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INVESTMENT ANALYSES USING FUZZY PROBABILITY CONCEPT

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Abstract. In an uncertain economic decision environment, an expert's knowledge about discounting cash flows consists of a lot of vagueness instead of randomness. Cash amounts and interest rates are usually estimated by using educated guesses based on expected values or other statistical techniques to obtain them. Fuzzy numbers can capture the difficulties in estimating these parameters. Fuzziness is one aspect of uncertainty. It is the vagueness found in the definition of a concept or the meaning of a term. Fuzzy memberships represent similarities of objects to imprecisely denned properties, while probabilities convey information about relative frequencies. In this paper, two types of investment analyses are made. First, fuzzy parameters are used in the stochastic investment analysis. Then, another investment analysis is examined by using the concept of *probability of a fuzzy event*.

Keywords: fuzzy number, fuzzy event, fuzzy probability, investment.

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1. Introduction

To deal with vagueness of human thought, Zadeh (1965) first introduced the fuzzy set theory, which was based on the rationality of uncertainty due to imprecision or vagueness. A major contribution of fuzzy set theory is its capability of representing vague knowledge. The theory also allows mathematical operators and programming to apply to the fuzzy domain.

The fundamental difference between fuzziness and probability is that fuzziness deals with deterministic plausibility, while probability concerns the likelihood of nondeterministic, stochastic events. Fuzziness is one aspect of uncertainty. It is the ambiguity (vagueness) found

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in the definition of a concept or the meaning of a term such as comfortable temperature or well cooked. However, the uncertainty of probability generally relates to the occurrence of phenomena, as symbolized by the concept of randomness. In other words, a statement is probabilistic if it expresses some kind of likelihood or degree of certainty or if it is the outcome of clearly defined but randomly occurring events. For example, the statements "There is a 50–50 chance that he will be there", and "Roll the dice and get a six" demonstrate the uncertainty of randomness. Hence, fuzziness and randomness differ in nature; that is, they are different aspects of uncertainty. The former conveys "subjective" human thinking, feelings, or language and the latter indicates an "objective" statistic in the natural sciences. From the modeling point of view, fuzzy models and statistical models also possess philosophically different kinds of information: Fuzzy memberships represent similarities of objects to imprecisely denned properties, while probabilities convey information about relative frequencies. In the following, fuzzy numbers and fuzzy sets are briefly introduced.

A fuzzy number, $\tilde{A}$ is a normal and convex fuzzy set with membership function $\mu_{\tilde{A}}(x)$ which both satisfies normality: $\mu_{\tilde{A}}(x)=1$, for at least one $x \in R$ and convexity: $\mu_{\tilde{A}}(x') \geq \mu_{\tilde{A}}(x_1) \Lambda \mu_{\tilde{A}}(x_2)$, where $\mu_{\tilde{A}}(x) \in [0,1]$ and $\forall x' \in [x_1, x_2]$. Λ stands for the minimization operator. A fuzzy number can be represented in various forms: (a, b, c) type representation of fuzzy numbers means that "a" is the least possible value, "b" is the most possible value, and "c" is the largest possible value. Another representation type is $\left(a, f_1\left(y\middle|\tilde{A}\right)/b, b/f_2\left(y\middle|\tilde{A}\right), c\right)$ where $f_1\left(y\middle|\tilde{A}\right)/b$ represents the linear function from the point "a" to "b" and $f_2\left(y\middle|\tilde{A}\right)/c$ represents the linear function from the point "b" to "c".

Quite often in finance future cash amounts and interest rates are estimated. One usually employs educated guesses, based on expected values or other statistical techniques, to obtain future cash flows and interest rates. Statements like approximately between \$12,000 and \$16,000 or approximately between 10% and 15% must be translated into an exact amount, such as \$14,000 or 12.5% respectively. Appropriate fuzzy numbers can be used to capture the vagueness of those statements.

A tilde is placed above a symbol if the symbol represents a fuzzy set. The membership function for a fuzzy set $\tilde{P}$ will be denoted by $\mu\left(x\middle|\tilde{P}\right)$. A fuzzy number is a special fuzzy subset of the real numbers.

