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EXTENDED HESITANT FUZZY SETS

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Abstract. Hesitant fuzzy sets (HFSs) are a useful tool to manage situations in which the decision makers (DMs) hesitate about several possible values for the membership to assess a variable, alternative, etc. However, HFSs have the information loss problem and cannot identify different DMs, which interferes with the application of HFSs in decision making. To overcome these limitations, we develop the extended hesitant fuzzy sets (EHFSs) in this paper. As an extension of HFSs, EHFSs have close relationships with existing fuzzy sets including intuitionistic fuzzy sets (IFSs), fuzzy multisets (FMSs), type-2 fuzzy sets (T2FSs), dual hesitant fuzzy sets (DHFSs), and especially HFSs. We propose a concept of extended hesitant fuzzy elements (EHFEs), then study the basic operations and the desirable properties of EHFEs in detail. Some extended hesitant distance measures are developed to illustrate their advantages comparing with the existing hesitant distance measures. To extend EHFSs to decision making, we combine the proposed distance measures with the Dempster-Shafer belief structure.

Keywords: extended hesitant fuzzy sets (EHFSs), extended hesitant fuzzy elements (EHFEs), hesitant fuzzy sets (HFSs), hesitant fuzzy elements (HFEs), distance measure, decision making.

JEL Classification: D81, D7.

Introduction

Zadeh (1965) introduced fuzzy sets (FSs) as a powerful tool to address fuzziness, which have wide applications in practice (Baležentis *et al.* 2012; Stankevičienė, Mencaitė 2012). Then researchers developed some extensions of FSs, such as intuitionistic fuzzy sets (IFSs) (Atanassov 1986), type-2 fuzzy sets (T2FSs) (Zadeh 1975; Mizumoto, Tanaka 1976; Dubois 1980), fuzzy multisets (FMSs) (Yager 1986), interval-valued fuzzy sets (IVFSs) (Zadeh 1975), interval-valued intuitionistic fuzzy sets (IVIFSs) (Atanassov, Gargov 1989), hesitant fuzzy sets (HFSs) (Torra 2010) and dual hesitant fuzzy sets (DHFSs) (Zhu *et al.* 2012b).

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Among these sets, IFSs should be one of the most famous extensions of FSs due to the simultaneous consideration of the membership and non-membership. From a mathematical point of view, IFSs, modeled with two functions that define an interval, can also be considered as IVFSs. Actually, IFSs and IVFSs are equipollent generalizations of the FSs. Atanassov and Gargov (1989) then proposed IVIFSs as another generalization of IFSs (Xu *et al.* 2008; Wang *et al.* 2009; Xu 2010; Chen *et al.* 2011). Be distinct from FSs and IFSs, FMSs, also known as bags, permit multiple occurrences of an element, and have wide applications in information retrieval (Miyamoto 2003). But, the basic operations of FMSs are not applied to FSs or IFSs. Furthermore, FSs, IFSs and FMSs all can be considered as particular cases of T2FSs. T2FSs, described by membership functions that are characterized by more parameters, permit the fuzzy membership as a FS improving the modeling capability of FSs (Hagras 2004; Doctor *et al.* 2005). However, T2FSs have some difficulties in establishing the secondary membership functions, and difficulties in manipulation (Karnik, Mendel 2001; Greenfield *et al.* 2009; Rickard *et al.* 2009).

HFSs, originally introduced by Torra (2010), have close relationships with IFSs and FMSs, and can also be considered as a particular case of T2FSs. The motivation to propose HFSs is that when defining the membership of an element, the difficulty of establishing the membership is not a margin of error (as in IFSs), or some possibility distributions (as in T2FSs) on the possible values, but a set of possible values. Torra (2010) gave an "membership problem" to illustrate this situation: two decision makers (DMs) discuss the membership of x into A, and one wants to assign 0.5 and the other 0.6, which can be denoted by a hesitant fuzzy element (HFE), $h = \{0.5, 0.6\}$. In such a case, two values given by the two DMs for the membership can be collected into a single HFE.

Moreover, Zhu *et al.* (2012b) developed DHFSs as a new extension of HFSs. DHFSs are a comprehensive tool encompassing several existing fuzzy sets with certain conditions, whose membership and nonmembership are represented by a set of possible values respectively. For example, DHFSs permit the DMs consider as possible values for the membership a few different values as 0.1, 0.2 and 0.3, and for the nonmembership as 0.4 and 0.5. In particular cases, DHFSs can reduce to FSs, IFSs, HFSs or FMSs.

