Mathematical Modelling and Analysis
Volume 8 Number 2, 2003, pages 165–174
 ISSN 1392-9292

NECESSARY OPTIMALITY CONDITIONS FOR NONSMOOTH VECTOR OPTIMIZATION PROBLEMS

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Received September 1, 2002; revised February 13, 2003

ABSTRACT

In this paper we introduce a notion of generalized derivative for nonsmooth vector functions in order to obtain necessary optimality conditions for vector optimization problems. This definition generalizes to the vector case the notion introduced by Michel and Penot and extended by Yang and Jeyakumar. This generalized derivative is contained in the Clarke subdifferential and then the corresponding optimality conditions are sharper than the Clarke's ones.

Key words: vector optimization, generalized derivatives

1. INTRODUCTION

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be locally Lipschitzian (or of class $C^{0,1}$) at $x_0 \in \mathbb{R}^n$ when there exist constants K_{x_0} and δ_{x_0} such that

$$\|f(x_1) - f(x_2)\| \leq K_{x_0} \|x_1 - x_2\|$$

$\forall x_1, x_2 \in \mathbb{R}^n$, $\|x_1 - x_0\| < \delta_{x_0}$, $\|x_2 - x_0\| < \delta_{x_0}$. For this type of functions, Rademacher theorem states that f is almost everywhere differentiable (in the sense of Lebesgue measure). Then the first order Clarke subdifferential of f at x_0, denoted by $\partial f(x_0)$ is defined as ([2])

$$\partial f(x_0) = \operatorname{cl}\operatorname{conv}\{l = \lim \nabla f(x_k), x_k \to x_0, \nabla f(x_k) \text{ exists}\},$$

where $\operatorname{cl}\operatorname{conv}\{\dots\}$ is the closure of the convex hull of all limit points. Similarly for a function $f : \mathbb{R}^n \to \mathbb{R}^m$ of class $C^{1,1}$, that is a differentiable function with a locally Lipschitzian jacobian, the second order Clarke subdifferential of f at x_0, denoted by $\partial^2 f(x_0)$, is defined as

$$\partial^2 f(x_0) = \mathrm{cl\,conv}\,\{l = \lim \nabla^2 f(x_k), x_k \to x_0, \nabla^2 f(x_k) \text{ exists}\}.$$

Thus $\partial^2 f(x_0)$ is a subset of the finite dimensional space $L(\mathbb{R}^n; L(\mathbb{R}^n; \mathbb{R}^m))$ of linear operators from $\mathbb{R}^n$ to the space $L(\mathbb{R}^n; \mathbb{R}^m)$ of linear operators from $\mathbb{R}^n$ to $\mathbb{R}^m$. By the previous construction $\partial^2 f(x_0)$ is a nonempty convex compact set of the space $L(\mathbb{R}^n; L(\mathbb{R}^n; \mathbb{R}^m))$ and the set valued map $x \to \partial^2 f(x)$ is upper semicontinuous. Let $u \in \mathbb{R}^n$; in the following we will denote by Lu the value of a linear operator $L : \mathbb{R}^n \to \mathbb{R}^m$ at the point $u \in \mathbb{R}^n$ and by $H(u, v)$ the value of a bilinear operator $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ at the point $(u, v) \in \mathbb{R}^n \times \mathbb{R}^n$. So we will set

$$\partial f(x_0)(u) = \{Lu : L \in \partial f(x_0)\}$$

and

$$\partial f(x_0)^2(u, v) = \{H(u, v) : H \in \partial^2 f(x_0)\}.$$

Some important properties are listed in the following [2].

- *Mean value theorem.* Let f be of class $C^{0,1}$ and $a, b \in \mathbb{R}^n$, then

$$f(b) - f(a) \in \mathrm{cl\,conv}\,\{\partial f(x)(b - a) : x \in [a, b]\},$$

  where $[a, b] = \mathrm{conv}\,\{a, b\}$.

