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ON THE ASYMPTOTIC BEHAVIOR OF AN APPROXIMATION OF SIEs DRIVEN BY p-SEMIMARTINGALES

K. KUBILIUS

Institute of Mathematics and Informatics

Akademijos 4,Vilnius 2600, Lithuania

E-mail: kubilius@ktl.mii.lt

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ABSTRACT

We study the convergence rate of a Milstein-type approximation of the solution of an integral equation driven by a special continuous p-semimartingale.

1. INTRODUCTION

In this paper, we consider the stochastic integral equation (SIE)

$$X_t = \xi + \int_0^t f(X_s)\, dW_s + \int_0^t g(X_s)\, dB_s^H, \qquad 0 \leqslant t \leqslant T, \tag{1.1}$$

where W is a standard Brownian motion and B^H is a fractional Brownian motion (fBm) with Hurst index $1/2 < H < 1$. A more detailed definition of an fBm will be formulated in the next section. The process B^H, $1/2 < H < 1$, is not a semimartingale but almost all its sample paths have bounded p-variation for $p > 1/H$. In (1.1), the first integral is stochastic and the second one is the defined pathwise Riemann–Stieltjes (RS) integral. The existence of the RS integral follows by the Love–Young inequality, which we will formulate in the next section.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$, $\mathbb{F} = \{\mathcal{F}_t, 0 \leqslant t \leqslant T\}$, be a stochastic basis satisfying the usual conditions. If f is a Lipschitz function and $g \in C^2(\mathbb{R})$, then (see [7]) there exists a unique adapted solution of equation (1.1) having almost all sample paths in the space $C\mathcal{W}_q([0,T])$, $q > 2$, where $C\mathcal{W}_q([0,T])$ is the class of all continuous functions defined on $[0,T]$ with bounded q-variation.

Equation (1.1) differs from the ordinary SDE by its last term on the right side. By an ordinary SDE we understand an equation of the form

$$\begin{cases} \widehat{X}_t = \sigma(t,\widehat{X}_t)\,dW_t + b(t,\widehat{X}_t)\,dt, & 0 \leqslant t \leqslant T, \\ \widehat{X}_0 = \xi. \end{cases} \tag{1.2}$$

It is well known that the convergence rate of the strong Euler–Peano approximation of the solution of SDE (1.2) has upper bound $\delta_n^{1/2}$, where δ_n is the mesh of a partition of $[0,T]$ of the nth approximation. For the strong continuous Milstein approximation, this bound is δ_n. Our equation (1.1) is more general than the SDE (1.2). Thus, to obtain the rate of convergence comparable with the strong Euler–Peano approximation we shall use a more complicated approximation.

Let $\varkappa^n = \{t_k^n\colon\ 0 \leqslant k \leqslant m(n)\}$, $n \geqslant 1$, be a sequence of partitions of the interval $[0,T]$, i.e., $0 = t_0^n < t_1^n < \cdots < t_{m(n)}^n = T$, and let $\delta_n = \max_k(t_k^n - t_{k-1}^n) \to 0$ as $n \to \infty$.

For each $n \geqslant 1$, we define the approximation

$$\begin{aligned} X^n(t) \quad = \quad & \xi + \int_0^t f(X^n(\tau_s^n))\,dW_s + \int_0^t g(X^n(\tau_s^n))\,dB_s^H \\ & + \int_0^t \int_{\tau_s^n}^s g'(X^n(\tau_s^n)) f(X^n(\tau_s^n))\,dW_u\,dB_s^H \\ & + \int_0^t \int_{\tau_s^n}^s g'(X^n(\tau_s^n)) g(X^n(\tau_s^n))\,dB_u^H\,dB_s^H, \end{aligned} \tag{1.3}$$

where $\tau_s^n = t_{k-1}^n$ and $X^n(\tau_s^n) = X^n(t_{k-1}^n)$ if $s \in (t_{k-1}^n, t_k^n]$, $1 \leqslant k \leqslant m(n)$. For fixed $n \in \mathbb{N}$, by definition, the process X^n is sample continuous and its sample paths belong to $C\mathcal{W}_q([0,T])$, $q > 2$.

Note that approximation (1.3) becomes the strong continuous Milstein approximation if $f \equiv 0$ and becomes the strong Euler–Peano approximation if $g \equiv 0$.

Theorem 1.1. *Let f and g be bounded functions, f be a Lipschitz function and $g \in C^2(\mathbb{R})$. Then*

$$\alpha_n V_q\bigl(X^n - X; [0,T]\bigr) \xrightarrow{\mathbf{P}} 0, \qquad n \to \infty,$$

where $V_q(X^n - X; [0,T]) = v_q^{1/q}(X^n - X; [0,T])$, $v_q(X^n - X; [0,T])$ is the q-variation of $X^n - X$, $\alpha_n = \delta_n^{-1/q} |\ln \delta_n|^{-1/2}$, $\delta_n < 1$, $q > 2$.

Corollary 1.1. Let the conditions of Theorem 1.1 be satisfied. Then, for all $\varepsilon > 0$,

$$\hat{\alpha}_n\Big(\mathbf{E}\sup_{0\leqslant t\leqslant T}\big|X_t^n - X_t\big|^r\Big)^{1/r} \longrightarrow 0, \qquad n\to\infty,$$

where $\hat{\alpha}_n = \delta_n^{-1/2+\varepsilon}$, $1\leqslant r<\infty$.