Comparing with FSs and IFSs, HFSs can be used to collect discrete data from the mathematical point of view. However, as the "membership problem" described above, if the two DMs both assign the value 0.5, we can only save one value by the HFE, and loss the other one, which appears to be an information loss problem of HFSs. Further, since generallly the DMs have different importance in group decision making (Wei *et al.* 2012; Wu *et al.* 2012) due to their different social importance, position in the group, previous merits etc., a leading DM in a group for example, the loss of information provided by the leading DM may lead to ineffective results. Therefore, we should extend HFSs to overcome these limitations, which is also the topic we should articulate in this paper.

To resolve the information loss problem, we can collect the possible values provided by the DMs by several possible value-groups. For example, continued with the "membership problem", if one DM assigns 0.5, another assigns 0.5 or 0.6, we collect one membership provided by each DM together resulting in two possible value-groups as (0.5, 0.5) and (0.5, 0.6). In such a case, all the memberships provided by the DMs are saved and distinguished clearly in the value-groups. Motivated by this idea, we develop a concept of extended hesitant fuzzy sets (EHFSs) in this paper, which considers possible value-groups for the membership of x into the set A. Furthermore, EHFSs have close relationships with IFSs, T2FSs, FMSs, DHFSs and especially HFSs. EHFSs are an extension of HFSs, on the contrary, HFSs can also be considered as a particular case of EHFSs. EHFSs increase the richness of numerical representation based on the value-groups, enhance the modeling abilities of HFSs, and can identify different DMs in decision making, which expand the applications of HFSs in practice.

On the other hand, the distance measure has received more and more attentions over the last decades and has wide applications in practice, such as pattern recognition (Li, Cheng 2002), cluster analysis (Yager 1988), approximate reasoning (Wang *et al.* 2004) and decision making (Wang, Xin 2005; Xu 2010). It is a significant research topic with respect to fuzzy theories. For example, some famous distance measures including Hamming distance, Euclidean distance and Hausdorff metric (Diamond, Kloeden 1994; Kacprzyk 1997; Chaudhuri, Rosenfeld 1999) have been extended to IFSs (Bustince, Burillo 1995; Szmidt, Kacprzyk 2000; Xu 2007b). For HFSs, Xu and Xia (2011) proposed some distance measures under the hesitant fuzzy environment, called hesitant distance measures, and gave some examples to show their applications in decision making.

As Xu and Xia (2011) explained, the hesitant distance measures can only be used in decision making with the conditions that all the DMs give their preferences anonymously or have the same importance so as to collect their preferences with no difference. This precondition interferes with the application of HFSs in practice, because the DMs usually have different importance in decision making. To deal with this problem, we develop some extended hesitant distance measures which take full advantages of EHFSs so as to show the advantages of HFESs comparing with HFSs. Furthermore, in order to take the DMs' knowledge (Yager *et al.* 1994) and risk preference (Liu, Wang 2007; Merigó, Casanovas 2009; Merigó, Gil-Lafuente 2009) into account, we combine the extended hesitant distance measures with Dempster-Shafer belief structure, and develop an approach to deal with group decision making problems. An energy policy example is also given to illustrate our results.

1. Preliminaries

Atanassov (1986) originally introduced the concept of intuitionistic fuzzy sets (IFSs) below.

Definition 1 (Atanassov, Gargov 1989). Let X be a fixed set, an IFS A on X is represented in terms of two functions μ: $X \rightarrow [0,1]$ and ν: $X \rightarrow [0,1]$, with the condition, $0 \leq \mu(x) + \nu(x) \leq 1$, $\forall\ x \in X$, where μ represents the membership and ν the nonmembership of x into the set A. IFSs are often represented as $< x, \mu_A, \nu_A >$, for all $x \in X$.

For convenience, Xu and Yager (2006) called (μ_A, ν_A) an intuitionistic fuzzy number (IFN).