- *Taylor expansion.* Let f be of class $C^{1,1}$ and $a, b \in \mathbb{R}^n$, then

$$f(b) - f(a) \in f'(a)(b - a) + \frac{1}{2}\mathrm{cl\,conv}\,\left\{\partial^2 f(x)(b - a, b - a) : x \in [a, b]\right\}.$$

In [2], Guerraggio and Luc have given necessary and sufficient optimality conditions for $C^{1,1}$ vector optimization problems, expressed by means of Clarke subdifferential. However, in literature, several alternative definitions for generalized subdifferentials has been proposed for the scalar case including, in particular, the Michel-Penot subdifferential ([8]). The Michel-Penot subdifferential of a locally Lipschitzian function at a point is a nonempty, convex, compact set, contained in the Clarke subdifferential. The smallness of the Michel-Penot subdifferential makes the corresponding results, like optimality conditions, sharper than the Clarke's ones. In [9] Yang and Jeyakumar used the notion of Michel-Penot subdifferential for $C^{1,1}$ scalar functions, proving also second order optimality conditions.

The aim of this paper is to generalize to the vector case the notions of Michel-Penot subdifferential and Yang-Jeyakumar subdifferential (sections 2 and 3) and to use them in order to obtain necessary optimality conditions for nonsmooth vector optimization problems involving $C^{1,1}$ data (section 4).

2. PRELIMINARY DEFINITIONS AND RESULTS

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $C^{0,1}$ function at $x_0 \in \mathbb{R}^n$. For such a function, the definition of Michel-Penot generalized derivative $\overline{f}'_M$ at x_0 in the direction $u \in \mathbb{R}^n$ is given by ([9])

$$\overline{f}'_M(x_0; u) = \sup_{z \in \mathbb{R}^n} \limsup_{s \downarrow 0} \frac{f(x_0 + su + sz) - f(x_0 + sz)}{s}.$$

Now let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a $C^{0,1}$ vector function at $x_0 \in \mathbb{R}^n$. We can define a generalized derivative at $x_0 \in \mathbb{R}^n$ in the sense of Michel-Penot as follows

$$f'_M(x_0; u) = \left(\bigcup_{z \in \mathbb{R}^n} A(z) \right)^c,$$

where

$$A(z) = \left\{ l = \lim_{k \to +\infty} \frac{f(x_0 + s_k u + s_k z) - f(x_0 + s_k z)}{s_k}, s_k \downarrow 0 \right\}.$$

and $()^c$ denote the closure of the set. It is trivial to prove that the previous set is nonempty and compact. The following lemma states the relations between the scalar and the vector case.

Lemma 2.1. $\overline{\xi f}'_M(x_0; u) \in \xi f'_M(x_0; u)$, $\forall \xi \in \mathbb{R}^m$.

Proof. We have

$$\overline{\xi f}'_M(x_0; u) = \sup_{z \in \mathbb{R}^n} \limsup_{s \downarrow 0} \frac{(\xi f)(x_0 + su + sz) - (\xi f)(x + sz)}{s} = \lim_{k \to +\infty} g(z_k),$$

where $z_k \in \mathbb{R}^n$ and

$$g(z_k) = \limsup_{s \downarrow 0} \frac{(\xi f)(x_0 + su + sz_k) - (\xi f)(x_0 + sz_k)}{s}.$$

By trivial calculations and eventually by extracting subsequences, we obtain

$$\begin{aligned} g(z_k) &= \lim_{j \to +\infty} \frac{(\xi f)(x_0 + s_{j,k} u + s_{j,k} z_k) - (\xi f)(x_0 + s_{j,k} z_k)}{s_{j,k}} \\ &= \sum_{i=1}^m \xi_i \lim_{j \to +\infty} \frac{f_i(x_0 + s_{j,k} u + s_{j,k} z_k) - f_i(x_0 + s_{j,k} z_k)}{s_{j,k}} \\ &= \sum_{i=1}^m \xi_i l_{i,k} = \xi l_k \end{aligned}$$

with $l_k \in A(z_k)$. Since $l_k \to l$ and $l \in f'_M(x_0;u)$, then $\overline{\xi f}'_M(x_0;u) \in \xi f'_M(x_0;u)$. ■