As we will see below, the rate of convergence in Corollary 1.1 mainly depends on the stochastic integral with respect to the Wiener process. Thus, the result obtained in Corollary 1.1 is close to the optimal one if the function $f\not\equiv 0$.

Consider the integral equation

$$x_t = x_0 + \int_0^t f(x_s)\,ds + \int_0^t g(x_s)\,dh_s, \tag{1.4}$$

where h is a continuous function with bounded p-variation, $1<p<2$. In [8], the uniform distance between the solution of (1.4) and its Milstein-type approximation was estimated by $\delta_n\wedge\gamma_p(n)\wedge\gamma_p^2(n)$, where $a\wedge b$ is the minimum of a and b, $\gamma_p(n) = \max_k V_p(h;[t_{k-1}^n, t_k^n])$. In particular, if h is a Lipschitz function of order α, then the uniform distance has the bound δ_n^α for $\delta_n<1$.

Define, for $t_k^n = kT/n$, $k=0,\dots,n-1$, the Euler approximation

$$\begin{cases} \widehat{X}^{(n)}(t_{k+1}^n) = \widehat{X}^{(n)}(t_k^n) + \sigma\big(t_k^n, \widehat{X}^{(n)}(t_k^n)\big)\big(W(t_{k+1}^n) - W(t_k^n)\big) \\ \qquad\qquad\qquad +b\big(t_k^n, \widehat{X}^{(n)}(t_k^n)\big)\big(t_{k+1}^n - t_k^n\big), \\ \widehat{X}^{(n)}(0) = x \end{cases}$$

and, for $t\in[t_k^n, t_{k+1}^n)$, $k=0,\dots,n-1$, put

$$\widehat{X}^{(n)}(t) = \widehat{X}^{(n)}(t_k^n) + \frac{t-t_k^n}{t_{k+1}^n - t_k^n}\big(\widehat{X}^{(n)}(t_{k+1}^n) - \widehat{X}^{(n)}(t_k^n)\big).$$

It is known (see [1], p. 277) that

$$\Big(\mathbf{E}\sup_{0\leqslant t\leqslant T}\big|\widehat{X}_t^n - \widehat{X}_t\big|^r\Big)^{1/r} \leqslant C\sqrt{\frac{1+\log n}{n}}, \qquad n\geqslant 1,$$

where C is a constant depending on r, T, and the functions b and σ.

Moreover, let $\sigma(t,x)\equiv\sigma(t)$ in (1.2). Then (see [6]) we have

$$\lim_{n\to\infty}\big(n/\ln n\big)^{1/2}\Big(\mathbf{E}\sup_{0\leqslant t\leqslant T}\big|\widehat{X}_t^n - \widehat{X}_t\big|^r\Big)^{1/r} = 1/\sqrt{2}\sup_{t\leqslant T}|\sigma(t)|.$$

In a special case, we can get the optimal convergence rate. Consider SIE

$$Y_t = \xi + \int_0^t f(Y_s)\, dW_s + \int_0^t g(s)\, dB_s^H,$$

and its Euler–Peano approximation

$$Y^n(t) = \xi + \int_0^t f(Y^n(\tau_s^n))\, dW_s + \int_0^t g(\tau_s^n)\, dB_s^H.$$

Then we obtain the following result.

Theorem 1.2. *Let f and g be bounded functions. Suppose that f is a Lipschitz function and g has a bounded derivative. Then, for all r, $1 \leqslant r < \infty$, there exists a constant C independent of n such that*

$$\mathbf{E} \sup_{0\leqslant t\leqslant T} \left|Y_t^n - Y_t\right|^r \leqslant C\delta_n^{r/2}.$$

2. BASIC NOTIONS AND AUXILIARY RESULTS

All facts mentioned below on the p-variation are taken from [3], [4].

Let f be a real-valued function defined on a closed interval $[a,b]$. For $0 < p < \infty$, denote by $v_p(f) := v_p(f;[a,b])$ the p-variation of f on $[a,b]$.

Define $V_p(f) := V_p(f;[a,b]) = v_p^{1/p}(f)$, which is a seminorm on the class $\mathcal{W}_p([a,b])$ of all functions of bounded p-variation on $[a,b]$. For each f, $V_p(f)$ is a non-increasing function of p, i.e., if $q < p$, then $V_p(f) \leqslant V_q(f)$. Thus, $\mathcal{W}_q([a,b]) \subseteq \mathcal{W}_p([a,b])$ if $1 \leqslant q < p < \infty$.

Let $p \geqslant 1$ and

$$V_{p,\infty}(f) = V_{p,\infty}\big(f;[a,b]\big) = V_p(f,[a,b]) + |f|_{\infty,[a,b]},$$

where $|f|_{\infty,[a,b]} = \sup_{a\leqslant x\leqslant b} |f(x)|$. Then $V_{p,\infty}(f)$ is a norm and $\mathcal{W}_p([a,b])$ equipped with the p-variation norm is a Banach space.