For three IFNs α, α_1 and α_1, Xu (2007a) gave some operations on them, shown as follows:

1) $\alpha^c = (\nu_\alpha, \mu_\alpha)$;

2) $\alpha_1 \oplus \alpha_2 = (\mu_{\alpha_1} + \mu_{\alpha_2} - \mu_{\alpha_1}\mu_{\alpha_2}, \nu_{\alpha_1}\nu_{\alpha_2})$;

3) $\alpha_1 \otimes \alpha_2 = (\mu_{\alpha_1}\mu_{\alpha_2}, \nu_{\alpha_1} + \nu_{\alpha_2} - \nu_{\alpha_1}\nu_{\alpha_2})$;

4) $\lambda\alpha = (1-(1-\mu_\alpha)^\lambda, {\nu_\alpha}^\lambda)$, $\lambda > 0$;

5) $\alpha^\lambda = ({\mu_\alpha}^\lambda, 1-(1-\nu_\alpha)^\lambda)$, $\lambda > 0$.

Torra (2010) defined hesitant fuzzy sets (HFSs) as follows.

Definition 2 (Torra 2010). Let X be a fixed set, a HFS on X is in terms of a function that when applied to X returns a subset of [0,1], which can be represented as the following mathematical symbol:

$$E = \{< x, h(x) > | \, x \in X\}, \tag{1}$$

Where *h(x)* is a finite set of some values in [0,1], denoting the possible memberships of the element $x \in X$ to the set *E*.

For convenience, *h(x)* is called a hesitant fuzzy element (HFE).

Given three HFEs h, h_1 and h_2, and let h_i^- and h_i^+ be the minimum and maximum memberships in h_i $(i = 1,2)$ respectively, Torra (2010) defined some operations which can be represented as follows:

1) $h^c = \cup_{\gamma \in h}\{1-\gamma\}$;

2) $h_1 \cup h_2 = \{h \in (h_1 \cup h_2) \,|\, h \geq \max(h_1^-, h_2^-)\}$;

3) $h_1 \cap h_2 = \{h \in (h_1 \cap h_2) \,|\, h \leq \min(h_1^+, h_2^+)\}$.

Xia and Xu (2011) developed some new operations as below:

1) $h^\lambda = \cup_{\gamma \in h}\{\gamma^\lambda\}$, $\lambda > 0$;

2) $\lambda h = \cup_{\gamma \in h}\{1-(1-\gamma)^\lambda\}$, $\lambda > 0$;

3) $h_1 \oplus h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$;

4) $h_1 \otimes h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\gamma_1\gamma_2\}$.

Zhu *et al.* (2012a) further developed the following relationships for HFEs:

1) $\lambda(h_1 \oplus h_2) = \lambda h_1 \oplus \lambda h_2$;

2) $(h_1 \otimes h_2)^\lambda = h_1^\lambda \otimes h_2^\lambda$.

Xia and Xu (2011) gave a method to rank any two HFEs as follows.

Definition 3 (Xia, Xu 2011). For a HFE h, $s(h) = (1/\#h)\sum_{\gamma \in h}\gamma$ is called the score function of h, where $\#h$ is the number of the elements in h. Moreover, for two HFEs h_1 and h_2, if $s(h_1) > s(h_2)$, then $h_1 > h_2$; if $s(h_1) = s(h_2)$, then $h_1 = h_2$.

Torra (2010) gave a definition to the envelope of HFEs as follows.

Definition 4 (Torra 2010). Given a HFE h, an IFN $A_{env(h)}$ is defined as the envelope of h. This number, which will be denoted by $A_{env}(h)$, is represented by (μ, ν) with μ and ν defined as $\mu = h^-$ and $\nu = 1 - h^+$, where $h^+ = \max\{\gamma \,|\, \gamma \in h\}$ and $h^- = \min\{\gamma \,|\, \gamma \in h\}$.

Furthermore, some properties of $A_{env}(h)$ are shown as follows:

1) $A_{env}(h^c) = (A_{env}(h))^c$;

2) $A_{env}(h_1 \cup h_2) = A_{env}(h_1) \cup A_{env}(h_2)$;

3) $A_{env}(h_1 \cap h_2) = A_{env}(h_1) \cap A_{env}(h_2)$.

Zhu *et al.* (2012b) originally introduced the concept of dual hesitant fuzzy sets (DHFSs).

