Mathematical Modelling and Analysis
Volume 7 Number 1, 2002, pages 159–168

ON CLASSICAL FORMULATION OF HELE-SHAW MOVING BOUNDARY PROBLEM FOR POWER-LAW FLUID[1]

S.V. ROGOSIN

Department of Mathematics and Mechanics Belarusian State University
F. Skaryny av. 4, 220050 Minsk, Belarus
E-mail: rogosin@bsu.by

Received September 9, 2001

ABSTRACT

A model of the Hele-Shaw flow for power-law fluid is proposed. A classical formulation of the corresponding moving boundary value problem is given.

1. INTRODUCTION

The Hele-Shaw approximation constitutes a well-known approach in the study of viscous flow problems. The Hele-Shaw flow means the injection/suction of a viscous fluid into a narrow channel between two closely situated plates. The fluid is supposed to be surrounded by another fluid of a small viscosity or even inviscid (i.e., we have the one-phase Stefan problem). In the framework of this model different situation were discussed intensively (see e.g., [8; 16; 22; 25]). Recently the Hele-Shaw approximation has been used for the study of flow of non-Newtonian fluids (cf. [3; 9; 10; 12].)

In the Newtonian case the filtration law can be presented in the form

$$-\nabla\phi = \Phi(w)\frac{\vec{w}}{w},\ w = |\vec{w}|, \tag{1.1}$$

where ϕ is a pressure, $\vec{w}$ is a velocity field, and the function Φ is linear with respect to w (i.e., (1.1) is the Darcy law). Non-Newtonian behaviour of fluids leads to different form of nonlinear filtration law. The most investigated case

[1]The work is partially supported by Belarusian Fund for Fundamental Scientific Research. The author expresses his deep gratitude to Prof. Stasys Rutkauskas for the warm hospitality during the Conference MMA-2001.

is the situation of power-type liquid for which the function Φ in (1.1) satisfies the relation

$$\Phi(w) = Cw^s. \tag{1.2}$$

The case $0 < s < 1$ corresponds to pseudo-plastic behaviour of a fluid, and the case $s > 1$ corresponds to dilatant rheological behaviour of a fluid (see e.g. [12]). Thus the incompressible non-Newtonian (power-law) fluid have to satisfy the following system

$$\begin{cases} -\nabla\phi &= Cw^s\dfrac{\vec{w}}{w}, \\ \nabla\cdot\vec{w} &= 0, \end{cases} \tag{1.3}$$

where the second relation is simply the continuity equation. Straightforward calculations show that (1.3) is equivalent to the system

$$\begin{cases} -\nabla\phi &= Cw^s\dfrac{\vec{w}}{w}, \\ \operatorname{div}\left(|\nabla\phi|^{1/s-1}\nabla\phi\right) &= 0. \end{cases} \tag{1.4}$$

Therefore, the pressure inside the domain occupied by the power-law fluid should be p-harmonic function, i.e. satisfies p-Laplace equation

$$\operatorname{div}\left(|\nabla\phi|^{p-2}\nabla\phi\right) = 0, \tag{1.5}$$

with $p = \frac{1}{s} + 1$. The equation (1.5) is a nonlinear elliptic equation for all $1 < p \le 2$ (i.e., for $s \ge 1$). When $p > 2$ (i.e., $0 < s < 1$) this equation degenerates at all points at which $\vec{w} = 0$.

The p-Laplace equation (1.5) has been studied from different points of view. We have to mention here the papers [2; 3; 17; 23], and the books [5; 15].

Let us briefly outline the goals of this paper. First we describe a physical mechanism of the behaviour of non-Newtonian fluid, conditions for which were formulated above rigorously. Then we recall some properties of p-harmonic functions. On their base we introduce a concept of "classical solution" to corresponding moving boundary value problem. We have left proof of local existence and uniqueness of such solution for the next consideration.