Let $a < c < b$, and let $f \in \mathcal{W}_p([a,b])$ with $0 < p < \infty$. Then

$$\begin{aligned} &v_p(f;[a,c]) + v_p(f;[c,b]) \leqslant v_p(f;[a,b]),\\ &V_p(f;[a,b]) \leqslant V_p(f;[a,c]) + V_p(f;[c,b]). \end{aligned} \tag{2.1}$$

It is known that

$$V_{p,\infty}(fg;[a,b]) \leqslant V_{p,\infty}(f;[a,b])V_{p,\infty}(g;[a,b]). \tag{2.2}$$

Let $f \in \mathcal{W}_q([a,b])$ and $g \in \mathcal{W}_p([a,b])$. For any partition $\varkappa = \{x_i\colon\ i = 0,\dots,n\}$ of $[a,b]$ such that $a = x_0 < x_1 < \cdots < x_n = b$ and for $p^{-1}+q^{-1} \geqslant 1$, we have by the Hölder inequality (see [10])

$$\sum_i V_q\bigl(f;[x_{i-1},x_i]\bigr)V_p\bigl(g;[x_{i-1},x_i]\bigr) \leqslant V_q\bigl(f;[a,b]\bigr)V_p\bigl(g;[a,b]\bigr). \tag{2.3}$$

Let $f \in \mathcal{W}_q([a,b])$ and $h \in \mathcal{W}_p([a,b])$ with $p>0$, $q>0$, $1/p+1/q>1$. If f and h have no common discontinuities, then the RS integral $\int_a^b f\,dh$ exists and the Love–Young inequality

$$\left|\int_a^b f\,dh - f(y)\bigl[h(b)-h(a)\bigr]\right| \leqslant C_{p,q}V_q(f;[a,b])V_p(h;[a,b]) \tag{2.4}$$

holds for every $y \in [a,b]$, where $C_{p,q} = \zeta(p^{-1}+q^{-1})$, $\zeta(s)$ denotes the Riemann zeta function, i.e., $\zeta(s) = \sum_{n\geqslant 1} n^{-s}$. If $h \in C\mathcal{W}_p([a,b])$, then the indefinite integral $\int_a^y f\,dh$, $y \in [a,b]$, is a continuous function.

Let $f \in \mathcal{W}_q([a,b])$ and $h \in \mathcal{C}W_p([a,b])$. From (2.4) it follows that

$$V_p\left(\int_a^{\cdot} f\,dh;[a,b]\right) \leqslant C_{p,q}V_{q,\infty}(f;[a,b])V_p(h;[a,b]). \tag{2.5}$$

An fBm with Hurst index $0<H<1$ is a centered Gaussian process $X = \{X_t, t \geqslant 0\}$ with $X_0 = 0$ and covariance

$$\operatorname{Cov}\bigl(X_t,X_s\bigr) = \frac{1}{2}\operatorname{Var}(X_1)\bigl(t^{2H}+s^{2H}-|t-s|^{2H}\bigr)$$

for all $t,s \geqslant 0$. If $\operatorname{Var}(X_1)=1$, we write $X = B^H$. The case $H = 1/2$ corresponds to the standard Brownian motion.

Since almost all sample paths of the processes B^H, $1/2 \leqslant H < 1$, are Hölder continuous, we have

$$V_r\bigl(B^H;[s,t]\bigr) \leqslant L^{H,1/r}\,(t-s)^{1/r}, \tag{2.6}$$

where $s<t$, $r>1/H$,

$$L^{H,\gamma} = \sup_{\substack{s\neq t\\ s,t\leqslant T}} \frac{\bigl|B_t^H - B_s^H\bigr|}{|t-s|^{\gamma}}, \quad 0<\gamma<H, \qquad \mathbf{E}\bigl(L^{H,\gamma}\bigr)^k < \infty, \quad \forall\ k \geqslant 1.$$

Any local martingale is locally of bounded q-variation for each $q>2$. Moreover (see [7]), for $q>2$ and $0<r\leqslant 2$, there exist finite constants $K_{p,r}$ and

ℓ_r such that, for continuous martingale $M = \{M(t), 0 \leqslant t \leqslant T\}$ and stopping times $\sigma < \tau \leqslant T$,

$$\begin{aligned} \mathbf{E}\big\{v_q\big(M;[\sigma,\tau]\big)\big\}^{r/q} &\leqslant K_{q,r}\mathbf{E}\Big\{\sup_{\sigma\leqslant t\leqslant\tau}\big|M(t)\big|\Big\}^r \\ &\leqslant K_{q,r}\ell_r\mathbf{E}\big\{\langle M\rangle_\tau - \langle M\rangle_\sigma\big\}^{r/2}. \end{aligned} \tag{2.7}$$

3. PROOFS

The following lemma is very useful in the proof of Theorem 1.1.

Lemma 3.1 ([2], see also [8]). *Let $p \geqslant 1$, $g \in C^2(\mathbb{R})$, and $x, y \in \mathcal{W}_p([a,b])$, $a < b$. Then*

$$V_p\big(g(x)-g(y);[a,b]\big) \leqslant \big\{|g'|_\infty + |g''|_\infty V_p\big(y;[a,b]\big)\big\} V_{p,\infty}\big(x-y;[a,b]\big).$$

Denote

$$\Gamma(X,\sigma,t) := C_{p,q}\max\big\{|g'|_\infty, |g''|_\infty\big\}\big[1+V_q\big(X;[\sigma,t]\big)\big]V_p\big(B^H;[\sigma,t]\big).$$

Define, for each $k \geqslant 1$, the stopping time

$$\sigma_k = \inf\Big\{t > \sigma_{k-1}^n:\ \Gamma(X,\sigma_{k-1},t) > \frac{1}{4}\Big\} \wedge T,$$

where $\sigma_0^n = 0$.