Definition 5 (Zhu *et al.* 2012b). Let X be a fixed set, then a DHFS D on X is described as:

$$D = \{< x, h(x), g(x) > | \, x \in X\}, \tag{2}$$

in which *h(x)* and *g(x)* are two sets of some values in [0,1], denoting the possible memberships and nonmemberships of the element $x \in X$ to the set *D* respectively, with the conditions $0 \le \gamma, \eta \le 1$, $0 \le \gamma^+ + \eta^+ \le 1$, where $\gamma \in h(x)$, $\eta \in g(x)$, $\gamma^+ = \bigcup_{\gamma \in h(x)} \max\{\gamma\}$, and $\eta^+ = \bigcup_{\eta \in g(x)} \max\{\eta\}$ for each $x \in X$.

For convenience, the pair $d(x) = (h(x), g(x))$ is called a dual hesitant fuzzy element (DHFE) denoted by $d = (h, g)$.

2. EHFSs and basic operations and properties

2.1. EHFSs

Given several HFSs, we use a Cartesian product of HFSs to construct an extended hesitant fuzzy set (EHFS) as follows.

Definition 6. Let *X* be a fixed set, $h_D(x) = \bigcup_{\gamma_D \in h_D(x)} \{\gamma_D\}$ $(D = 1, \ldots, m)$ be HFSs on *X*. Then, an EHFS, that is H_{h_D}, is defined as:

$$H_{h_D}(x) = h_1(x) \times \ldots \times h_m(x) = \bigcup_{\gamma_1 \in h_1(x), \ldots, \gamma_m \in h_m(x)} \left\{< x, (\gamma_1(x), \ldots, \gamma_m(x)) > | \, x \in X\right\}. \tag{3}$$

For convenience, we call:

$$H = h_1 \times \ldots \times h_m = \bigcup_{\gamma_1 \in h_1, \ldots, \gamma_m \in h_m} \{(\gamma_1, \ldots, \gamma_m)\}, \tag{4}$$

an extended hesitant fuzzy element (EHFE).

For $H = \bigcup_{\gamma_1 \in h_1, \ldots, \gamma_m \in h_m} \{(\gamma_1, \ldots, \gamma_m)\}$, let $u = (\gamma_1, \ldots, \gamma_m)$, then we call u a membership unit (MU). Based on u, an EHFE *H*, can also be indicated by:

$$H = \bigcup_{u \in H} \{u\} = \bigcup_{(\gamma_1, \ldots, \gamma_m) \in H} \{(\gamma_1, \ldots, \gamma_m)\}. \tag{5}$$

HFSs can be used to construct EHFSs. On the contrary, EHFSs can reduce to HFSs. To investigate the relationship between HFSs and EHFSs, we develop a concept of reduced EHFEs.

Definition 7. Given an EHFE $H = \bigcup_{(\gamma_1, \ldots, \gamma_m) \in H} \{(\gamma_1, \ldots, \gamma_m)\}$, then

$$h_H = \bigcup_{(\gamma_1, \ldots, \gamma_m) \in H} \{\gamma_1, \ldots, \gamma_m\} = \bigcup_{\gamma \in H} \{\gamma\} \tag{6}$$

is called a reduced EHFE.

Furthermore, if there is only one MU in H, i.e., $H = \{(\gamma_1, \ldots, \gamma_m)\}$ and satisfying $(\gamma_1 \ne, \ldots, \ne \gamma_m)$, we consider that the EHFE *H* is equivalent to a HFE, denoted by $h_H = \{\gamma_1, \ldots, \gamma_m\}$.

Proposition 1. The HFS *h(x)* is a particular case of the EHFS *H(x)*, where *H(x)* $= \{(\gamma_1(x), \ldots, \gamma_m(x))\}$ $(\gamma_1(x) \ne, \ldots, \ne \gamma_m(x))$, for each element in the domain.

Consider that IFSs are a particular case of HFSs, where HFSs are nonempty closed intervals (Torra 2010). By an operation of envelope for EHFSs defined in the rest of the paper, we can also transform EHFSs into closed intervals. Therefore, IFSs can also be considered as a particular case of EHFSs. We state this below.

Proposition 2. IFSs are a particular case of EHFSs, where EHFSs are nonempty closed intervals.