2. DERIVATION OF THE FLOW EQUATION

In this section we recall the derivation of physical model and make corresponding simplifications led us to the Hele-Shaw type approximation. We suppose first that the non-Newtonian fluid is situated between two large plates separated by a narrow gap. The width of the gap we denote $2h$ supposing that it is essentially smaller than the scale of the plates. It is customary to consider a fully developed laminar flow far away from possibly existing moving boundary. We introduce the system of Cartesian co-ordinates (x_1, x_2, x_3) with (x_1, x_2)-plane lying between the plates $x_3 = -h, x_3 = h$. Denote also by $\vec{v} = (v_1, v_2, v_3)$ the velocity vector of the flow. To derive the model equation we use the law of the conservation of mass and the law of the conservation of momentum. The fluid under consideration is supposed to be non-Newtonian (power-law) and incompressible. The later assumption gives us the continuity equation in the form

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0. \tag{2.1}$$

Hence the rate-of-strain tensor $\mathbf{D}$ can be presented in the form [21]

$$\mathbf{D} = \frac{1}{2}\begin{pmatrix} 2\frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} & \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} & 2\frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} \\ \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} & \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} & 2\frac{\partial v_3}{\partial x_3} \end{pmatrix}.$$

This tensor is symmetric, satisfies the relation $\mathrm{tr}\mathbf{D} = 0$, and is insensitive to the rigid-body motion. The invariant $\left(\mathrm{tr}(2\mathbf{D} : \mathbf{D})^2\right)^{\frac{1}{2}} = \left(2\sum_{i=1}^{n} d_{i,i}^2\right)^{\frac{1}{2}}$ is denoted by $\dot{\gamma}$ and is commonly used as a scalar measure for the rate of deformation. The stress equation for non-Newtonian fluid reads $\mathbf{T}' = 2\eta(\dot{\gamma}, T)\mathbf{D}$, where $\mathbf{T}'$ is the deviatoric stress, T is the temperature, and η is the viscosity function of the fluid depending in general on the rate of deformation $\dot{\gamma}$ and the temperature T, having a specific form for every fluid. In the case of isothermal flow (which is assumed to be discussed) the viscosity does not depend on the temperature, i.e., $\mathbf{T}' = 2\eta(\dot{\gamma})\mathbf{D}$. Then the total stress tensor $\mathbf{T}$ can be represented in the form

$$\mathbf{T} = -\phi\mathbf{I} + 2\eta(\dot{\gamma})\mathbf{D}, \tag{2.2}$$

where $\mathbf{I}$ is the identity tensor, ϕ is the pressure.

The equation of the conservation of momentum can be taken in the form

$$\rho\frac{D\underline{\vec{v}}}{Dt} = \mathrm{div}\mathbf{T} + \rho\underline{g},$$

where ρ is the density, $\frac{D}{Dt}$ is the material derivative, $\underline{\vec{v}}$ is the (averaged with respect to gap's width) velocity vector, $\underline{g}$ is the (averaged) gravity.

Applying (2.2) one can get the following form of momentum conservation law:

$$\rho\frac{D\underline{\vec{v}}}{Dt} = -\nabla\phi + \mathrm{div}\,(2\eta(\dot{\gamma})\mathbf{D}) + \rho\underline{g}. \tag{2.3}$$

If the gravity vector $\underline{g}$ is potential $\underline{g} = \nabla U$, then one can introduce the effective pressure $\phi^* = \phi - \rho \underline{U}$. Therefore the gravity term can be neglected, and (2.3) can be rewritten as

$$\rho \frac{D\vec{\underline{v}}}{Dt} = -\nabla\phi + \operatorname{div}\left(2\eta(\dot{\gamma})\mathbf{D}\right). \tag{2.4}$$

In the case of power-law liquid its viscosity η is supposed to satisfy the relation $\eta(\dot{\gamma}) = K\dot{\gamma}^{s-1}$, where $K > 0$ is called the consistency index, $s > 0$ is called power-law index. Both magnitudes are material constants [4; 12].

Let us now propose certain assumptions to simplify the considered model. These conditions seems to be natural and are close to that from [2; 3].

- (i) suppose from now on that the flow is parallel to x_1x_2-plane. It is natural to assume that the "area" occupied by the fluid is essentially larger than the width between the plates. In this case x_3-component of the viscosity forces can be also neglected;
- (ii) inertia and body forces are supposed to be essentially smaller than viscosity forces and pressure difference. Thus $\frac{D\vec{\underline{v}}}{Dt} = 0$ in (2.4);
- (iii) the thickness of the gap is small and no-slip conditions on $x_3 = \pm h$ are presumed. Hence the derivatives of v_1 and v_2 in the direction of x_3-axes is much bigger than those in x_1- and x_2-directions.