Proof of Theorem 1.1. For short, we further write $hk(Z_s)$ instead of $h(Z_s)k(Z_s)$ for any process Z. Since V_q, $q > 2$, is a seminorm, we have

$$\begin{aligned} &V_q\big(X - X^n;[\sigma_{k-1},\sigma_k]\big) \\ &\quad\leqslant V_q\bigg(\int_0^\cdot \big[f(X_s)-f(X_s^n)\big]\,dW_s;[\sigma_{k-1},\sigma_k]\bigg) \\ &\qquad + V_q\bigg(\int_0^\cdot \big[f(X_s^n)-f(X^n(\tau_s^n))\big]\,dW_s;[\sigma_{k-1},\sigma_k]\bigg) \\ &\qquad + V_q\bigg(\int_0^\cdot \big[g(X_s)-g(X_s^n))\big]\,dB_s^H;[\sigma_{k-1},\sigma_k]\bigg) \\ &\qquad + V_q\bigg(\int_0^\cdot \Big\{g(X_s^n)-g(X^n(\tau_s^n)) - g'f(X^n(\tau_s^n))\int_{\tau_s^n}^s dW_u \\ &\qquad - g'g(X^n(\tau_s^n))\int_{\tau_s^n}^s dB_u^H\Big\}dB_s^H;[\sigma_{k-1},\sigma_k]\bigg) = \sum_{i=1}^4 I_i. \end{aligned} \tag{3.1}$$

Now we shall estimate the q-variation of every term in the previous equality. We first estimate the terms I_k, $3 \leqslant k \leqslant 4$. By the Love–Young inequality (2.4) we have

$$\begin{aligned} I_3 &= V_q\Big(\int_{\sigma_{k-1}}^{\cdot} \big[g(X_s) - g(X^n(\tau_s^n))\big]\, dB_s^H; [\sigma_{k-1}, \sigma_k]\Big) \\ &\leqslant C_{p,q} V_{q,\infty}\big(g(X) - g(X^n(\tau^n)); [\sigma_{k-1}, \sigma_k]\big) V_p\big(B^H; [\sigma_{k-1}, \sigma_k]\big). \end{aligned}$$

By Lemma 3.1 and by the inequality

$$|X - X^n|_{\infty,[\sigma_{k-1},\sigma_k]} \leqslant V_q(X - X^n; [\sigma_{k-1}, \sigma_k]) + V_q(X - X^n; [0, \sigma_{k-1}])$$

we have

$$\begin{aligned} *I_3 &\leqslant C_{p,q}\big\{\big|g'\big|_\infty + \big|g''\big|_\infty V_q\big(X; [\sigma_{k-1}, \sigma_k]\big)\big\} \\ &\quad \times V_{q,\infty}\big(X - X^n; [\sigma_{k-1}, \sigma_k]\big) V_p\big(B^H; [\sigma_{k-1}, \sigma_k]\big) \\ &\leqslant 0.5\, V_q\big(X - X^n; [\sigma_{k-1}, \sigma_k]\big) + 0.25\, V_q\big(X - X^n; [0, \sigma_{k-1}]\big). \end{aligned}$$

Thus,

$$\begin{aligned} V_q\big(X - X^n; [\sigma_{k-1}, \sigma_k]\big) &\leqslant \sum_{\substack{k=1\\k\neq 3}}^{4} I_k + 0.5\, V_q\big(X - X^n; [\sigma_{k-1}, \sigma_k]\big) \\ &\quad + 0.25\, V_q\big(X - X^n; [0, \sigma_{k-1}]\big). \end{aligned} \tag{3.2}$$

Carrying the third term from the right-side of inequality (3.2) to the left-side, by the inequality

$$\Big(\sum_{i=1}^{n} a_i\Big)^2 \leqslant n \sum_{i=1}^{n} a_i^2, \tag{3.3}$$

which is valid for any $a_i \in \mathbb{R}$, we get

$$V_q^2\big(X - X^n; [\sigma_{k-1}, \sigma_k]\big) \leqslant 16\Big\{\sum_{\substack{k=1\\k\neq 3}}^{4} I_k^2 + (0.25)^2 V_q^2\big(X - X^n; [0, \sigma_{k-1}]\big)\Big\}. \tag{3.4}$$

We now estimate I_4. First we note that by (2.1) and (2.5)

$$\begin{aligned} I_4 &\leqslant \sum_{k=1}^{m(n)} V_q\Big(\int_{t_{k-1}^n}^{\cdot} \Big[g(X_s^n) - g\big(X^n(t_{k-1}^n)\big) - g'f\big(X^n(t_{k-1}^n)\big)\int_{t_{k-1}^n}^{s} dW_u \\ &\quad -g'g\big(X^n(t_{k-1}^n)\big)\int_{t_{k-1}^n}^{s} dB_u^H\Big] dB_s^H; [t_{k-1}^n, t_k^n]\Big) \\ &\leqslant 2C_{p,q}\sum_{k=1}^{m(n)} V_q\Big(g(X^n) - g\big(X^n(t_{k-1}^n)\big) - g'f\big(X^n(t_{k-1}^n)\big)\int_{t_{k-1}^n}^{\cdot} dW_u \\ &\quad -g'g\big(X^n(t_{k-1}^n)\big)\int_{t_{k-1}^n}^{\cdot} dB_u^H; [t_{k-1}^n, t_k^n]\Big) V_p\big(B^H; [t_{k-1}^n, t_k^n]\big). \end{aligned} \tag{3.5}$$