A trapezoidal fuzzy number (*TrFN*) is shown in Fig. 1. The membership function of a *TRFN*, $\tilde{V}$ is defined in Eq. (1) as follows:

$$\mu\left(x\middle|\tilde{V}\right) = \left(m_1, f_1\left(y\middle|\tilde{V}\right)/m_2, m_3/f_2\left(y\middle|\tilde{V}\right), m_4\right), \tag{1}$$

where $m_1 \prec m_2 \prec m_3 \prec m_4$, $f_1\left(y\middle|\tilde{V}\right)$ is a continuous monotone increasing function of y for $0 \leq y \leq 1$ with $f_1\left(0\middle|\tilde{V}\right) = m_1$ and $f_1\left(1\middle|\tilde{V}\right) = m_2$ and $f_2\left(y\middle|\tilde{V}\right)$ is a continuous monotone decreasing function of y for $0 \leq y \leq 1$ with $f_2\left(1\middle|\tilde{V}\right) = m_3$ and $f_2\left(0\middle|\tilde{V}\right) = m_4$. $\mu\left(y\middle|\tilde{V}\right)$ is denoted simply as $(m_1/m_2, m_3/m_4)$.

In case of $m_2 = m_3$, trapezoidal fuzzy number $\tilde{V}$ becomes a triangular fuzzy number (TFN) as shown in Fig. 2.

In the literature, many works have been published on fuzzy capital budgeting. These works generally use fuzzy numbers instead of crisp numbers in the known formulas. Ward (1985), Buckley (1987), Chiu and Park (1994), Wang and Liang (1995), Kahraman (2001),

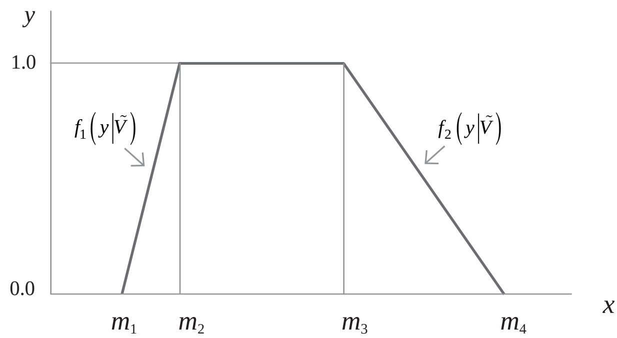

Fig. 1. A flat fuzzy number, $\tilde{V}$

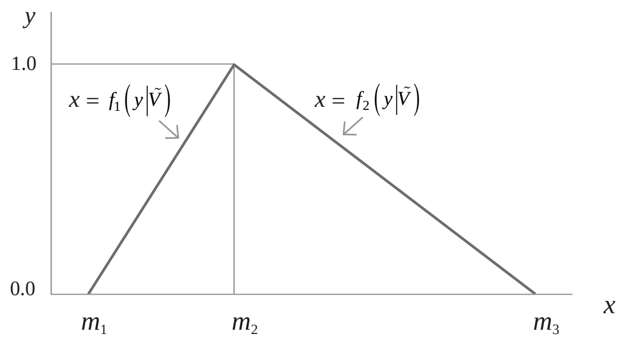

Fig. 2. A triangular fuzzy number, $\tilde{V}$

Kahraman *et al.* (1995; 2000; 2002), Kuchta (2000) are among the authors who deal with the fuzzy capital budgeting techniques. Kahraman and Kaya (2008a; 2008b) developed fuzzy equivalent annual worth and fuzzy benefit/cost analyses. This paper aims at calculating the fuzzy probabilities of events when the parameters of the capital budgeting problems are incomplete and vague.

The rest of the paper is organized as follows. Section 2 summarizes the stochastic investment analysis and this analysis is extended to the fuzzy case using fuzzy numbers. Finally, the possibilities of probabilities are obtained. The analysis in Section 3 is based on the concept "*fuzzy probability*". The fuzzy probabilities of a negative net present worth (NPW) and a positive NPW are formulated and a numerical example is presented. Section 4 gives the conclusions.

2. Probabilistic investment analysis

A typical investment may involve several factors such as the initial cost, expected life of the investment, the market share, the operating cost and so on. Values for factors are projected at the time the investment project is first proposed and are subject to deviations from their expected values. Such variations in the outcomes of future events, often termed risk, have been of primary concern to most decision makers in evaluating investment alternatives.

Typical parameters for which conditions of risk can reasonably be expected to exist include the initial investment, yearly operating and maintenance expenses, salvage values, the life of an investment, the planning horizon, and the minimum attractive rate of return. The parameters can be statistically independent, correlated with time and/or correlated with each other.