Applying these assumptions to our system (2.1)-(2.4) we obtain the following description of "pure" plane non-Newtonian flow:

$$\operatorname{div}\vec{\underline{v}} = 0, \;\; \nabla\phi = \operatorname{div}\left(2\eta(\dot{\gamma})\mathbf{D}\right), \tag{2.5}$$

where (cf. assumption (i)) $\vec{\underline{v}} = (\underline{v}_1, \underline{v}_2)$, $\underline{v}_i = \frac{1}{2h}\int_{-h}^{h} v_i(x_1, x_2, x_3)dx_3, \; i = 1, 2;$ $\dfrac{\partial\phi}{\partial x_3} = 0$, and

$$\mathbf{D} = \frac{1}{2}\begin{pmatrix} 0 & 0 & \dfrac{\partial v_1}{\partial x_3} \\ 0 & 0 & \dfrac{\partial v_2}{\partial x_3} \\ \dfrac{\partial v_1}{\partial x_3} & \dfrac{\partial v_2}{\partial x_3} & 0 \end{pmatrix}, \; \dot{\gamma} = \sqrt{\left(\frac{\partial v_1}{\partial x_3}\right)^2 + \left(\frac{\partial v_2}{\partial x_3}\right)^2}.$$

Due to symmetry of the flow with respect to $x_3 = 0$ we have $\eta(\dot{\gamma})\frac{\partial v_i}{\partial x_3} = \frac{\partial\phi}{\partial x_i}x_3, i = 1, 2$. Therefore from the no-slip condition on $x_3 = \pm h$ we obtain $\underline{v}_1 = -\frac{\partial\phi}{\partial x_1}\frac{S}{h}$, $\underline{v}_2 = -\frac{\partial\phi}{\partial x_2}\frac{S}{h}$, with $S = \int_0^h \frac{\zeta^2}{\eta(\zeta)}d\zeta$. Then the continuity equation (2.5) becomes

$$\frac{\partial}{\partial x_1}\left(S\frac{\partial\phi}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(S\frac{\partial\phi}{\partial x_2}\right) = 0. \tag{2.6}$$

In the case of a power-law fluid it follows from (2.4), (2.6) that $\dot{\gamma} = \frac{|x_3|}{\eta(\dot{\gamma})}|\nabla\phi|$. Hence $\eta = K^{\frac{1}{s}}|x_3\nabla\phi|^{1-\frac{1}{s}}$, $S = C|\nabla\phi|^{1-\frac{1}{s}}$. Therefore the continuity equation for power-law fluid is in fact p-Laplace equation (cf. (1.5))

$$\operatorname{div}\left(|\nabla\phi|^{p-2}\nabla\phi\right) = 0, \tag{2.7}$$

with $p = \frac{1}{s} + 1$. In what follows we restrict our attention to the case $p > 2$, i.e. on the pseudo-plastic flow.

Let us now describe the mathematical model of the discussed physical model. Let Ω be a bounded plane domain containing origin. Let us additionally supposed that the boundary Γ of Ω consists of finite number of smooth curves. Let $\mu \neq 0$ be a finite positive measure with compact support in Ω. We consider the plane domain D such that supp $\mu \subset\subset D \subset \Omega$. It means that the supp μ is compactly embedded into D, but ∂D and $\partial\Omega$ can have nonempty intersection. Let us denote $\Gamma_0 := \partial D \cap \Omega$, $\Gamma_1 := \partial D \cap \partial\Omega$, and both sets consists of finite number of smooth arcs.

It is supposed that the non-Newtonian fluid is injected through the "border" of the domain supp μ (in particular through a number of point sources). At certain moment the fluid occupies the domain D. Γ_0 is then an unknown free boundary (moving front of the fluid), and Γ_1 is an impermeable wall. The flow is supposed to be isothermal (cf. [9; 10; 22]). Different type of boundary conditions are posed on Γ_0 and Γ_1. On Γ_1 there exists no normal flow (impermeable wall). It means $\frac{\partial\phi}{\partial n} = 0$ on Γ_1. On the moving boundary Γ_0 we have surface-tension condition since the rest of the Hele-Shaw cell (i.e. the rest of the domain Ω) is supposed to be filled in by the fluid of essentially smaller viscosity, or even by the air. Without loss of generality one can suppose $\phi = 0$ on Γ_0. At last there is a kinematic condition on Γ_0 which follows from the mass balance. It appears when Γ_0 is moving. It reads that the fluid velocity of a particle at Γ_0 should coincide with the velocity of this "point" as a geometric object, i.e. $\vec{v} = -\frac{1}{C}|\nabla\phi|^{p-2}\nabla\phi = \frac{\partial\Gamma_0}{\partial t}$.