Theorem 2.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function at $x_0 \in \mathbb{R}^n$. Then $f'_M(x_0;u) = \nabla f(x_0)u$, $\forall u \in \mathbb{R}^n$.*

Proof. Let $l \in f'_M(x_0;u)$; then by the definition of f'_M there exist sequences $z_k \in \mathbb{R}^n$ and $s_k \downarrow 0$ such that

$$\begin{aligned} l &= \lim_{k\to+\infty} \frac{f(x_0+s_ku+s_kz_k)-f(x_0+s_kz_k)}{s_k} \\ &= \lim_{k\to+\infty} \frac{f(x_0+s_ku+s_kz_k)-f(x_0)+f(x_0)-f(x_0+s_kz_k)}{s_k} \\ &= \lim_{k\to+\infty} \frac{\nabla f(x_0)(s_ku+s_kz_k)-\nabla f(x_0)s_kz_k+o(s_k)}{s_k} \\ &= \nabla f(x_0)u + \lim_{k\to+\infty} \frac{o(s_k)}{s_k} = \nabla f(x_0)u. \end{aligned}$$

■

We now prove a generalized mean value theorem for f'_M. To do this, we scalarize the function and we use a mean value theorem for scalar functions proved in [6].

Lemma 2.2. *Let $A \subset \mathbb{R}^n$ be a closed and convex subset of $\mathbb{R}^n$ such that $\xi A \cap \mathbb{R}_- \neq \emptyset$, $\forall \xi \in \mathbb{R}^n$. Then $0 \in A$.*

Proof. Trivial. ■

Proposition 2.1. [6] *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $C^{0,1}$ function. Then $\forall a,b \in \mathbb{R}^n$, $\exists \alpha \in [a,b]$ such that*

$$f(b) - f(a) \le \overline{f}'_M(\alpha; b-a).$$

Theorem 2.2. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a given $C^{0,1}$ vector function. Then the following generalized mean value theorem holds*

$$0 \in f(b) - f(a) - \operatorname{cl}\operatorname{conv}\{f'_M(x;b-a) : x \in [a,b]\}.$$

Proof. For each $\xi \in \mathbb{R}^m$ we have

$$(\xi f)(b) - (\xi f)(a) \le \overline{\xi f}'_M(\alpha; b-a) = \xi l_\xi, \quad l_\xi \in f'_M(\alpha; b-a),$$

where $\alpha \in [a,b]$ and then

$$\xi(f(b) - f(a) - l_\xi) \le 0, \quad l_\xi \in f'_M(\alpha; b-a),$$

$$\xi\left(f(b)-f(a)-\operatorname{cl}\operatorname{conv}\{f'_M(x;b-a):x\in[a,b]\}\right)\cap\mathbb{R}_- \neq \emptyset,\ \forall \xi\in\mathbb{R}^m$$

and the previous lemma implies

$$0\in f(b)-f(a)-\operatorname{cl}\operatorname{conv}\{f'_M(x;b-a):x\in[a,b]\}.$$

■

The obtained result states that f'_M is a subset of $\partial f(x_0)(u)$. The next example shows that inclusion may be strict.

Theorem 2.3. *Let $f:\mathbb{R}^n\to\mathbb{R}^m$ be a $C^{0,1}$ vector function at x_0. Then $f'_M(x_0;u)\subset\partial f(x_0)(u)$.*

Proof. Let $l\in f'_M(x_0;u)$. Then there exist sequences $z_k\in\mathbb{R}^n$ and $s_k\downarrow 0$ such that

$$l=\lim_{k\to+\infty}\frac{f(x_0+s_ku+s_kz_k)-f(x_0+s_kz_k)}{s_k}.$$

So, by the upper semicontinuity of ∂f, we have

$$\frac{f(x_0+s_ku+s_kz_k)-f(x_0+s_kz_k)}{s_k}\in\operatorname{cl}\operatorname{conv}\{\partial f(x)(u);$$
$$x\in[x_0+s_kz_k,x_0+s_ku+s_kz_k]\}\subset\partial f(x_0)(u)+\epsilon B,$$

where B is the unit ball of $\mathbb{R}^m$, $\forall n\geq n_0(\epsilon)$. So $l\in\partial f(x_0)u+\epsilon B$. Taking the limit when $\epsilon\to 0$, we obtain $l\in\partial f(x_0)(u)$. ■

