Mathematical Modelling and Analysis
Volume 7 Number 2, 2002, pages 241–252

ON THE SOLUTION OF ILL-POSED PROBLEMS BY PROJECTION METHODS WITH A POSTERIORI CHOICE OF THE DISCRETIZATION LEVEL[1]

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Received October 1, 2002; revised October 30, 2002

ABSTRACT

We consider linear ill-posed problems $Au = f$ with minimum-norm solution u_*. Instead of f noisy data f^δ are given satisfying $\|f^\delta - f\| \leq \delta$ with known noise level δ. The projection methods for finding approximation u_n to u_* are discussed in assumptions guaranteeing in case $f^\delta = f$ the monotone convergence $u_n \to u_*$ $(n \to \infty)$. In noisy case $\delta > 0$ we propose for two projection methods a posteriori rules for choice $n = n(\delta)$ as largest $n = 1, 2 \ldots$, for which inequality $\|u_n - u_*\| \leq \|u_{n-1} - u_*\|$ can be proved. Numerical results are given.

Key words: ill-posed problems, projection methods, a posteriori rule.

1. INTRODUCTION

In this paper we consider linear ill-posed problems

$$Au = f \tag{1.1}$$

where $A \in \mathcal{L}(H, F)$ is a bounded operator with the non-closed range $\mathcal{R}(A)$ and H, F are infinite dimensional real Hilbert spaces with inner products $(\cdot, \cdot)$ and norms $\|\cdot\|$. We suppose that $f \in \mathcal{R}(A)$. The null-space $\mathcal{N}(A)$ of operator A can be non-trivial. We are interested in the minimum-norm solution u_* of problem (1.1). It is assumed that instead of exact data f there are given noisy data $f^\delta \in F$ with $\|f^\delta - f\| \leq \delta$ and known noise level δ. The typical

[1]This work was supported by the Estonian Science Foundation, grant 4345.

feature of ill-posed problems is unstability of the solution of the problem $Au = f^\delta$ with respect to noise in data and therefore solution procedure must use noise level. Ill-posed problems are typically solved by iteration methods or by special regularization methods (the Tikhonov method etc, see [3; 14; 21; 27]). For using computers in solution procedures the discretization of the problem is unavoidable. If certain problems are successfully discretized, additional regularization is not needed. Namely, if the discretization method converges in the case of exact data f, then in the case of noisy data f^δ this method can be viewed as regularization method, when the discretization step as a regularization parameter is properly chosen according to the noise level δ. This phenomena is called as self-regularization by discretization (see [25]).

In this paper we consider the projection methods for problem (1.1). For well-posed problems projection methods are thoroughly investigated (see [16]), corresponding convergence conditions are not very restricting. For ill-posed problems the convergence conditions of projection methods in the case of exact data $f^\delta = f$ were stated in [1; 2; 4; 5; 7; 13; 18; 19; 20; 22; 25; 26; 28]. These conditions for traditional projection methods (the least square method, the Galerkin method, the collocation method) are quite restricting.

The plan of this paper is the following. Section 2 is a short review (based on paper [25]) of the results of the projection methods for ill-posed problems in the Hilbert spaces, concerning convergence conditions in the case of exact data $f^\delta = f$, but also self-regularization conditions, if the dimension of the discretized equation is chosen a priori or by the discrepancy principle. In Sections 3, 4 we consider the least error method and a special collocation method for integral equations of the first kind respectively. These methods have a specific feature: they converge in the case of exact data by very mild conditions. For the case of noisy data we propose a new a posteriori rule (the monotone error rule) for the choice of dimension of the discretized equation. In the final section of this paper numerical examples are given.