In order to determine analytically the probability distribution for the measure of effectiveness, a number of simplifying assumptions are normally made. The simplest situation is one involving a known number of random and statistically independent cash flows. As an example, suppose the random variable A_j denotes the net cash flow occurring at the end of period j, $j = 0, 1, ..., N$. Hence, the present worth (PW) is given in Eq. (2) as follows:

$$PW = \sum_{j=0}^{N} A_j (1+i)^{-j} . \quad (2)$$

Since the expected value, $E[.]$, of a sum of random variables equals the sum of the expected values of the random variables, then the expected present worth is given in Eq. (3) as follows:

$$E[PW] = \sum_{j=0}^{N} E[A_j](1+i)^{-j} . \quad (3)$$

Furthermore, since the A_j's are statistically independent, then the variance, $V(.)$, of present worth is given in Eq. (4) as follows:

$$V(PW) = \sum_{j=0}^{N} V(A_j)(1+i)^{-j} . \quad (4)$$

The central limit theorem, from probability theory, establishes that the sum of independently distributed random variables tends to be normally distributed as the number of terms in the summation increases. Hence, as N increases, PW tends to be normally distributed with a mean value of $E[PW]$ and a variance of $V(PW)$.

In recent years, several authors have studied the evaluation of the expected net present value and the variance of the net present value of probabilistic cash flows under random timings. Buck and Askin (1986) define partial means and related measures, and show their relationships to several less general risk measures. Schlaifer (1961) and Morris (1968) developed Gaussian linear loss integrals. These integrals are measures over a partial domain of X in contrast to full domain measures given by traditional statistical moments. Hillier (1963) introduced an analytical method which determines the probability distribution function of the net present value and the internal rate of return of a series of random discrete cash flows which occur at constant times. Wagle (1967) introduced a method similar to Hillier's. This method did not require the means and variances of cash flows to be supplied by the user. These two measures are computed from the data. Young and Contreras (1975) considered random lump-sum cash flows occurring at random times and uniform cash flows with random starting and cessation times. Spahr (1982) considers the uncertainty of future capital investment and reinvestment rates. Necessary formulae for the variance of the future worth are presented. He assumes a constant timing of cash flows. Giacotto (1984) considers serially correlated cash flows occurring at constant times. He presents necessary formulae for the mean and variance of the net present value. Park (1984) presents necessary formulae for the benefit/cost ratios when the cash flows are probabilistic. The formulae also assume constant timings of cash flows. Zinn *et al.* (1977) introduced formulae for the expected net present value, variance and semivariance of net present values for different cash flow profiles with random time. Tufekci and Young (1987) present the moments of the net present value of probabilistic investment alternatives. Benzion and Yagil (1987) compare two discounting methods for evaluating multi-period stochastic income streams that are identical and independent over time.

3. Investment analysis under fuzziness

In the following, two types of investment analyses under fuzziness are made. First, a fuzzy probabilistic investment analysis based on fuzzy parameters is made. In this analysis, the crisp parameters in stochastic investment analysis are fuzzy numbers. Later, an investment analysis based on probabilities of fuzzy events is made.

3.1. Probabilistic investment analysis based on fuzzy parameters

The Bayesian probability theorists argue that it is inherently meaningless to be uncertain about a probability since this appears to violate the subjectivists' assumption that individual can develop unique and precise probability judgments. However, many others have found the concept of uncertainty about the probability to be both intuitively appealing and potentially useful. This can be potentially relevant to a decision-making under very uncertain data and imprecise knowledge. In this case, the analysis should be able to capture and characterize uncertain and imprecise values. One of the natural ways of quantifying these uncertainties is to use interval values. It is possible to choose a best-estimate value from the interval values by applying reasonable engineering judgment (Yu and Park 2000).

Fuzzy probability is characterized by a possibility distribution of probability, which represents an imprecise probability by means of a subjective possibility measure associated with judgmental uncertainty. This can simultaneously model the probability and its degree of confidence in that probability expressed by an expert.

To use the concept, a consistent way of assigning values for the degree of possibility has to be developed. This can be done by introducing a membership function or possibility distribution. The possibility distributions are determined from an expert's experience and intuition, it can be generally considered a satisfactory approximation if a membership function or a possibility distribution has a simple and mathematically tractable shape. This study uses a triangular representation of the possibility distribution, which is the most popular membership function.

An illustrative example

A new cost reduction proposal is expected to have annual expenses of \$20,000 with a standard deviation of \$3,000, and it will likely save \$24,000 per year with a standard deviation of \$4,000. The proposed operation will be in effect for 3 years, and a rate of return of 20 percent before taxes is required. Determine the probability that implementation of the proposal will actually result in an overall loss and the probability that the PW of the net savings will exceed \$10,000.