Example 2.1. *Let $f:\mathbb{R}\to\mathbb{R}^2$, $f(x)=\left(x^2\sin\left(\frac{1}{x}\right)+x^2,x^2\right)$. f is of class $C^{0,1}$ at $x_0=0$ and $f'_M(0;1)=(0,0)\in\partial f(0)(1)=[-1,1]\times\{0\}$.*

3. A GENERALIZED DERIVATIVE FOR $C^{1,1}$ VECTOR FUNCTIONS

Let $f:\mathbb{R}^n\to\mathbb{R}^m$ be a $C^{1,1}$ vector function at $x_0\in\mathbb{R}^n$. Then the second order generalized derivative at $x_0\in\mathbb{R}^n$ in the directions $u,v\in\mathbb{R}^n$ in the sense of Michel-Penot is the following

$$f''_M(x_0;u,v)=\left(\bigcup_{z\in\mathbb{R}^n}A(z)\right)^c,$$

where

$$A(z)=\{l=\lim_{k\to+\infty}\frac{\nabla f(x_0+s_kv+s_kz)u-\nabla f(x+s_kz)u}{s_k},s_k\downarrow 0\}.$$

It is trivial to prove that $f''_M(x_0;u,v)$ is nonempty and compact subset and that $f''_M(x_0;u,v) = (\nabla f(\cdot)u)'_M(x_0;v)$. The results given above can be extended to the second order.

Proposition 3.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a $C^{1,1}$ vector function at $x_0 \in \mathbb{R}^n$. Then $\overline{\xi f}''_M(x_0;u,v) \in \xi f''_M(x_0;u,v)$, $\forall \xi \in \mathbb{R}^m$.*

Proof. In fact, we have

$$\overline{(\xi \nabla f(\cdot)u))}'_M(x_0;v) = \sup_{z\in\mathbb{R}^n} \limsup_{s\downarrow 0} \frac{\xi\nabla f(x_0+sv+sz)u - \xi\nabla f(x_0+sz)u}{s}$$

$$= \sup_{z\in\mathbb{R}^n} \limsup_{s\downarrow 0} \frac{\nabla(\xi f)(x_0+sv+sz)u - \nabla(\xi f)(x_0+sz)u}{s} = \overline{\xi f}''_M(x_0;u,v)$$

and then the thesis follows by using lemma 2.1. ∎

Proposition 3.2. [9] *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $C^{1,1}$ scalar function. Then, $\forall a, b \in \mathbb{R}^n$, there exists $\alpha \in [a,b]$ such that the following generalized Taylor formula holds*

$$f(b) - f(a) - \nabla f(a)(b-a) \le \frac{1}{2}\overline{f}''_M(\alpha; b-a, b-a),\ \ \alpha \in [a,b].$$

Theorem 3.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a $C^{1,1}$ vector function. Then the following generalized Taylor formula holds*

$$0 \in f(b) - f(a) - \nabla f(a)(b-a) - \frac{1}{2}\mathrm{cl\,conv}\,\{f''_M(x;b-a,b-a) : x \in [a,b]\}.$$

Proof. For each $\xi \in \mathbb{R}^m$, we have

$$(\xi f)(b) - (\xi f)(a) - \nabla(\xi f)(a)(b-a) \le \frac{1}{2}\overline{\xi f}''_M(\alpha; b-a, b-a) = \frac{1}{2}\xi l_\xi$$

where $l_\xi \in f''_M(\alpha; b-a, b-a)$. So

$$\xi\left(f(b) - f(a) - \nabla f(a)(b-a) - \frac{1}{2}l_\xi\right) \le 0, \quad \forall \xi \in \mathbb{R}^m$$

$$\xi\Big(f(b) - f(a) - \nabla f(a)(b-a) - \frac{1}{2}\mathrm{cl\,conv}\,\{f''_M(x;b-a,b-a)$$
$$: x \in [a,b]\}\big) \cap \mathbb{R}_- \neq \emptyset,$$

$$0 \in f(b) - f(a) - \nabla f(a)(b-a) - \frac{1}{2}\mathrm{cl\,conv}\,\{f''_M(x;b-a,b-a) : x \in [a,b]\}.$$