By the Itô formula for p-semimartingales (see [9]) we have

$$\begin{aligned} g(X^n(t)) - g\big(X^n(t_{k-1}^n)\big) &= \int_{t_{k-1}^n}^{t} g'(X^n(s))\, dX_s^n \\ &\quad +\frac{1}{2}\int_{t_{k-1}^n}^{t} g''(X^n(s)) f^2\big(X^n(t_{k-1}^n)\big) ds. \end{aligned}$$

Thus,

$$\begin{aligned} &V_q\Big(g(X^n) - g\big(X^n(t_{k-1}^n)\big) - g'f\big(X^n(t_{k-1}^n)\big)\int_{t_{k-1}^n}^{\cdot} dW_u \\ &\quad -g'g\big(X^n(t_{k-1}^n)\big)\int_{t_{k-1}^n}^{\cdot} dB_u^H; [t_{k-1}^n, t_k^n]\Big) \\ &\leqslant |f|_\infty V_q\Big(\int_{t_{k-1}^n}^{\cdot} \big[g'(X^n(s)) - g'(X^n(t_{k-1}^n))\big] dW_s; [t_{k-1}^n, t_k^n]\Big) \\ &\quad +|g|_\infty V_q\Big(\int_{t_{k-1}^n}^{\cdot} \big[g'(X^n(s)) - g'(X^n(t_{k-1}^n))\big] dB_s^H; [t_{k-1}^n, t_k^n]\Big) \\ &\quad +|g'|_\infty |f|_\infty V_q\Big(\int_{t_{k-1}^n}^{\cdot} g'(X^n(s))\int_{t_{k-1}^n}^{s} dW_u\, dB_s^H; [t_{k-1}^n, t_k^n]\Big) \\ &\quad +|g'|_\infty |g|_\infty V_q\Big(\int_{t_{k-1}^n}^{\cdot} g'(X^n(s))\int_{t_{k-1}^n}^{s} dB_u^H\, dB_s^H; [t_{k-1}^n, t_k^n]\Big) \\ &\quad +\frac{1}{2}|f|_\infty^2 V_1\Big(\int_{t_{k-1}^n}^{\cdot} g''(X^n(s))\, ds; [t_{k-1}^n, t_k^n]\Big) := \sum_{i=1}^{5} J_k^{(i,n)}. \end{aligned}$$

Using (3.5), (3.3), the Cauchy inequality, and (2.3), we obtain

$$\begin{aligned}\mathbf{E}I_4^2 &\leqslant 4C_{p,q}^2\,\mathbf{E}\Big\{\sum_{i=1}^{5}\sum_{k=1}^{m(n)} J_k^{(i,n)} V_p\big(B^H;[t_{k-1}^n,t_k^n]\big)\Big\}^2\\ &\leqslant 20C_{p,q}^2\sqrt{\mathbf{E}V_p^4\big(B^H;[0,T]\big)}\sum_{i=1}^{5}\sqrt{\mathbf{E}\Big(\sum_{k=1}^{m(n)}\big(J_k^{(i,n)}\big)^2\Big)^2}. \qquad (3.6)\end{aligned}$$

We further have

$$\mathbf{E}\Big(\sum_{k=1}^{m(n)}\big(J_k^{(1,n)}\big)^2\Big)^2 \leqslant \sum_{k=1}^{m(n)}\mathbf{E}\big(J_k^{(1,n)}\big)^4 + \sum_{\substack{i,j=1\\ i\neq j}}^{m(n)}\sqrt{\mathbf{E}\big(J_i^{(1,n)}\big)^4\,\mathbf{E}\big(J_j^{(1,n)}\big)^4}.$$

By inequality (2.7) we obtain

$$\begin{aligned}\mathbf{E}\big(J_k^{(1,n)}\big)^4 &\leqslant |f|_\infty^4 K_{q,4}\ell_4\mathbf{E}\Big(\int_{t_{k-1}^n}^{t_k^n}\big[g'(X^n(s))-g'(X^n(t_{k-1}^n))\big]^2 ds\Big)^2\\ &\leqslant |f|_\infty^4 K_{q,4}\ell_4|g''|_\infty^4\big(t_k^n-t_{k-1}^n\big)^2\mathbf{E}V_q^4\big(X^n;[t_{k-1}^n,t_k^n]\big).\end{aligned}$$

Thus,

$$\begin{aligned}&\mathbf{E}\Big(\sum_{k=1}^{m(n)}\big(J_k^{(1,n)}\big)^2\Big)^2\\ &\qquad\leqslant |f|_\infty^4 K_{q,4}\ell_4|g''|_\infty^4 T^2\max_{1\leqslant k\leqslant m(n)}\mathbf{E}V_q^4\big(X^n;[t_{k-1}^n,t_k^n]\big). \qquad (3.7)\end{aligned}$$