The expected value of the present worth of savings and cost is

$E[PW] = (\$24{,}000 - \$20{,}000)(P/A, 20\%, 3) = 8{,}426.$

The variance is calculated from the relation as shown in Eq. (5):

$$\sigma^2_{\text{saving-costs}} = \sigma^2_{\text{saving}} + \sigma^2_{\text{costs}} \tag{5}$$

to obtain

$\text{Var}[PW] = (\$3{,}000)^2 (P/F, 20{,}2) + (\$3{,}000)^2 (P/F, 20{,}4) + (\$3{,}000)^2 (P/F, 20{,}6) + (\$4{,}000)^2 (P/F, 20{,}2) + (\$4{,}000)^2 (P/F, 20{,}2) + (\$4{,}000)^2 (P/F, 20{,}2) = 37{,}790{,}000$

from which

$$\sigma_{PW} = \sqrt{\text{Var}[PW]} = \$6{,}147.$$

Assuming that the PW is normally distributed, we find that

$$P(\text{loss}) = P\left(Z < \frac{0 - \$8{,}426}{\$6{,}147}\right) = P(Z < 1.37) = 0.0853$$

and

$$P(PW > \$10{,}000) = P\left(Z > \frac{\$10{,}000 - \$8{,}426}{\$6{,}147}\right) = P(Z > 0.256) = 0.40.$$

Under fuzziness, the fuzzy expected net present value in a triangular fuzzy number form is calculated as in the following way. The expected fuzzy annual expenses are around \$20,000 with a fuzzy standard deviation of around \$3,000, and it will possibly save around \$24,000 per year with a fuzzy standard deviation of around \$4,000. The proposed operation will be in effect for 3 years, and a rate of return of around 20 percent before taxes is required. Determine the possibility that implementation of the proposal will actually result in an overall loss and the possibility that the PW of the net savings will exceed around \$10,000. The fuzzy annual expenses are

$$E_e\left[\tilde{X}\right] = (\$19{,}000;\ \$20{,}000;\ \$24{,}000)$$

$$\sigma_e\left(\tilde{X}\right) = (\$2{,}500;\ \$3{,}000;\ \$3{,}500)$$

The fuzzy annual savings are

$$E_s\left[\tilde{X}\right] = (\$23{,}000;\ \$24{,}000;\ \$25{,}000)$$

$$\sigma_s\left(\tilde{X}\right) = (\$3{,}500;\ \$4{,}000;\ \$4{,}500)$$

The required fuzzy rate of return is $\tilde{i}_{annual} = (18\%,\ 20\%,\ 22\%)$.

The fuzzy variance of the cash flows is calculated by using Eq. (6):

$$\tilde{\sigma}_{s-e}^2 = \tilde{\sigma}_s^2 + \tilde{\sigma}_e^2 \tag{6}$$

and it is equal to $\tilde{\sigma}_{s-e}^2 = (18,500,000;\ 25,000,000;\ 32,500,000)$ with the left side representation $f_1\left[y|\tilde{\sigma}_{s-e}\right] = \sqrt{(18,500,000 + 6,500,000y)}$ and the right side representation $f_r\left[y|\tilde{\sigma}_{s-e}\right] = \sqrt{(32,500,000 + 7,500,000y)}$ where y shows the degree of membership.

The fuzzy present worth is calculated by using Eq. (7):

$$P\tilde{W} = \left[E_s\left[\tilde{X}\right] - E_e\left[\tilde{X}\right]\right]\left(P/A,\ \tilde{i}_{annual}, n\right). \tag{7}$$

The left side representation of the difference between savings and expenses is $f_l\left(y\middle|\tilde{X}_{s-e}\right) = (-1,000 + 5,000y)$ and the right side representation is $f_r\left(y\middle|\tilde{X}_{s-e}\right) = (6,000 - 2,000y)$ where y shows the degree of membership. Similarly, the left side representation of the fuzzy interest rate is $f_l\left(y\middle|\tilde{i}_{annual}\right) = (0.18 + 0.02y)$ and the right side representation is $f_r\left(y\middle|\tilde{i}_{annual}\right) = (0.22 - 0.02y)$. Using these representations we find the left side representation of the $P\tilde{W}$ as

$$f_l\left(y\middle|P\tilde{W}\right) = (-1,000 + 5,000y) \times \left[\frac{(1.22 - 0.02y)^3 - 1}{(1.22 - 0.02y)^3 (0.22 - 0.02y)}\right]$$

and the right side representation as

$$f_r\left(y\middle|P\tilde{W}\right) = (6,000 - 2,000y) \times \left[\frac{(1.18 + 0.02y)^3 - 1}{(1.18 + 0.02y)^3 (0.18 + 0.02y)}\right].$$

When these two functions are combined on x-y axes, we obtain the graph of fuzzy PW.