3. STATIONARY PROBLEM

Here we discuss the stationary formulation of our problem. Our concern is connected with the following three facts. First of all, the basic equation describing the behaviour of the fluid in the domain is a *nonlinear* partial differential equation. Second, the *topology* of the moving boundary Γ_0 can be changed in accordance with the given configuration of the Hele-Shaw cell. At last, the moving boundary is defined not only by the fluid's properties but also by the *geometry* of the given domain Ω, as well as by the geometry of the given *nonlocal source* $\omega_0 =$ supp μ.

In order to restrict our attention on analytical problems we study only *local in time* formulation of conception of a classical solution.

Let as before Ω be a given (bounded or not) open plane domain with the piece-wise smooth boundary $\partial\Omega$. Let $\mu \neq 0$ be a given finite positive measure with compact support containing in $\overline{\Omega}$. Denote by D a domain satisfying the inclusion $\omega_0 := \text{supp}\,\mu \subset D \subset \Omega$ and introduce the function ϕ_D as a solution of the following boundary value problem:

$$-\text{div}\,\left(|\nabla\phi_D|^{p-2}\nabla\phi_D\right) = \mu \ \text{ in } \ D, \tag{3.1}$$

$$\phi_D = 0 \ \text{ on } \ \Gamma_0 = \partial D \cap \Omega, \tag{3.2}$$

$$|\nabla\phi_D|^{p-2}\frac{\partial\phi_D}{\partial n} = 0 \ \text{ on } \ \Gamma_1 = \partial D \cap \Omega. \tag{3.3}$$

The boundary condition (3.3) can be rewritten in the form

$$\frac{\partial\psi_D}{\partial\tau} = 0 \ \text{ on } \ \Gamma_1 = \partial D \cap \Omega,$$

where ψ_D is so called p-harmonic conjugate to ϕ_D, i.e. the function connected with ϕ_D by the relation

$$\begin{cases} \dfrac{\partial\psi_D}{\partial x_1} &= -C'|\nabla\phi_D|^{p-2}\dfrac{\partial\phi_D}{\partial x_2}, \\[2ex] \dfrac{\partial\psi_D}{\partial x_2} &= C'|\nabla\phi_D|^{p-2}\dfrac{\partial\phi_D}{\partial x_1}. \end{cases}$$

We note that in the case of the finite number of sources (in particular, one source) the measure μ in (3.1) can be taken in the form $\sum\limits_{k=1}^{n} c_k\delta_k$, where δ_k is the Dirac measure at the points $(x_1(k), x_2(k)) \in \Omega$. For $p = 2$ it is known [19] that for any piece-wise smooth domain there exists a unique solution to the problem (3.1)-(3.3) which is in fact the solution of a mixed boundary value problem for harmonic functions. The only additional assumption is that the linear measure of Γ_0 is not equal to zero. The boundary condition (3.2) on Γ_0 should be understood in this case in the following way: for each $\varepsilon > 0$ there exists a compact set $K \subset \overline{D}$ such that $|\phi_D| < \varepsilon$ on $D \setminus K$.

For certain simple domains the corresponding solution to mixed problem (3.1)-(3.3) for $p = 2$ can be presented in closed form [18]. For more general domains the conformal mapping technique can be applied.