By (2.5) we further obtain

$$\begin{aligned}J_k^{(2,n)} &\leqslant |g|_\infty C_{p,q}V_{q,\infty}\big(g'(X^n)-g'(X^n(t_{k-1}^n));[t_{k-1}^n,t_k^n]\big)\\ &\quad\times V_p\big(B^H;[t_{k-1}^n,t_k^n]\big)\\ &\leqslant 2|g|_\infty|g''|_\infty C_{p,q}V_q\big(X^n;[t_{k-1}^n,t_k^n]\big)V_p\big(B^H;[t_{k-1}^n,t_k^n]\big). \qquad (3.8)\end{aligned}$$

From inequalities (2.5) and (2.2) we get

$$\begin{aligned}J_k^{(3,n)} &\leqslant |g'|_\infty|f|_\infty C_{p,q}V_{q,\infty}\big(g'(X^n)\big(W-W(t_{k-1}^n)\big);[t_{k-1}^n,t_k^n]\big)\\ &\quad\times V_p\big(B^H;[t_{k-1}^n,t_k^n]\big)\\ &\leqslant |g'|_\infty|f|_\infty C_{p,q}V_{q,\infty}\big(g'(X^n);[t_{k-1}^n,t_k^n]\big)\\ &\quad\times V_{q,\infty}\big(W-W(t_{k-1}^n);[t_{k-1}^n,t_k^n]\big)V_p\big(B^H;[t_{k-1}^n,t_k^n]\big)\\ &\leqslant 4|g'|_\infty|f|_\infty|g''|_\infty C_{p,q}V_q\big(X^n;[0,T]\big)V_q\big(W;[t_{k-1}^n,t_k^n]\big)\\ &\quad\times V_p\big(B^H;[t_{k-1}^n,t_k^n]\big) \qquad (3.9)\end{aligned}$$

and

$$J_k^{(4,n)} \leqslant 4|g'|_\infty |g|_\infty |g''|_\infty C_{p,q} V_q\big(X^n;[0,T]\big) V_p^2\big(B^H;[t_{k-1}^n,t_k^n]\big). \qquad (3.10)$$

It is evident that

$$J_k^{(5,n)} \leqslant |f|_\infty^2 |g''|_\infty \big(t_k^n - t_{k-1}^n\big). \qquad (3.11)$$

By the inequality

$$\begin{aligned} V_q\big(X^n;[t_{k-1}^n,t_k^n]\big) \quad \leqslant \quad & |f|_\infty V_q\big(W;[t_{k-1}^n,t_k^n]\big) + |g|_\infty V_p\big(B^H;[t_{k-1}^n,t_k^n]\big) \\ & +2C_{p,q}|g'|_\infty |f|_\infty V_q\big(W;[t_{k-1}^n,t_k^n]\big) V_p\big(B^H;[t_{k-1}^n,t_k^n]\big) \\ & +2C_{p,q}|g'|_\infty |g|_\infty V_p^2\big(B^H;[t_{k-1}^n,t_k^n]\big), \end{aligned}$$

inequalities (3.6), (3.7)-(3.11), and (2.3), (2.6) we get

$$\begin{aligned} \mathbf{E}I_4^2 \quad \leqslant \quad & 20C_{p,q}^2 \sqrt{\mathbf{E}V_p^4\big(B^H;[0,T]\big)} \Big\{ \Big(\mathbf{E} \max_{1\leqslant k\leqslant m(n)} V_q^4\big(X^n;[t_{k-1}^n,t_k^n]\big)\Big)^{1/2} \\ & \vee \Big(\mathbf{E} \max_{1\leqslant k\leqslant m(n)} V_q^8\big(X^n;[t_{k-1}^n,t_k^n]\big)\Big)^{1/4} \\ & \vee \Big(\mathbf{E} \max_{1\leqslant k\leqslant m(n)} V_q^8\big(W;[t_{k-1}^n,t_k^n]\big)\Big)^{1/4} \\ & \vee \Big(\mathbf{E} \max_{1\leqslant k\leqslant m(n)} V_p^8\big(B^H;[t_{k-1}^n,t_k^n]\big)\Big)^{1/4} \vee \delta_n \Big\} \\ & \times \Big[\sqrt{K_{q,4}\ell_4}\, |f|_\infty^2 |g''|_\infty^2 T + 4C_{p,q}^2 |g|_\infty^2 |g''|_\infty^2 \sqrt[4]{\mathbf{E}V_p^8\big(B^H;[0,T]\big)} \\ & +2^4 C_{p,q}^2 |g'|_\infty^2 |g''|_\infty^2 |f|_\infty^2 \sqrt[4]{\mathbf{E}V_q^8\big(X^n;[0,T]\big) V_p^8\big(B^H;[0,T]\big)} \\ & +2^4 C_{p,q}^2 |g'|_\infty^2 |g''|_\infty^2 |g|_\infty^2 \sqrt[4]{\mathbf{E}V_p^8\big(X^n;[0,T]\big) V_p^8\big(B^H;[0,T]\big)} \\ & +|f'|_\infty^4 |g''|_\infty^2 T \Big] \leqslant \delta_n^{2/q} R, \end{aligned} \qquad (3.12)$$