The possibility of loss can be calculated by using Eqs. (8–9).

$$P_1(\text{loss}) = P_1\left(\tilde{Z} \prec \frac{\tilde{0} - f_r\left(y\middle|P\tilde{W}\right)}{f_r\left(y|\tilde{\sigma}_{s-e}\right)}\right), \tag{8}$$

$$P_r(\text{loss}) = P_r\left(\tilde{Z} \prec \frac{\tilde{0} - f_l\left(y\middle|P\tilde{W}\right)}{f_l\left(y|\tilde{\sigma}_{s-e}\right)}\right), \tag{9}$$

where $f_r\left(y\middle|P\tilde{W}\right)$ represents the right side membership function of $P\tilde{W}$ depending on the membership degree, "y". And $f_r\left[y|\tilde{\sigma}_{s-e}\right]$ represents the right side membership function of $\tilde{\sigma}_{s-e}$ depending on the membership degree, "y".

The graph of these two functions is illustrated in Fig. 3.

To calculate the possibility that the PW of the net savings will exceed around \$10,000, the following equations can be used.

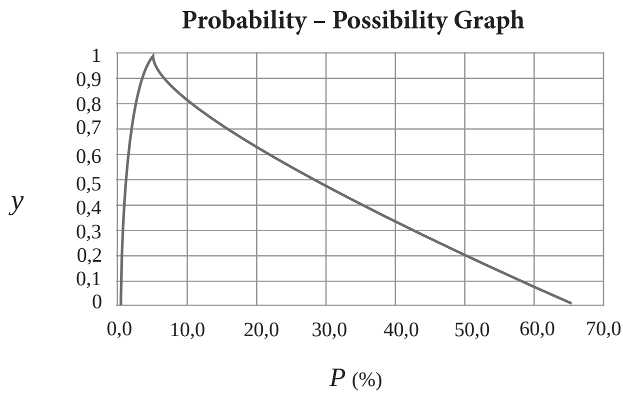

Fig. 3. The possibilities of probabilities P(Loss)

To calculate the possibility that the PW of the net savings will exceed around \$10,000, the following equations can be used.

$$P_1\left(P\tilde{W} > \$10,000\right) = P_1\left(\tilde{Z} \prec \frac{10,\tilde{0}00 - f_r\left(y \middle| P\tilde{W}\right)}{f_r\left(y \middle| \tilde{\sigma}_{s-e}\right)}\right)$$

and

$$P_r\left(P\tilde{W} > \$10,000\right) = P_r\left(\tilde{Z} \prec \frac{10,\tilde{0}00 - f_l\left(y \middle| P\tilde{W}\right)}{f_l\left(y \middle| \tilde{\sigma}_{s-e}\right)}\right),$$

where $\$10,\tilde{0}00$ is accepted as (\$9,000; \$10,000; \$11,000) with the left side representation (\$9,000 + \$1,000y) and the right side representation (\$11,000 – \$1,000y). The graph of these two functions is illustrated in Fig. 4.

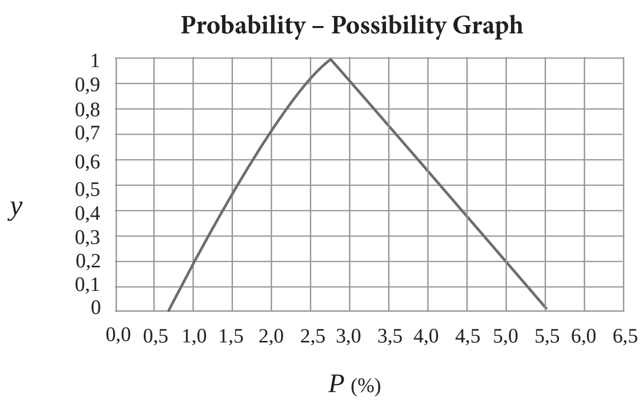

Fig. 4. The possibilities of probabilities P(Profit > 10,000)

3.2. Investment analysis using the probability of a fuzzy event

The formula for calculating the probability of a fuzzy event A is a generalization of the probability theory as shown in Eq. (10) (Zimmermann 1994):

$$P(A)=\begin{cases}\int \mu_A(x)\,f(x)\,dx, & \text{if } X \text{ is continuous,}\\ \sum_i \mu_A(x_i)\,f(x_i), & \text{if } X \text{ is discrete,}\end{cases} \tag{10}$$

where f_x denotes the probability distribution function of X.