Another remark is connected with the following: the arcs Γ_0 and Γ_1 meet each other in general not in a smooth way. The same is true for $\Gamma_K := \partial K \cap D$ and Γ_1. To avoid this difficulty one can use the standard balayage approach [6]. We first fix the value of $\varepsilon > 0$ and the corresponding compact domain K_ε. Then on Γ_{K_ε} the boundary values of ϕ_D are equal to $a_\varepsilon(t)$, $t \in \Gamma_{K_\varepsilon}$, where $|a_\varepsilon(t)| = \varepsilon$, $a_\varepsilon \in C^2$. This function can be continuously extended into the streep $\{(x_1, x_2) \in K_\varepsilon : \rho\left((x_1, x_2); \Gamma_{K_\varepsilon}\right) \leq b\}$. Then there exists a sequence of domains D_j, $D_j \subseteq d_{j+1}$, $\cup_j D_j = K_\varepsilon$ such that their boundaries are smooth curves. Then it is known that for any given continuous arc Γ_{K_ε} the corresponding solution of the problem

$$\begin{cases} -\Delta\phi_\varepsilon &= \mu, \\ \phi_\varepsilon|_{D_j} &= a^j = A_\varepsilon|_{D_j}, \\ \dfrac{\partial\phi_\varepsilon}{\partial n}\Big|_\Gamma &= 0 \end{cases}$$

converges to the solution of problem (3.1)-(3.3) in the domain K_ε in the case $p = 2$.

We use the above discussed considerations in the next sections. As for the case $p \neq 2$ the situation with the solvability of problem (3.1)-(3.3) is more complicated. Therefore we need to recall some properties of p-harmonic functions to formulate the corresponding results.

4. PROPERTIES OF P-HARMONIC FUNCTIONS

There are two meaning of p-harmonic functions. First one (*classical* p-harmonic functions) is the following: for a given plane domain D any function $\phi \in \mathcal{C}^2(D)$, $\nabla\phi \not\equiv 0$ in D satisfying the nonlinear differential equation

$$\operatorname{div}\left(|\nabla\phi|^{p-2}\nabla\phi\right) = 0 \tag{4.1}$$

for a given $p \in (1, \infty)$ is called p-harmonic function in the domain D. The crucial things determined the behavior of p-harmonic functions are their singular points, i.e. the points of D at which $\nabla\phi = 0$. It is known [2] that in a neighbourhood of any internal point of D at which $\nabla\phi \neq 0$ the solution of (4.1) belongs even to $\mathcal{C}^\infty$. But near singular point its behaviour is less regular. The highest level of regularity one can prove is $\mathcal{C}^{k,\alpha}_{loc}(D)$, where an integer number $k \geq 1$ and a real number $\alpha \in (0, 1]$ are determined by the equation [17]

$$k + \alpha = \frac{1}{6}\left(7 + 1/(p-1) + \sqrt{1 + 14/(p-1) + 1/(p-1)^2}\right).$$

In [1; 17] some examples are presented showing that for $p \neq 2$ the above described class is an optimal one. It should be noted that singular points of p-harmonic functions in two-dimensional case are isolated points.

As in classical case $p = 2$ of harmonic functions it is possible to introduce for any $p \neq 2$ a function ψ dual to a p-harmonic function ϕ via relations

$$\begin{cases} \psi_{x_1} &= -|\nabla\phi|^{p-2}\phi_{x_2}, \\ \psi_{x_2} &= |\nabla\phi|^{p-2}\phi_{x_1}. \end{cases} \tag{4.2}$$

This function has the properties of stream function for the solution of (4.1) and does satisfy the equation

$$\operatorname{div}\left(|\nabla\psi|^{p'-2}\nabla\psi\right) = 0,$$

where $p' \in (1,\infty), \frac{1}{p}+\frac{1}{p'}=1$, i.e. ψ is p'-harmonic function in D (cf. [3]). Another family of classical solutions to (4.1) (so called spiral solutions) is described in [3].

Introducing a complex p-potential $w=\phi+i\psi$ and using the standard notation for the formal complex derivatives (cf. e.g. [11]) one can rewrite (4.2) in the form of complex p-Cauchy-Riemann equations

$$\begin{cases} w_z - \overline{w}_z &= (w_z+\overline{w}_z)^{\frac{p}{2}}(w_{\bar z}+\overline{w}_{\bar z})^{\frac{p-2}{2}}, \\ w_{\bar z}-\overline{w}_{\bar z} &= (w_z+\overline{w}_z)^{\frac{p-2}{2}}(w_{\bar z}+\overline{w}_{\bar z})^{\frac{p}{2}}. \end{cases}$$

The later equations can be rewritten as a nonlinear analog of Beltrami equations (see e.g., [20]).