2. CONVERGENCE OF PROJECTION METHODS

Let H, F are Hilbert spaces and $H_n \subset H$, $F_n \subset F$ $(n \in N)$ are corresponding finite-dimensional subspaces with $\dim H_n = \dim F_n$. We denote the corresponding orthoprojectors by P_n and Q_n, i.e. $P_n H = H_n$, $Q_n F = F_n$. In the projection method we take as approximation to solution u_* of equation (1.1) element $u_n \in H_n$, satisfying conditions

$$u_n \in H_n, \quad (Au_n - f^\delta, z_n) = 0 \quad (\forall\, z_n \in F_n)\,. \tag{2.1}$$

The last conditions are equivalent to the equation

$$Q_n A u_n = Q_n f^\delta, \quad u_n \in H_n. \tag{2.2}$$

In the following we give from [25] two theorems about the convergence conditions of projection methods in the case of exact data (Theor. 2.1) and also in the case of noisy data with a priori choice of n (Theor. 2.1) or with the choice of n by the discrepancy principle (Theor. 2.2). We denote by A^* the adjoint operator of A and use also notions

$$\kappa_n = \sup_{\omega_n \in H_n} \frac{\|\omega_n\|}{\|A\omega_n\|}, \quad \kappa_n^* = \sup_{z_n \in F_n} \frac{\|z_n\|}{\|A^* z_n\|}.$$

Theorem 2.1. *Suppose that* $\mathcal{N}(A^*) \cap F_n = 0$ $(\forall n \in N)$,

$$\|u - P_n u\| \to 0 \quad for \quad n \to \infty \quad (\forall u \in H), \tag{2.3}$$

$$\|P_n A^* z_n\| \geq \tau^* \|A^* z_n\| \quad (\forall z_n \in F_n, n \in N), \quad \tau^* = \text{const} > 0. \tag{2.4}$$

Then equation (1.1) *has the unique solution* $u_* \in H$ *and equation* (2.2) *has the unique solution* $u_n \in H_n$. *If* $\delta = 0$, *then* $u_n \to u_*$ *for* $n \to \infty$. *If* $\delta > 0$, *then* $u_{n(\delta)} \to u_*$ *as* $\delta \to 0$ *for such a choice of* $n = n(\delta)$ *that*

$$n(\delta) \to \infty, \quad \delta \kappa_{n(\delta)}^* \to 0 \quad for \quad \delta \to 0. \tag{2.5}$$

Theorem 2.2. *Suppose that* (2.3), (2.4) *hold and*

$$\mathcal{N}(A) \cap H_n = 0 \quad (\forall n \in N), \tag{2.6}$$

$$\|Q_n A \omega_n\| \geq \tau \|A \omega_n\| \quad (\forall \omega_n \in H_n, n \in N), \quad \tau = \text{const} > 0,$$

$$\kappa_{n+1} \|(I - Q_n')A\| \leq \gamma' = \text{const} \ (\forall n \in N),$$

where Q_n' *is the orthoprojector in* F *to* AH_n. *Then equation* (1.1) *has the unique solution* $u_* \in H$ *and equation* (2.2) *has the unique solution* $u_n \in H_n$. *The convergence* $u_{n(\delta)} \to u_*$ *for* $\delta \to 0$ *holds by the choice of* $n = n(\delta)$ *by the discrepancy principle:* $n(\delta)$ *is first index* $n \in N$ *satisfying*

$$\|Au_n - f^\delta\| \leq b\delta, \quad b = \text{const}, \quad b > \tau^{-1}. \tag{2.7}$$

In the following three theorems the previous two theorems are concretized for specific projection methods (2.2), which are characterized by different relations between subspaces H_n and F_n. We consider the least error method ($H_n = A^* F_n$), the least square method ($F_n = AH_n$) and in the case $F = H$, $A = A^* > 0$ also the Galerkin method ($F_n = H_n$). Proofs can be found in [7; 8; 25]. The name of the least error method can be explained as follows. Let $f^\delta = f$. Then element $u_n \in A^* F_n$ which minimizes $\|u_n - u_*\|$, satisfies condition $(u_n - u_*, A^* z_n) = 0$ $(\forall z_n \in F_n)$, which is the same condition as (2.1).

Theorem 2.3. *Let $\mathcal{N}(A) = 0$, $\mathcal{N}(A^*) = 0$ and $\|z - Q_n z\| \to 0$ for $n \to \infty$ $(\forall z \in F)$. Then the least error method determines for all $n \in N$ the unique approximation u_n. If $\delta = 0$, then $u_n \to u_*$ for $n \to \infty$. If $\delta > 0$ and $n = n(\delta)$ is chosen a priori by conditions* (2.5)*, then $u_{n(\delta)} \to u_*$ for $\delta \to 0$. The last convergence holds also in the case, if there exists $\alpha \in R$, $\alpha > 0$ such that*

$$\begin{aligned} \left(\kappa_n^*\right)^\alpha \left\|(I - Q_n)(AA^*)^{\alpha/2}\right\| &\le \gamma = \text{const} \quad (\forall n \in N)\,, \\ \left(\kappa_{n+1}^*\right)^\alpha \left\|(I - Q_n)(AA^*)^{\alpha/2}\right\| &\le \text{const} \qquad (\forall n \in N) \end{aligned} \tag{2.8}$$

and $n = n(\delta)$ is chosen by the discrepancy principle with $b > (1+\gamma^2)^{\alpha/2}$ (see (2.7)*).*

