c_i\}$ are unknown coefficients, determinated by the Ritz method from the minimum of functional

$$I(u_1) = \int_{\Omega} [(gradu_1)^2 + 2(gradu_1)(gradu_0)]dxdy.$$

Numerical realization is carried out using the programming system POLYE, created at the department of applied mathematics and computational methods under the guidance of V.L. Rvachev. The isolines are showed in Fig. 6 for various cases of the anode block of resonator.

The presented results have a good coincidence with results obtained by variable separation method [1]. However, changing only one parameter n we can conduct computational experiments in modeling of the given problems.

In the design of impulse magnetrons of centimeter range, the anode block with slot-hole resonator type is widely used [11]. For such case, the isolines for various number of resonator cells are shown in Fig. 7.

3. CONCLUSION

We considered the underlying principles of the RFM for construction locus equation, possessing symmetry. The RFM allows us to use variational methods for solving boundary value problems in domains with a complicate geometry. The RFM is unique among meshfree methods because RFM solution can be constructed to satisfy all boundary conditions exactly.

Note that, the RFM and programming system POLYE provide the possibility to work with alphabetic parameters *a, b, c, d, n,* r_1, r_2, r_3, to conduct multivariable calculations and analyse electrostatic fields. At the same time, the CPU time and numbers of the R-operations are decreased essentially.

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R – funkcijų metodas kraštiniam uždaviniui sudėtingose srityse, turinčiose simetriją

Yu.S. Semerich

Staipsnyje nagrinėjamas ciklinio tipo tikslios simetrijos sritys. R – funkcijų metodas taikomas sudarant simetrinių objektų kontūrų lygtis. Šis metodas, kombinuojant jį su Ritz'o metodu, panaudotas sprendžiant Laplaso lygties Dirichlė kraštinį uždavinį sudėtingoje simetrinėje srityje. Uždavinys aprašo elektrostatinio lauko, gaunamo susikertant magnetinių laukų sistemai, potencialo pasiskirstymą, t.y., jis modeliuoja reversinės konstrukcijos cilindrinį magnetroną. Šis uždavinys realizuotas skaitiškai kompiuterinės programos POLYE pagalba.