3.2.1. The fuzzy probability of a negative NPW

The risk of a negative NPW is introduced using the concept of probability of a fuzzy event (Zadeh 1965). The risk of a negative NPW is defined as the probability of occurrence of the fuzzy event of a negative NPW. Mathematically this can be stated as shown in Eq. (11):

$$FR = P\ (\text{fuzzy event of a negative NPW}) = \tilde{P}(\text{a negative NPW}), \tag{11}$$

where FR and $\tilde{P}$ denote fuzzy risk and the probability of a fuzzy event respectively. Zadeh (1968) has defined the probability of a fuzzy event $\tilde{A}$ as in Eq. (12):

$$\tilde{P}(\tilde{A}) = \int_{R^n} \mu_{\tilde{A}}(x)\,dP, \tag{12}$$

where R^n is Euclidean n space, $\mu_{\tilde{A}}(x)$ is the membership function of the fuzzy event $\tilde{A}$ and P is a probability measure over R^n. A point in R^n is denoted by x. $\tilde{P}(\tilde{A})$ may be rewritten using $dP = f(x)\,dx$ as shown in Eq. (13):

$$\tilde{P}(\tilde{A}) = \int_{R^n} \mu_{\tilde{A}}(x)\,f(x)\,dx, \tag{13}$$

where $f(x)$ is the probability density function (PDF) of the random variable X. The fuzzy risk $r_{nn\tilde{p}w}$ of a negative NPW can then be defined as in Eq. (14):

$$r_{nn\tilde{p}w} = \int \mu_{nn\tilde{p}w}(x)\,f(x)\,dx, \tag{14}$$

where $\mu_{nn\tilde{p}w}(x)$ is the membership function of the fuzzy set $nn\tilde{p}w$ of a negative NPW and $f(x)$ is the PDF of the negativity level of NPW in the system. Based on the PDF $f(x)$ of the negative NPW and the membership function $\mu_{nn\tilde{p}w}(x)$ of the fuzzy set $nn\tilde{p}w$ of a negative NPW, direct or numerical integration may be performed to evaluate the fuzzy risk $r_{nn\tilde{p}w}$. $\mu_{nn\tilde{p}w}(x)$ in Eq. (14) may be defined as in Fig. 5 and Eq. (15):

$$\mu_{nn\tilde{p}w} = \begin{cases}1.0, & a \le x \le b,\\ \dfrac{x-c}{b-c}, & b \le x \le c.\end{cases} \tag{15}$$

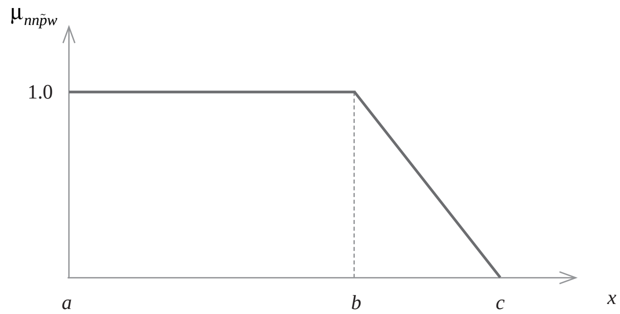

Fig. 5. Membership function of the fuzzy set of a negative NPW

Based on the central limit theorem and for the simplicity, the *PDF* of the negativity level of NPW in the system can be obtained as follows. The parameters in both functions are assumed to be same since this avoids us handling many parameters. Fig. 6 illustrates the *PDF*.

Eq. (16) is valid to define a *PDF*:

$$k_1 \int_a^b \frac{x-a}{b-a} dx + k_2 \int_b^c \frac{x-c}{b-c} dx = 1.0. \tag{16}$$

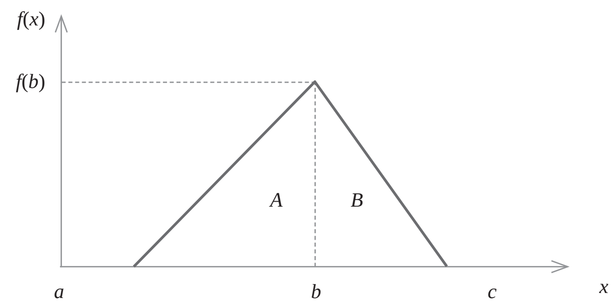

Fig. 6. Triangular probability distribution for a negative NPW

Then, the area (A) of the left side of the triangular probability distribution can be calculated by using Eq. (17):

$$k_1 \int_a^b \frac{x-a}{b-a} dx = A, \tag{17}$$

where $k_1 = 2A/(b-a)$ and for the area (B) of the right side can be calculated by using Eq. (18)

$$k_2 \int_b^c \frac{x-c}{b-c} dx = B, \tag{18}$$

where $k_2 = -2B/(b-c)$ and $A + B = 1.0$.