Theorem 2.4. *Let $\mathcal{N}(A) = 0$, (*(2.3)*) holds and there exists $\alpha \in R$, $\alpha > 0$ such that*

$$(\kappa_n + \kappa_{n+1})^\alpha \left\|(I - P_n)(A^*A)^{\alpha/2}\right\| \le \text{const} \quad (\forall n \in N)\,. \tag{2.9}$$

Then the least square method determines for all $n \in N$ the unique approximation u_n. If $\delta = 0$, then $u_n \to u_*$ for $n \to \infty$. If $\delta > 0$, then convergence $u_{n(\delta)} \to u_*$ as $\delta \to 0$ holds for the a priori choice of $n = n(\delta)$ by the rule

$$n(\delta) \to \infty\,, \qquad \delta\kappa_{n(\delta)} \to 0 \quad \text{for} \quad \delta \to 0 \tag{2.10}$$

and also by choice of $n = n(\delta)$ by the discrepancy principle with $b > 1$ (see (2.7)*).*

Theorem 2.5. *Let $H = F$, $A = A^* > 0$ and (*(2.3)*) holds. Suppose that there exists $\alpha \in R$, $\alpha > 0$ such that*

$$\kappa_n^\alpha \left\|(I - P_n)A^\alpha\right\| \le \gamma\,, \quad \kappa_{n+1}^\alpha \left\|(I - P_n)A^\alpha\right\| \le \text{const} \ (\forall n \in N)\,. \tag{2.11}$$

Then the Galerkin method determines for all $n \in N$ the approximate solution u_n. If $\delta = 0$, then $u_n \to u_*$ for $n \to \infty$. If $\delta > 0$, then convergence $u_{n(\delta)} \to u_*$ as $\delta \to 0$ holds for the a priori choice of $n = n(\delta)$ by rule (2.10) *and also by choice of $n = n(\delta)$ by the discrepancy principle with $b > 1 + \gamma$ (see* (2.7)*).*

Note that the self-regularization of ill-posed problems by projection methods was studied also in [1; 2; 5; 6; 7; 8; 13; 15; 17; 18; 20; 22; 23; 26; 28]. From considered methods particular interest is presented by the least error method, converging in the case of exact data by mild conditions. This convergence was stated already in [4]. In the case of noisy data for the least error method convergence $u_{n(\delta)} \to u_*$ as $\delta \to 0$ has been proved in [2; 5; 17; 20; 25] for the a priori choice of $n = n(\delta)$, and in [6; 15; 25] for choosing $n = n(\delta)$ by the discrepancy principle. Note that the a priori choice of $n = n(\delta)$ by condition (2.5) is problematic – we need to calculate or estimate κ_n^*. The choice of $n = n(\delta)$ by the discrepancy principle is more practical, but has a limited sphere of applicability (see the strong conditions (2.8)).

Note that condition (2.8) and analogous severe conditions (2.9), (2.11) are fulfilled by the solution of integral equations of the first kind with the kernels of the Green type function, if spline subspaces H_n or F_n are used (see [25]).