Another meaning of p-harmonic functions (*weak* p-harmonic functions) is the following: these are any nonnegative weak solution $\phi \in W^{1,p}_{loc}(D)$ of the equation

$$\int\limits_D |\nabla\phi|^{p-2}\langle\nabla\phi,\nabla\psi\rangle dx_1 dx_2 = 0,\ \forall\psi\in W_0^{1,p'}(D).$$

5. ON CLASSICAL FORMULATION OF THE EVOLUTIONARY PROBLEM FOR THE HELE-SHAW FLOW OF NON-NEWTONIAN FLUID

Our purpose here is to describe the classical type Hele-Shaw model for non-Newtonean fluid. We use here the notation presented in Sec. 2. As it was already mentioned the geometry of the domain plays an important role in our considerations. Therefore, we assume for simplicity that the curve Γ_0 (moving front of the flow) consists of the only one simple Jordan arc.

Let Ω be a fixed plane domain as before, and ω be a fixed open neighbourhood of the domain $\omega_0 = \text{supp}\ \mu$ and the set $\mathcal{S}_{\omega,\Omega}$ be a class of simply connected domains D such that $\omega_0 \subset \omega \subset\subset D \subset \Omega$, such that $\Gamma_0 = \partial D \cap \Omega$ is an open simple Jordan arc of class $\mathcal{C}^2$.

Let there exists a function ϕ_D satisfied the system of equations (3.1)-(3.3) (the solution of stationary problem). Let us suppose also that this solution can be continuously extended up to Γ_0. Let $\mathbb{I} \subset \mathbb{R}$ be an open interval.

Definition 5.1. A map $\mathbb{I} \ni t \to D_t \in \mathcal{S}_{\omega,\Omega}$ is called a (local in time on $\mathbb{I}$) classical solution to the Hele-Shaw moving boundary value problem for a non-Newtonian fluid if there exists a map $\zeta : (0,1)\times\mathbb{I} \to \mathbb{R}^2$ of class $\mathcal{C}^2$ such that

- (i) $\zeta(s,t) \in \Gamma_{0,t}$ for all s,t;
- (ii) for any fixed $t\in\mathbb{I}$ the function $\zeta(\cdot,t)$ is a diffeomorphism of class $\mathcal{C}^2$ on the interval $(0,1)$;
- (iii) $\frac{\partial\zeta}{\partial t}(s,t) = -\left|\nabla\phi_{D_t}\right|^{p-2}\nabla\phi_{D_t}|_{\zeta(s,t)}$ for all s,t.

We have to note (cf. [14]) that conditions (i), (ii) mean that for each t the function $\zeta(\cdot,t)$ parameterizes the moving arc $\Gamma_{0,t}$. The condition (iii) says that the point $\zeta(s,t)$ moves with the velocity $-\left|\nabla\phi_{D_t}(\zeta(s,t))\right|^{p-2}\nabla\phi_{D_t}(\zeta(s,t))$, where $\nabla\phi_{D_t}$ means a continuous extension of the gradient $\nabla\phi_{D_t}$ to $\Gamma_{0,t}$.

We have to stress once more that the above definition is in any case only local in time because of possible changing of topology of the curve $\Gamma_{0,t}$. Global variant of this definition can appear only in sense of life-time estimate of certain topologically stable situation, say e.g. an estimate of the length of time-interval $\mathbb{I}$ such that for each $t\in\mathbb{I}$ the curve $\Gamma_{0,t}$ consists of the same (finite) number of open arcs. It should be noted also that instability appears either at the physical modelling of the flow (e.g. as in [13]) or at its numerical analysis [2; 3].

Let us make a remark also about the measure μ. As in [14] (cf. also [7; 24]) at the solving the problem in the classical sense one can always suppose that μ is a smooth function. It can be shown by using mollification technique. It is natural to suppose it because the only behavior of ϕ_D near the moving front has an influence in the above formulated definition. Therefore the function ϕ_D can be smoothed in a neighbourhood of supp μ. Hence μ itself can be supposed to be smooth.

REFERENCES

[1] G. Aronsson. On certain p-harmonic functions in the plane. *Manuscr. Math.*, **61**(1), 79, 1988.

[2] G. Aronsson. On p-harmonic functions, convex duality and an asymptotic formula for injection mold fluid. *Euro. J. Appl. Math.*, **7**(5), 417, 1996.