Then $f(x)$ can be given as shown in Eq. (19).

$$f(x) = \begin{cases} \dfrac{2A(x-a)}{(b-a)^2}, & a \le x \le b \\ \dfrac{2B(c-x)}{(b-c)^2}, & b \le x \le c \end{cases} \tag{19}$$

and finally $r_{nn\tilde{p}w}$ is obtained by using Eq. (20):

$$r_{nn\tilde{p}w} = \begin{cases} \int_a^b \dfrac{2A(x-a)}{(b-a)^2} dx, & a \le x \le b, \\ \int_b^c -\dfrac{2B(c-x)^2}{(b-c)^3} dx, & b \le x \le c. \end{cases} \tag{20}$$

3.2.2. The fuzzy probability of a positive NPW

In this case, the fuzzy probability $r_{nn\tilde{p}w}$ of a positive NPW can be defined by Eq. (21):

$$r_{nn\tilde{p}w} = \int \mu_{nn\tilde{p}w}(x) f(x) dx, \tag{21}$$

where $\mu_{nn\tilde{p}w}(x)$ is the membership function of the fuzzy set $nn\tilde{p}w$ of a positive NPW and $f(x)$ is the *PDF* of the positivity level of NPW in the system. Based on the *PDF* $f(x)$ of the positive NPW and the membership function $\mu_{nn\tilde{p}w}(x)$ of the fuzzy set $nn\tilde{p}w$ of a positive NPW, direct or numerical integration may be performed to evaluate the fuzzy probability $r_{nn\tilde{p}w}$. $\mu_{nn\tilde{p}w}(x)$ in Eq. (21) may be defined as in Fig. 7 and Eq. (22):

$$\mu_{nn\tilde{p}w} = \begin{cases} \dfrac{x-a}{b-a}, & a \le x \le b, \\ 1.0, & b \le x \le c. \end{cases} \tag{22}$$

Based on the central limit theorem and for the simplicity, the *PDF* of the positivity level of NPW in the system can be obtained as follows. Fig. 8 illustrates the *PDF* of a positive NPW.

The Eqs. (16–19) are still valid to define the *PDF*.

And finally $r_{nn\tilde{p}w}$ is obtained by using Eq. (23) as follows:

$$r_{nn\tilde{p}w} = \begin{cases} \int_a^b \dfrac{2A(x-a)^2}{(b-a)^3} dx, & a \le x \le b, \\ \int_b^c \dfrac{2B(c-x)}{(b-c)^2} dx, & b \le x \le c. \end{cases} \tag{23}$$

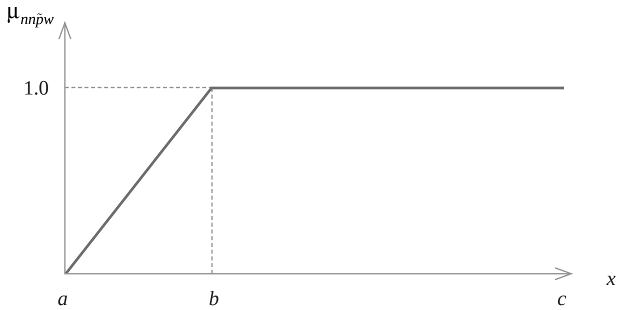

Fig. 7. Membership function of the fuzzy set of a positive NPW

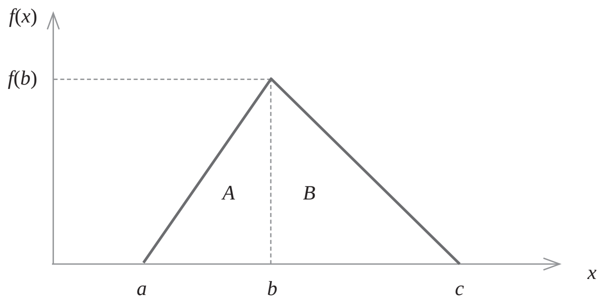

Fig. 8. Triangular probability distribution of a positive NPW

An illustrative example

The risk of obtaining a negative NPW of an investment is being investigated. The possibility of having a negative NPW is given as in Fig. 9. On the X axis, the negative NPW is given as an index dividing the real amount by $\$10^5$.