3. MONOTONE ERROR RULE IN LEAST ERROR METHOD

The choice of the regularization parameter is an actual problem in all regularization methods. Lately for the choice of regularization parameter $r = r(\delta)$ in some regularized approximation u_r the monotone error rule (ME-rule) was proposed. For resulting parameter r_{ME} convergence $u_{r_{\mathrm{ME}}} \to u_*$ as $\delta \to 0$ was shown and order optimal error estimates were given (see [9; 10; 11; 12; 24]). The ME-rule is applicable in algorithms, where in the case of exact data $f^\delta = f$ the monotone convergence $u_r \to u_*$ for $r \to \infty$ holds. The idea of the ME-rule is to choose in the case of noisy data for regularization parameter $r_{\mathrm{ME}} = r(\delta)$ the largest r-value, for which under information $\|f^\delta - f\| \le \delta$ we can prove that $\|u_r - u_*\|$ is monotonically decreasing for $r \in (0, r_{\mathrm{ME}}]$. For continuous regularization methods as the Tikhonov method, where u_r is differentiable with respect to r, this means that $\frac{d}{dr}\|u_r - u_*\|^2 \le 0$ for all $r \in (0, r_{\mathrm{ME}})$. For iteration methods where regularization parameter r is the stopping index $n \in N$, this means that

$$\|u_n - u_*\| < \|u_{n-1} - u_*\| \quad \text{for} \quad n = 1, 2, \ldots, n_{\mathrm{ME}}. \tag{3.1}$$

Let us consider now the problem of developing the ME-rule for projection methods. In these methods the regularization parameter $n \in N$ as in iteration methods and the ME-rule should give $n_{\mathrm{ME}} = n(\delta)$ satisfying condition (3.1).

The aim of this section is to develop the ME-rule for the least error method. We assume that the subspaces F_n fulfill the condition

$$F_n \subset F_{n+1} \quad (n = 0, 1, \ldots) \tag{3.2}$$

and we show that then the ME-rule is applicable in the following form.

ME-rule: choose $n_{ME} = n(\delta)$ in the least error approximation $u_n = A^* v_n$ ($v_n \in F_n$) as the first index $n = 1, 2, \ldots$, for which

$$d_{\mathrm{ME}}(n) := \frac{(v_{n+1} - v_n, f^\delta)}{2\|v_{n+1} - v_n\|} \le \delta\,. \tag{3.3}$$

Note that we get the element $v_n \in F_n$ in a computational procedure automatically without extra work. The function $d_{\mathrm{ME}}(n)$ can be represented also in the form

$$d_{\mathrm{ME}}(n) = \frac{\|u_{n+1}\|^2 - \|u_n\|^2}{2\|v_{n+1} - v_n\|}\,. \tag{3.4}$$

Namely, for the approximation $u_n = A^*v_n$ ($v_n \in F_n$) in the least error method we have

$$\|u_n\|^2 = \|A^*v_n\|^2 = (AA^*v_n - f^\delta + f^\delta, v_n) = (Au_n - f^\delta, v_n) + (f^\delta, v_n)\,.$$

So as $(Au_n - f^\delta, v_n) = 0$ due to conditions (2.1), the previous equality attains the form

$$\|u_n\|^2 = (f^\delta, v_n)\,. \tag{3.5}$$

From the last equality follows the equality of functionals $d_{\mathrm{ME}}(n)$ in (3.3) and (3.4).

In the following theorem some properties of approximations u_n and functional $d_{\mathrm{ME}}(n)$ are given.

Theorem 3.1. *Let subspaces F_n in the least error method satisfy condition* (3.2). *Then:*

1) $\|u_n\| \le \|u_{n+1}\|$ $(\forall\, n \in N)$,

2) $0 \le d_{\mathrm{ME}}(n) \le \|Au_n - f^\delta\|/2$ $(\forall\, n \in N)$,

3) if $f^\delta = f$, *then* $\|u_n - u_*\| \le \|u_{n-1} - u_*\|$ $(n = 1, 2, \ldots)$,

4) if $f^\delta \ne f$, *then* (3.1) *holds.*

Proof. From the equality

$$\mathcal{N}(Q_nA) = (\mathcal{R}((Q_nA)^*))^\perp = (\mathcal{R}(A^*Q_n))^\perp = (A^*F_n)^\perp \tag{3.6}$$