∎

Theorem 3.2. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a $C^{1,1}$ vector function at $x_0 \in \mathbb{R}^n$. Then $f''_M(x_0; u, v) \subset \partial^2 f(x_0)(u, v)$.*

Proof. We have

$$f''_M(x_0; u, v) = (\nabla f(\cdot)u)'_M(x_0; v) \subset \partial(\nabla f(\cdot)u)(x_0)(v) = \partial^2 f(x_0)(u, v).$$

■

The following example shows that the inclusion may be strict.

Example 3.1. *Let f is of class $C^{1,1}$ at $x_0 = 0$*

$$f : \mathbb{R} \to \mathbb{R}^2, \quad f(x) = \left(x^4 \sin\left(\frac{1}{x}\right) + x^4, x^4\right)$$

and

$$f''_M(0; 1, 1) = (0, 0) \in \partial^2 f(0)(1, 1) = [-1, 1] \times \{0\}.$$

4. NECESSARY OPTIMALITY CONDITIONS FOR $C^{1,1}$ VECTOR FUNCTIONS

Consider the following Pareto set constrained optimization problem

$$\min_{\mathbb{R}^m_+} f(x) \qquad \text{subject to} \qquad x \in X, \tag{4.1}$$

where X is a subset of $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^m$. A point $x_0 \in \mathbb{R}^n$ is called a *local minimum point* (*local weak minimum point*) of (4.1) if there exists a neighbourhood N of x_0 such that no $x \in N \cap X$ satisfies $f(x_0) - f(x) \in \mathbb{R}^m_+ \backslash \{0\}$ ($f(x_0) - f(x) \in \operatorname{int} \mathbb{R}^m_+$).

We remember that the following set

$$WF(X, x_0) = \{d : \exists s_k \downarrow 0, x_0 + s_k d \in X\}$$

is called cone of weak feasible directions to X at x_0.

Lemma 4.1. *Let $x_0 \in X$ be a local weak minimum point. Then $\forall d \in WF(X, x_0)$, $\exists s_k \downarrow 0$ such that $\nabla f(x_0 + s_k d)d \notin -\operatorname{int} \mathbb{R}^m_+$.*

Proof. Ab absurdo, $d \in WF(X, x_0)$ such that $\forall s_k \downarrow 0$, $\nabla f(x_0 + s_k d)d \in -\operatorname{int} \mathbb{R}^m_+$. Let $s_k \downarrow 0$ such that $x_0 + s_k d \in X$. By the mean value theorem we have

$$f_i(x_0 + s_k d) - f_i(x_0) = \nabla f_i(x_0 + \alpha_{i,k} d)d < 0,$$

if k is large enough. This contradicts the local optimality of x_0. ■

Theorem 4.1. *Let f be a $C^{1,1}$ vector function. If x_0 is local weak minimum point then*

$$f''_M(x_0;d,d) \not\subset -\operatorname{int}\mathbb{R}^m_+,$$

$\forall d \in WF(X,x_0)$, $\nabla f(x_0)d \in -(\mathbb{R}^m_+ \setminus \operatorname{int}\mathbb{R}^m_+)$.