The triangular probability distribution of the random variable *negative* NPW is given as in Fig. 10. We want to calculate the fuzzy risk of having a negative NPW.

$$r_{nn\tilde{p}w} = \begin{cases} \int_{-1.25}^{-0.5} \frac{2A(x+1.25)}{0.5625} dx, & -1.25 \le x \le -0.5, \\ \int_{-0.5}^{0.0} \frac{2Bx^2}{0.125} dx, & -0.5 \le x \le 0.0, \end{cases}$$

where $A + B = 1.0$ and $B = 1/2.5 = 0.4$. The result is

$$r_{nn\tilde{p}w} = \begin{cases} 0.600, & -1.25 \le x \le -0.5, \\ 0.267, & -0.5 \le x \le 0.0. \end{cases}$$

This means that the fuzzy risk of having a negative NPW between –1.25 and –0.5 is 0.6 while it is 0.267 between –0.5 and 0.0.

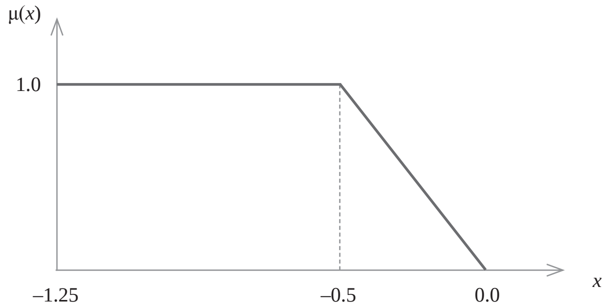

Fig. 9. Membership function of negative NPW

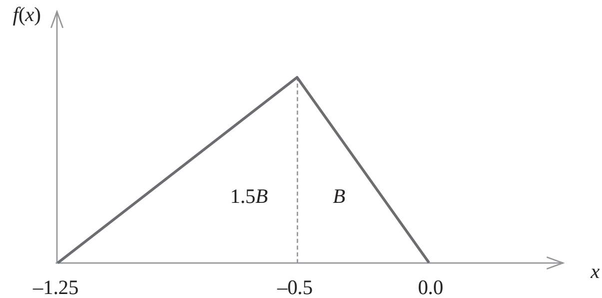

Fig. 10. Triangular probability distribution for the example

4. Conclusions

Fuzziness and randomness differ in nature; that is, they are different aspects of uncertainty. The former conveys "subjective" human thinking, feelings, or language and the latter indicates an "objective" statistic in the natural sciences. Fuzzy memberships represent similarities of objects to imprecisely denned properties, while probabilities convey information about relative frequencies.

To handle the uncertainty of net cash flows due to a lack of knowledge, this study uses the concept of fuzzy probability. Fuzzy probability is characterized by a possibility distribution of probability, which represents an imprecise probability by means of a subjective possibility measure associated with judgmental uncertainty. Two approaches in this study to obtain probabilities of fuzzy events have been applied. The first one is directly to use the fuzzy numbers in the crisp calculations. And the second one is to use Zadeh's (1968) probability of a fuzzy event.

For further research, L-R type fuzzy numbers instead of triangular fuzzy numbers used above may be an extension of this paper or type-2 fuzzy sets may be used instead of type-1 fuzzy sets used above.

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INVESTICIJŲ ANALIZĖ TAIKANT TIKIMYBINĘ NEAPIBRĖŽTŲJŲ AIBIŲ KONCEPCIJĄ

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Santrauka

Nepastovioje ekonominių sprendimų aplinkoje eksperto žinios apie diskontuotus pinigų srautus yra daugiau neapibrėžtos nei atsitiktinės. Pinigų srautų dydis ir palūkanų normos paprastai nustatomos remiantis prognozuojamomis tikėtinomis vertėmis ar pasitelkiant kitus statistinius metodus. Nustatant šiuos rodiklius, kylančius sunkumus gali padėti sumažinti neapibrėžtieji skaičiai. Tačiau neapibrėžtumas yra tik vienas iš nepastovumo aspektų. Sudėtinga nusakyti šio termino prasmę ir koncepciją. Neapibrėžtosios aibės išreiškia netiksliai įvardytas objektų savybes, o tikimybės perteikia informaciją apie santykinius įvykių dažnumus. Šiame straipsnyje atlikta dviejų tipų investicijų analizė. Pirmiausia, atliekant stochastinę investicijų analizę, panaudoti neapibrėžtųjų aibių elementai. Vėliau atlikta investicijų analizė, taikant tikimybinę neapibrėžtojo įvykio koncepciją.

Reikšminiai žodžiai: neapibrėžtasis skaičius, neapibrėžtasis įvykis, neapibrėžtoji tikimybė, investicijos.

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