follows that approximation $u_n \in A^*F_n$ in the least error method is the minimum-norm solution of the equation $Q_n(Au - f^\delta) = 0$. Indeed, due to (3.6) all solutions of equation $Q_n(Au - f^\delta) = 0$ have form $u = u_n + u_n'$ with $u_n \in A^*F_n$ and $u_n' \in (A^*F_n)^\perp$, but $\|u_n + u_n'\|^2 = \|u_n\|^2 + \|u_n'\|^2 \ge \|u_n\|^2$. From (3.2) follows that u_{n+1} solves both equations $Q_{n+1}(Au - f^\delta) = 0$ and $Q_n(Au - f^\delta) = 0$. This fact with property $u_n = \arg\min\{\|u\| : u \in H,\ Q_n(Au - f^\delta) = 0\}$ gives $\|u_n\| \le \|u_{n+1}\|$, hence the assertion 1) is proved. The assertion 1) with equality (3.4) gives the left inequality $d_{\mathrm{ME}}(n) \ge 0$ in assertion 2). So as u_n and u_{n+1} both solve equation $Q_n(Au - f^\delta) = 0$, the equalities $(Au_{n+1} - f^\delta, v_n) = 0$, $(Au_n - f^\delta, v_n) = 0$ are true (see (2.1)). It yields equality $(A(u_{n+1} - u_n), v_n) = 0$, via relations

$$(Au_n, v_{n+1} - v_n) = (AA^*v_n, v_{n+1} - v_n) = (v_n, A(u_{n+1} - u_n))$$

also equality $(Au_n, v_{n+1} - v_n) = 0$. Hence the functional $d_{\mathrm{ME}}(n)$ in (3.3) can be written also in the form $d_{\mathrm{ME}}(n) = (Au_n - f^\delta, v_n - v_{n+1})/[2\|v_n - v_{n+1}\|]$ and has estimate $\|Au_n - f^\delta\|/2$ (the right inequality in assertion 2)).

To prove assertions 3), 4) we use equalities (3.5), $u_n = A^* v_n$, $Au_* = f$, inequality $\|f^\delta - f\| \le \delta$ and get

$$\begin{aligned}\|u_{n-1} - u_*\|^2 - \|u_n - u_*\|^2 &= \|u_{n-1}\|^2 - \|u_n\|^2 - 2(u_{n-1} - u_n, u_*) \\ &= (f^\delta, v_{n-1}) - (f^\delta, v_n) - 2(v_{n-1} - v_n, Au_*) \\ &= (v_n - v_{n-1}, f^\delta + 2(f - f^\delta)) \\ &\ge (v_n - v_{n-1}, f^\delta) - 2\|v_n - v_{n-1}\|\|f - f^\delta\| \\ &\ge 2\|v_n - v_{n-1}\|(d_{\mathrm{ME}}(n-1) - \delta)\,.\end{aligned}$$

Thus it holds the implication $d_{\mathrm{ME}}(n-1) > \delta \;\Rightarrow \|u_n - u_*\| < \|u_{n-1} - u_*\|$. It proves assertions 3), 4). Theorem 3.1 is proved. ■

From the estimate $d_{\mathrm{ME}}(n) \le \|Au_n - f^\delta\|/2$ follows that $n_{\mathrm{ME}} \le n_{D,2}$, where $n_{D,2}$ is parameter, get by the discrepancy principle (2.7) with $b = 2$. But it is worth to emphasize that according to Theorem 2.3 the discrepancy principle in the least error method requires using of constant b, which is large enough ($b > (1+\gamma^2)^{\alpha/2}$ with α and γ from conditions (2.8)). For example, the problem, solved in the numerical experiments of Section 5, requires $b > 1+2\sqrt{3}$ (see [7]).

It is an open problem whether $u_{n_{\mathrm{ME}}(\delta)} \to u_*$ for $\delta \to 0$.

4. ME-RULE IN SPECIAL COLLOCATION METHOD FOR INTEGRAL EQUATIONS OF THE FIRST KIND

In this section we consider the integral equation of the first kind

$$(Au)(t) \equiv \int_0^1 K(t,s)u(s)ds = f(t) \quad (0 \le t \le 1)$$

with operator $A : L_2(0,1) \to L_2(0,1)$ and $f \in C[0,1]$. We assume that for all discretization levels $n = 0, 1, \ldots$ some sets of $m = m(n)$ knots $\{t_i \in [0,1],\, i = 1, \ldots, m\}$ are given with $t_i \ne t_j$ for $i \ne j$. Consider an analogue of the least error method, giving approximation $u_n \in H_n = SPAN\{K(t_1,s), \ldots, K(t_m,s)\}$, which minimizes $\|u_n - u_*\|_{L_2(0,1)}$. Then $(u_n - u_*, K(t_i,s)) = 0$ $(i = 1, \ldots m)$ in case $f^\delta = f$. The exact solution u_* satisfies

$$(u_*, K(t_i,s)) = \int_0^1 K(t_i,s)u_*(s)ds = f(t_i) \quad (i = 1, \ldots, m)\,. \tag{4.1}$$