where R is a certain constant. By (2.7) we further obtain

$$\begin{aligned} \mathbf{E}I_2^2 \quad \leqslant \quad & K_{q,2}\ell_2 \mathbf{E} \int_0^T \big[g\big(X^n(s)\big) - g\big(X^n(\tau_s^n)\big)\big]^2 ds \\ \leqslant \quad & K_{q,2}\ell_2 T \mathbf{E} \sup_{t\leqslant T} \big|g\big(X^n(s)\big) - g\big(X^n(\tau_s^n)\big)\big|^2 \\ \leqslant \quad & K_{q,2}\ell_2 |g'|_\infty T \mathbf{E} \max_{1\leqslant k\leqslant m(n)} V_q^2\big(X^n;[t_{k-1}^n,t_k^n]\big) \end{aligned} \qquad (3.13)$$

and

$$\mathbf{E}I_1^2 \leqslant K_{q,2}\ell_2 |f'|^2 \mathbf{E} \int_{\sigma_{k-1}}^{\sigma_k} V_q^2\big(X - X^n;[0,s]\big) ds. \qquad (3.14)$$

Thus, by (3.1), (3.3), (3.4), and (3.12)-(3.14), we have

$$\begin{aligned}
\mathbf{E}V_q^2\big(X-X^n;[0,\sigma_j]\big) &\leqslant j\sum_{k=1}^{j}\mathbf{E}V_q^2\big(X^n-X^{n+1};[\sigma_{k-1},\sigma_k]\big)\\
&\leqslant 16j\sum_{k=1}^{j}\Big\{\delta_n^{2/q}R+(0.25)^2\mathbf{E}V_q\big(X-X^n;[0,\sigma_{k-1}]\big)\\
&\quad+K_{q,2}\ell_2|f'|^2\mathbf{E}\int_{\sigma_{k-1}^n}^{\sigma_k}V_q^2\big(X-X^n;[0,s]\big)ds\Big\}\\
&\leqslant 16j^2R\delta_n^{2/q}+j^2\mathbf{E}V_q^2\big(X-X^n;[0,\sigma_{j-1}]\big)\\
&\quad+16jK_{q,2}\ell_2|f'|^2\int_0^T\mathbf{E}V_q^2\big(X-X^n;[0,s\wedge\sigma_j]\big)ds.
\end{aligned}$$

Using this recurrent expression and the Gronwall lemma, one can show that there exists a constant $C(j)$ independent of n such that

$$\mathbf{E}V_q^2\big(X-X^n;[0,\sigma_j]\big)\leqslant C(j)R\delta_n^{2/q}.$$

Let $\alpha_n=\delta_n^{-1/q}|\ln\,\delta_n|^{-1/2}$. Then, for every $\gamma>0$,

$$\begin{aligned}
&\mathbf{P}\Big(\alpha_nV_q\big(X-X^n;[0,T]\big)>\gamma\Big)\\
&\quad\leqslant\mathbf{P}\Big(\alpha_nV_q\big(X-X^n;[0,\sigma_j]\big)>\frac{\gamma}{2}\Big)+\mathbf{P}\big(\sigma_j<T\big)\\
&\quad\leqslant\frac{4C(j)}{\gamma^2}\,|\ln\,\delta_n|^{-1}+\frac{4}{j}\,\mathbf{E}\Gamma(X,0,T).
\end{aligned}$$

Thus, we obtain the assertion of the theorem.■

Proof of Corollary 1.1. Let ε be any positive number and $\rho=2/(1-\varepsilon)$. Since, for every $q\geqslant 1$,

$$\sup_{t\leqslant T}|X(t)-X^n(t)|\leqslant V_q\big(X-X^n;[0,T]\big),$$

we have

$$\delta_n^{-1/2+\varepsilon}\sup_{t\leqslant T}|X(t)-X^n(t)|\leqslant\delta_n^{-1/2+\varepsilon}V_\rho\big(X-X^n;[0,T]\big).$$

For all $r\geqslant 1$ and $q>2$ (see [7]), we have

$$\mathbf{E}V_q^{2r}\big(X^n;[0,T]\big)<\infty\quad\text{and}\quad\mathbf{E}V_q^{2r}\big(X;[0,T]\big)<\infty.$$

Thus, the r.v.'s $V_\rho(X - X^n; [0,T])$ are uniformly integrable, and by Theorem 1.1 we get the assertion of Corollary 1.1.

Proof of Theorem 1.2. It is evident that the r.v.'s $\sup_{t\leqslant T}|Y_t|^r$ and $\sup_{t\leqslant T}|Y_t^n|^r$ are integrable for $r \geqslant 1$. By the Burkholder–Davis–Gundy (BDG) and Hölder's inequalities we get

$$\begin{aligned}\mathbf{E}\sup_{0\leqslant t\leqslant T}\big|Y(t) - Y^n(t)\big|^r &\leqslant 3^{r-1}C(r)T^{r/2-1}L^r\int_0^T \mathbf{E}\Big(\sup_{0\leqslant s\leqslant t}|Y(s) - Y^n(s)|^r\Big)dt \\ &\quad + 3^{r-1}C(r)T^{r/2-1}L^r\mathbf{E}\int_0^T |Y^n(s) - Y^n(\tau_s^n)|^r\,ds \\ &\quad + 3^{r-1}\mathbf{E}\sup_{0\leqslant t\leqslant T}\bigg|\int_0^t\big[g(s) - g(\tau_s^n)\big]\,dB_s^H\bigg|^r, \qquad (3.15)\end{aligned}$$

where L is the Lipschitz constant for the function f and $C(r)$ is the BDG constant.