Proof. Ab absurdo, suppose there exists some $d \in WF(X,x_0)$ with $\nabla f(x_0)d \in -(\mathbb{R}^m_+ \setminus \operatorname{int}\mathbb{R}^m_+)$ and $f''_M(x_0;d,d) \subset -\operatorname{int}\mathbb{R}^m_+$. Let s_k be a sequence that satisfies the previous lemma. Taking the suitable subsequence if necessary, we have

$$s_k^{-1}(\nabla f(x_0+s_kd)d - \nabla f(x_0)d) \to l \in f''_M(x_0;d,d) \subset -\operatorname{int}\mathbb{R}^m_+.$$

Thus, the assumption $\nabla f(x_0)d \in -(\mathbb{R}^m_+ \setminus \operatorname{int}\mathbb{R}^m_+)$ implies that

$$\nabla f(x_0+s_kd)d \in \nabla f(x_0)d - \operatorname{int}\mathbb{R}^m_+ \subset -(\mathbb{R}^m_+ \setminus \operatorname{int}\mathbb{R}^m_+) - \operatorname{int}\mathbb{R}^m_+ = -\operatorname{int}\mathbb{R}^m_+.$$

■

Corollary 4.1. Let $X = \mathbb{R}^n$ and f be a $C^{1,1}$ vector function. If x_0 is a local weak minimum point, then

$$f''_M(x_0;d,d) \not\subset -\operatorname{int}\mathbb{R}^m_+,$$

$\forall d \in S^1$, $\nabla f(x_0)d \in -(\mathbb{R}^m_+ \setminus \operatorname{int}\mathbb{R}^m_+)$.

Proof. It is trivial, recalling $WF(\mathbb{R}^n, x_0) = \mathbb{R}^n$. ■

So theorem 5.1 in [2] follows from corollary 4.1.

Corollary 4.2. Let $X = \mathbb{R}^n$ and f be a $C^{1,1}$ vector function. If x_0 is a local weak minimum point then

$$\partial^2 f(x_0)(d,d) \not\subset -\operatorname{int}\mathbb{R}^m_+,$$

$\forall d \in S^1$, $\nabla f(x_0)d \in -(\mathbb{R}^m_+ \setminus \operatorname{int}\mathbb{R}^m_+)$.

Proof. It is trivial, recalling $f''_M(x_0;d,d) \subset \partial^2 f(x_0)(d,d)$. ■

Example 4.1. *Let $f : \mathbb{R} \to \mathbb{R}^2$, $f(x) = \left(x^4 \sin\left(\frac{1}{x}\right) - \frac{x^2}{4}, -x^2\right)$. The point $x_0 = 0$ is not a local weak minimum point. We have $f''_M(0;d,d) = \left(-\frac{d^2}{2}, -2d^2\right) \in -\operatorname{int}\mathbb{R}^2_+$ (the necessary condition is not satisfied) but*

$$\partial^2 f(0)(d,d) = \left[-\frac{3d^2}{2}, \frac{d^2}{2}\right] \times \{-2d^2\},$$

i.e., the necessary condition is satisfied.

5. CONCLUSIONS

The study of the class of $C^{1,1}$ functions has been renewed since the work of Hiriart-Urruty, Strodiot and Hien Nguyen [5]. The need for investigate these functions, as pointed out in several papers on this topic (see [2] and the references therein) comes from the fact that several problems of applied mathematics including variational inequalities, semi-infinite programming, iterated local minimization, etc. involve differentiable functions with no hope to be twice differentiable. The notion of generalized derivative for this class of function is crucial in order to derive second order optimality conditions. For the scalar case several definitions have been proposed including, in particular, the Michel-Penot derivative and the Yang-Jeyakumar derivative. In this paper these notions have been extended to the vector case and then used for proving necessary optimality conditions for vector problems. These generalized derivatives are subsets of Clarke subdifferentials and this is important for establishing the "best" optimality conditions for nonsmooth vector problems. Example 4.1 shows that these new conditions can be used when Clarke's ones fail.

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Būtinos neglodžių vektorių optimizavimo uždavinių optimalios sąlygos

D. La Torre

Straipsnyje įvedama apibendrintos išvestinės sąvoka neglodžioms vektor-funkcijoms, kad galima būtų gauti optimalumo sąlygas vektorių optimizavimo uždaviniams. Šis apibrėžimas apibendrina Michel ir Penot įvestas sąvokas, kurias išplėtė Yang ir Jeyakumar. Išvestinės apibendrinimas įeina į Clarke subdiferencialą, tačiau optimalumo sąlygos yra jautresnės nei Clarko.