If instead of f noisy data f^δ are given, for determining u_n relations

$$(u_n, K(t_i,s)) = \int_0^1 K(t_i,s)u_n(s)ds = f^\delta(t_i) \quad (i = 1, \ldots, m) \tag{4.2}$$

can be used. Hence we get for determining the coefficients c_i^n in representation $u_n = \sum_{j=1}^{m(n)} c_j^n K(t_j, s)$ the system of linear algebraic equations

$$\sum_{j=1}^{m} \int_0^1 K(t_j, s) K(t_i, s) ds \cdot c_j^n = f^\delta(t_i) \quad (i = 1, \dots, m)\,. \tag{4.3}$$

We assume that system $\{K(t_i, s)\}_{i=1}^m$ is a linearly independent system in $L_2(0,1)$ for all $s \in [0,1]$. Then system (4.3) is uniquely solvable. This assumption is not very strong while otherwise for all $f \in \mathcal{R}(A)$ the set $\{f(t_1), \dots, f(t_m)\}$ would be linearly dependent (see (4.1)).

The special collocation method, discussed above, was considered in papers [1; 2; 18; 19; 26; 28]. We give for the case of exact data and $m = n+1$ the following convergence theorem from [26].

Theorem 4.1. *Let* $\int_0^1 |K(t,s)|^2 ds \le c = \text{const}$ $(0 \le t \le 1)$,

$$\int_0^1 |K(t', s) - K(t, s)|^2 ds \to 0 \quad \text{for} \quad t' \to t \quad (0 \le t,\, t' \le 1).$$

Then $\lim_{\Delta_n \to 0} \|u_n - u_*\| = 0$, *where* $\Delta_n = \sup_{t \in [0,1]} \inf_{1 \le i \le m} |t - t_i|$.

For case $f^\delta \ne f$ and $m = n+1$ in papers [1; 2] the following result about convergence by the a priori choice of $n = n(\delta)$ was proved.

Theorem 4.2. *Let* $\sum_{i=1}^{n} (f^\delta(t_i) - f(t_i))^2 \le \delta_n$. *Let* λ_n *be the least eigenvalue of the matrix* $Q(t_i, t_j) = \int_0^1 K(t_i, \tau) K(t_j, \tau) d\tau$ $(1 \le i,\, j \le n)$. *If*

$$\lim_{n\to\infty} \Delta_n = 0\,, \quad \lim_{n\to\infty} \delta_n = 0\,, \quad \lim_{n\to\infty} \delta_n^2 \lambda_n^{-1} = 0\,,$$

then $\|u_n - u_*\| \to 0$ *for* $n \to \infty$.

We do not know from literature any a posteriori rule for choice of $n = n(\delta)$ in this special collocation method. In the following our aim is to formulate the ME-rule. We assume that on discretization level n set of knots $\{t_i,\, i \in I_n\}$ is used, where index sets I_n satisfy

$$I_n \subset I_{n+1} \quad (n = 0, 1, \dots). \tag{4.4}$$

Let noise $\varepsilon(t_i) \equiv f^\delta(t_i) - f(t_i)$ in every knot t_i satisfy

$$\left|\varepsilon(t_i)\right| \le \delta_i \,. \tag{4.5}$$

From (4.1), (4.2) we get the equalities

$$(u_n, u_*) = \sum_{i\in I_n} c_i^n (K(t_i, s), u_*) = \sum_{i\in I_n} c_i^n f(t_i)\,,$$

$$\|u_n\|^2 = \sum_{i\in I_n} c_i (K(t_i, s), u_n) = \sum_{i\in I_n} c_i^n f^\delta(t_i)\,.$$