Note that

$$\begin{aligned}&\mathbf{E}\int_0^T |Y^n(s) - Y^n(\tau_s^n)|^r ds \\ &\quad\leqslant 2^{r-1}\mathbf{E}\int_0^T\bigg|\int_{\tau_s^n}^s f\big(Y^n(\tau_u^n)\big)\,dW_u\bigg|^r ds + 2^{r-1}\mathbf{E}\int_0^T\bigg|\int_{\tau_s^n}^s g(\tau_u^n)\,dB_s^H\bigg|^r ds \\ &\quad\leqslant 2^{r-1}C(r)|f|_\infty^r\delta_n^{r/2}T + 2^{r-1}|g|_\infty^r\mathbf{E}\int_0^T \big|B^H(s) - B^H(\tau_s^n)\big|^r ds. \qquad (3.16)\end{aligned}$$

By the chain rule we obtain

$$\int_0^t g(s)\,dB_s^H = g(t)B^H(t) - \int_0^t B_s^H\,dg(s).$$

Since

$$\int_0^t g(\tau_s^n)\,dB_s^H = g(\tau_t^n)B^H(t) - \int_0^{\tau_t^n} B^H(\hat\tau_s^n)g'(s)\,ds,$$

where $\hat\tau_s^n = t_{k+1}^n$ for $s \in [t_k^n, t_{k+1}^n)$, $0 \leqslant k \leqslant m(n) - 1$, we have

$$\begin{aligned}\int_0^t\big[g(s) - g(\tau_s^n)\big]\,dB_s^H &= \big[g(t) - g(\tau_t^n)\big]B_t^H - \int_{\tau_t^n}^t B^H(s)g'(s)\,ds \\ &\quad - \int_0^{\tau_t^n}\big[B^H(s) - B^H(\hat\tau_s^n)\big]\,ds. \qquad (3.17)\end{aligned}$$

Thus, we get

$$\sup_{0\leqslant t\leqslant T}\left|\int_0^t \big[g(s)-g(\tau_s^n)\big]dB_s^H\right|^r$$
$$\leqslant 2^r|g'|_\infty^r \sup_{t\leqslant T}|B_t^H|^r\delta_n^r + 2^{r-1}|g'|_\infty^r T^{r-1}\int_0^T \big|B^H(s)-B^H(\hat\tau_s^n)\big|^r ds. \quad (3.18)$$

By inequalities (3.15)-(3.18) we obtain

$$\begin{aligned}
&\mathbf{E}\sup_{0\leqslant t\leqslant T}|Y(t)-Y^n(t)|^r \\
&\quad\leqslant \big(6^{r-1}C^2(r)T^{r/2}L^r|f|_\infty^r + 2\cdot 6^{r-1}|g'|_\infty^r\delta_n^{r/2}\mathbf{E}\sup_{t\leqslant T}|B_t^H|^r\big)\delta_n^{r/2} \\
&\quad\quad +3^{r-1}C(r)T^{r/2-1}L^r\mathbf{E}\int_0^T\Big(\sup_{s\leqslant t}\big|Y(s)-Y^n(s)\big|^r\Big)dt \\
&\quad\quad +6^{r-1}C(r)T^{r/2-1}L^r|g|_\infty^r\mathbf{E}\int_0^T|B^H(s)-B^H(\tau_s^n)|^r ds \\
&\quad\quad +6^{r-1}T^{r/2-1}L^r|g'|_\infty^r\mathbf{E}\int_0^T|B^H(s)-B^H(\hat\tau_s^n)|^r ds. \qquad (3.19)
\end{aligned}$$

For each $\alpha\geqslant 1$, there is a constant K_α such that (see [5])

$$\mathbf{E}|B^H(t)-B^H(s)|^\alpha \leqslant K_\alpha|t-s|^{\alpha H}.$$

Thus, we have

$$\int_0^T \mathbf{E}\big|B^H(s)-B^H(\hat\tau_s^n)\big|^r ds \leqslant K_r\int_0^T\big(s-\hat\tau_s^n\big)^{rH}ds \leqslant K_r\delta_n^{rH}T \qquad (3.20)$$

and

$$\int_0^T \mathbf{E}\big|B^H(s)-B^H(\tau_s^n)\big|^r ds \leqslant K_r\delta_n^{rH}T \qquad \text{for} \quad \frac{1}{2}<H<1. \qquad (3.21)$$

By the Gronwall lemma and inequalities (3.19)-(3.21) we get

$$\mathbf{E}\sup_{0\leqslant t\leqslant T}\big|Y(s)-Y^n(s)\big|^r \leqslant C\delta_n^{r/2}e^{3L^2T},$$

where C is a certain constant.

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Apie stochastinių integralinių lygčių, valdomų p-semimartingalų, aproksimacijos asimptotinį elgesį

K. Kubilius

Nagrinėjamas Milšteino tipo aproksimacijos konvergavimo greitis į integralinės lygties, valdomos specialaus tolydaus p-semimartingalo, sprendinį. Įrodyta, kad nagrinėjamas stiprios Milšteino tipo aproksimacijos konvergavimo greitis yra artimas difuzinės lygties stiprios Eulerio-Peano aproksimacijos konvergavimo greičiui.