[3] G. Aronsson and U. Janfalk. On Hele-Shaw flow of power-law fluid. *Euro. J. Appl. Math.*, **3**, 343, 1992.

[4] G. Astarita and G. Marrucci. *Principles of non-Newtonian Fluid Mechanics.* McGraw-Hill, New York, 1974.

[5] P. Drabek, A. Kufner and F. Nicolosi. *Quasilinear Elliptic Equations with Degeneration and Singularities.* W. de Gruyter, Berlin, 1997.

[6] Yu.V. Egorov and M.A. Shubin. *Foundation of the Classical Theory of Partial Differential Equations.* Springer Verlag, Berlin, 1998.

[7] C.M. Elliot and V. Janovsky. A variational inequality approach to Hele-Shaw flow with a moving boundary. *Proc. R.Soc. Edinbourgh*, **88A**, 93, 1981.

[8] C.M. Elliot and J.R. Ockendon. *Variational Methods for Moving Boundary Value Problems.* Pitman, London, 1982.

[9] R.P. Gilbert and P. Shi. Nonisothermal, non-Newtonian Hele-Shaw flows, Part I. In: *Proceedings on Transport Phenomenon.* Technomic, Lancaster, 1992.

[10] R.P. Gilbert and P. Shi. Nonisothermal, non-Newtonian Hele-Shaw flows, Part II. *Nonlinear Analysis. Theory, Methods & Applications*, **27**(5), 1996.

[11] R.P. Gilbert and Zhenyuan Xu. Several Hele-Shaw flows associated with injection molding. In: *Complex Analysis and its Applications.* Longman, Harlow, 26, 1994.

[12] R.V. Gol'dstein and V.M. Entov. *Qualitative Methods in Continuous Mechanics.* Longman, Harlow, 1994.

[13] V.A. Gorodtsov and V.M. Entov. Instability of the displacement front for non-Newtonian fluids in the Hele-Shaw flow. *Prikl. Math. Mech.*, **61**(1), 115, 1997. (in Russian)

[14] B. Gustafsson. Applications of variational inequalities to a moving boundary value problem for Hele-Shaw flow. *SIAM J. Math. Anal.*, **16**(2), 279, 1985.

[15] J. Heinonen, T. Kilpelaeinen and O. Martio. *Nonlinear Potential Theory of Degenerate Elliptic Equations.* Clarendon Press, Oxford, 1993.

[16] S.D. Howison. Complex variables methods in Hele-Shaw moving boundary value problems. *Euro. J. Appl. Math.*, **3**, 209, 1992.

[17] T. Iwanec and J.J. Manfredi. Regularity of p-harmonic functions on the plane. *Rev. Mat. Iberoam.*, **5**(1/2), 1, 1989.

[18] A. Jeffrey. *Complex Analysis and Applications.* CRC Press, Boca Raton, 1992.

[19] O.A. Ladyzhenskaya. *Boundary Value Problems of Mathematical Physics.* Springer Verlag, Berlin, 1985.

[20] V.N. Monakhov. *Boundary Value Problems with Free Boundaries for Elliptic Systems of Equations.* AMS, Providence RI, 1983.

[21] N.I. Muskhelishvili. *Some Basic Problems of Elasticity Theory.* Wolters-Noordhof, Groningen, 1953.

[22] O.A. Oleinik, M. Primicerio and E.V. Radkevich. Stefan-like problems. *Mecanica*, **28**, 129, 1993.

[23] S.I. Pokhozhaev. The fibering method and its applications to nonlinear boundary value problem. *Rend. Mat. Univ. Trieste*, **XXXI**, 235, 1999.

[24] M. Primicerio and J.-F. Rodrigues. The Hele-Shaw problem with nonlocal injection condition. *GAKUTO Int. Math. Sci. Appl.*, **2**, 375, 1993.

[25] P.G. Saffman. Viscous fingering in a Hele-Shaw cells. *J. Fluid Mech.*, **73**, 173, 1986.

Hele-Shaw uždavinio ne-Niutono skysčiams klasikinis formulavimas

S.V. Rogosin

Pasiūlytas naujas ne-Niutono skysčių tekėjimo modelis. Pateiktas klasikinis formulavimas uždavinio, sprendžiamo srityse su laisvai judančiu paviršiumi. Sprendinio egzistencijos ir vienaties klausimai bus tiriami kitame darbe.