Using them and (4.4), (4.5), we have

$$\|u_{n-1} - u_*\|^2 - \|u_n - u_*\|^2 =$$

$$\begin{aligned}
&= \|u_n\|^2 - \|u_{n-1}\|^2 + 2(u_{n-1} - u_*, u_{n-1}) - 2(u_n - u_*, u_n)\\
&= \|u_n\|^2 - \|u_{n-1}\|^2 + 2 \sum_{i\in I_{n-1}} c_i^{n-1}\varepsilon_i - 2\sum_{i\in I_n} c_i^n \varepsilon_i\\
&= \|u_n\|^2 - \|u_{n-1}\|^2 - 2 \sum_{i\in I_n/I_{n-1}} c_i^n\varepsilon_i + 2\sum_{i\in I_{n-1}} (c_i^{n-1} - c_i^n)\varepsilon_i\\
&\ge \|u_n\|^2 - \|u_{n-1}\|^2 - 2\Big[\sum_{i\in I_n/I_{n-1}} |c_i^n|\delta_i + \sum_{i\in I_{n-1}} |c_i^{n-1} - c_i^n|\delta_i\Big]\,.
\end{aligned}$$

Hence condition (3.1) holds for the choice of n_{ME} by the following **ME rule:** choose $n_{\mathrm{ME}} = n(\delta)$ as the first index $n = 1, 2, \ldots$, for which

$$\|u_{n+1}\|^2 - \|u_n\|^2 \ge 2\Big[\sum_{i\in I_{n+1}/I_n} |c_i^{n+1}|\delta_i + \sum_{i\in I_n} |c_i^n - c_i^{n+1}|\delta_i\Big]\,.$$

Note that u_n and u_{n+1} are the minimum-norm solutions of equations $\int_0^1 K(t_i, s)u(s)ds = f(t_i)$ with $i \in I_n$ and $i \in I_{n+1}$ respectively (see [1; 2; 19]), hence due to (4.4) $\|u_{n+1}\| \ge \|u_n\|$ ($\forall\, n \in N$). It guarantees for exact data $f^\delta = f$ monotonicity of error: $\|u_n - u_*\| \le \|u_{n-1} - u_*\|$ ($n = 1, 2, \ldots$).

It is an open problem whether $u_{n_{\mathrm{ME}}} \to u_*$ by suitable assumptions.

5. NUMERICAL EXAMPLES

We consider a simple integral equation of the first kind

$$Au(t) \equiv \int_0^t u(s)ds = f(t) \tag{5.1}$$

in an L^2-space setting with $H = F = L_2(0,1)$. We assume that $f(0) = 0$ and $f' \in L_2(0,1)$. Then the equation (5.1) has the unique solution $u_* = f'$. The problem (5.1) was solved by the least error method with subspaces F_n consisting of piecewise constant functions on $k = k(n) = 2^n$ subintervals generated by $m = k+1$ uniform mesh points $t_i = (i-1)/(m-1)$, $i = 1, \dots, m$:

$$F_n = SPAN\{\Psi_i(t),\ i = 1, \dots, k(n)\}, \quad \Psi_i(t) = \begin{cases} 1, & t \in [t_i, t_{i+1}] \\ 0, & \text{otherwise}. \end{cases}$$

It is clear that these subspaces fulfill condition $F_n \subset F_{n+1}$ in (3.2) – by transition $n \to (n+1)$ every subinterval will be halved. The approximation u_n which we get by the least error method, is in the case of exact data the best approximation of u_* in subspace H_n, where

$$H_n = SPAN\{A^*\Psi_i(t),\ i = 1, \dots, k(n)\},$$

$$A^*\Psi_i(t) = \begin{cases} 1/k & \text{for} \quad t \in [0, t_i) \\ t_{i+1} - t & \text{for} \quad t \in [t_i, t_{i+1}) \\ 0 & \text{for} \quad t \in [t_{i+1}, 1]. \end{cases}$$

In computations instead of f noisy data f^δ with $\|f^\delta - f\| \le \delta$ were used. The parameter $n = n(\delta)$ which determines the number $k = 2^n$ of subintervals, was chosen by the monotone error rule (giving n_{ME}) and by the discrepancy principle (giving n_D) with $b = 1 + 2\sqrt{3}$: n_D is first index n satisfying $\|Au_n - f^\delta\| \le (1 + 2\sqrt{3})\delta$. Note that the discrepancy principle in the least error method for the problem (5.1) is theoretically justified only for $b > 1 + 2\sqrt{3}$ (see [7]).

We give the results of computation for two examples with different right-hand sides f and solution u_*.

Example 1: $u_*(t) = \frac{\pi}{2}\cos\left(\frac{\pi t}{2}\right)$, $f(t) = \sin\left(\frac{\pi t}{2}\right)$, $f^\delta(t) = f(t) + \delta$;

Example 2: $u_*(t) = \frac{7}{3}t^6 - \frac{6}{5}t^5 - \frac{32}{15}t^3 + 1$, $f(t) = \frac{1}{3}t^7 - \frac{1}{5}t^6 - \frac{8}{15}t^4 + t$, $f^\delta(t) = f(t) - \delta$.

We computed besides n_{ME} and n_{D} also n_{opt} as the last number for which inequality $\|u_n - u_*\| \le \|u_{n-1} - u_*\|$ was true. In all cases for $n_{\mathrm{opt}} + l$, $l = 1, 2, 3$ we had the opposite inequality $\|u_n - u_*\| > \|u_{n-1} - u_*\|$.

In the following Table 1 we give numbers of subintervals $k = k(n) = 2^n$, corresponding to the parameters n_{opt}, n_{ME} and n_{D}: $k_{\mathrm{opt}} = 2^{n_{\mathrm{opt}}}$, $k_{\mathrm{ME}} = 2^{n_{\mathrm{ME}}}$, $k_{\mathrm{D}} = 2^{n_{\mathrm{D}}}$. The errors $e_{k(n)} = \|u_n - u_*\|$, corresponding to k_{opt}, k_{ME} and k_{D} are also presented.

Note that due to (3.1) always holds $n_{\mathrm{ME}} \le n_{\mathrm{opt}}$. As Table 1 show, in both examples index n_{ME} was not very much smaller than the optimal index n_{opt}: for $\delta = 10^{-2}$ we had $n_{\mathrm{ME}} = n_{\mathrm{opt}} - 2$, for other δ we had $n_{\mathrm{ME}} = n_{\mathrm{opt}} - 1$. However, in practical problems n_{opt} is unknown. From the Table 1 we can also see that in both examples the ME-rule gave better results than the discrepancy principle, hence one can recommend to use the ME-rule for choice of n.

Table 1.

	δ	k_{opt}	k_{ME}	k_D	$e_{k_{opt}}$	$e_{k_{ME}}$	e_{k_D}
	10^{-1}	1	1	1	.219	.219	.219
Example 1	10^{-2}	4	1	1	.038	.219	.219
	10^{-3}	8	2	2	.0055	.0276	.0276
	10^{-4}	16	8	4	.0008	.0017	.0065
	10^{-5}	32	16	8	.0001	.0004	.0017

	δ	k_{opt}	k_{ME}	k_D	$e_{k_{opt}}$	$e_{k_{ME}}$	e_{k_D}
	10^{-1}	1	1	1	.202	.202	.202
Example 2	10^{-2}	4	1	1	.052	.202	.202
	10^{-3}	16	8	4	.0077	.0102	.0372
	10^{-4}	32	16	8	.0011	.0020	.0088
	10^{-5}	64	32	16	.0002	.0004	.0019

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Nekorektiškų uždavinių sprendimas projekciniais metodais su aposterioriniu diskretizacijos žingsnio parinkimu

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Darbe sprendžiamas nekorektiškas uždavinys $Au = f$ ir ieškomas normalusis sprendinys u_*. Vietoj f apibrėžiamas triukšmo paveiktas šaltinis f^δ, tenkinantis nelygybę $\|f^\delta - f\| \leq \delta$, čia δ yra žinomas trukšmo lygis. Analizuojami projekciniai metodai, leidžiantys rasti sprendinio u_* artinį u_n, apibrėžiamos sąlygos, garantuojančios, kad $u_n \to u_*(n \to \infty)$ monotoniškai, jei $f^\delta = f$. Jei $\delta > 0$, tai siūlomos dvi aposteriorinės taisyklės $n = n(\delta)$ parinkimui, leidžiančios įrodyti, kad projekcinio metodo sprendiniui dar galioja nelygybė $\|u_n - u_*\| \leq \|u_{n-1} - u_*\|$. Pateikti ir išanalizuoti skaitinio eksperimento rezultatai.