Moreover, as discussed by Zhu *et al.* (2012b), for a given DHFS $d(x) \neq \varnothing$, if $h(x)$ and $g(x)$ have only one value for each element in the domain, then DHFSs reduce to IFSs; if $g(x) = \varnothing$ and $h(x) \neq \varnothing$, then DHFSs reduce to HFSs. Thus, with the analyses above and according to Propositions 1 and 2, we conclude that DHFSs can also be considered as a particular case of EHFSs when DHFSs reduce to IFSs or HFSs.

2.2. Basic operations and properties

Definition 8. Let $H = \bigcup_{\gamma_1 \in h_1, \ldots, \gamma_m \in h_m} \{(\gamma_1, \ldots, \gamma_m)\}$ be an EHFE, then we define its complement as:

$$H^c = \bigcup_{(\gamma_1, \ldots, \gamma_m) \in H} \{(1-\gamma_1, \ldots, 1-\gamma_m)\}. \tag{7}$$

Since each u can be considered as a HFE, by the operation of HFEs, Eq.(7) can also be denoted as:

$$H^c = \bigcup_{u \in H} \{u^c\}. \tag{8}$$

Obviously, the complement of complement of an EHFE is itself, which can be concluded as below.

Proposition 3. The complement of an EHFE is involutive, denoted by $(H^c)^c = H$.

Given an EHFE $H = \bigcup_{u \in H} \{u\}$, we define the minimum and maximum memberships of H as follows:

1) The minimum membership of H: $\gamma_H^- = \min\{\gamma \mid \gamma \in h_H\}$;

2) The maximum membership of H: $\gamma_H^+ = \max\{\gamma \mid \gamma \in h_H\}$.

where h_H is the reduced EHFE of *H*.

Further, the minimum and maximum memberships of u can be defined as follows:

1) The minimum membership of *u*: $u^- = \min\{\gamma \mid \gamma \in u\}$;

2) The maximum membership of *u*: $u^+ = \max\{\gamma \mid \gamma \in u\}$.

For any two EHFEs, H_1 and H_2, we now define their union and intersection.

Definition 9. Given two EHFEs, H_1 and H_2, the union of them is defined as:

$$H_1 \cup H_2 = \bigcup_{u \in (H_1 \cup H_2)} \{u \mid u^- \geq \max(\gamma_{H_1}^-, \gamma_{H_2}^-)\}; \tag{9}$$

or equivalently:

$$H_1 \cup H_2 = \bigcup_{u_1 \in H_1, u_2 \in H_2} \{u_1, u_2 \mid u_1^-, u_2^- \geq \max(\gamma_{H_1}^-, \gamma_{H_2}^-)\}. \tag{10}$$

The intersection of them is defined as:

$$H_1 \cap H_2 = \bigcup_{u \in (H_1 \cap H_2)} \{u \mid u^+ \leq \min(\gamma_{H_1}^+, \gamma_{H_2}^+)\}; \tag{11}$$

or equivalently:

$$H_1 \cap H_2 = \bigcup_{u_1 \in H_1, u_2 \in H_2} \{u_1, u_2 \mid u_1^+, u_2^+ \leq \min(\gamma_{H_1}^+, \gamma_{H_2}^+)\}, \tag{12}$$

where $\gamma_{H_1}^+$ and $\gamma_{H_2}^+$ are the maximum memberships in H_1 and H_2 respectively.

Example 1. Let $H_1 = \{(0.2,0.3),(0.2,0.4)\}$ and $H_2 = \{(0.3,0.4)\}$ be two EHFEs, we have $\gamma_{H_1}^- = 0.2$, $\gamma_{H_2}^- = 0.3$, $\gamma_{H_1}^+ = 0.4$ and $\gamma_{H_2}^+ = 0.4$. By Definition 9, we can get:

$$H_1 \cup H_2 = \{(0.3,0.4)\},\ H_1 \cap H_2 = \{(0.2,0.3),(0.2,0.4),(0.3,0.4)\}.$$

The operations between EHFEs and HFEs have a close relationship.

Proposition 4. Assume two EHFEs, H_1 and H_2, and two reduced EHFEs of H_1 and H_2, h_{H_1} and h_{H_2}, the following are valid:

1) $h_{(H_1 \cup H_2)} = h_{H_1} \cup h_{H_2}$;

2) $h_{(H_1 \cap H_2)} = h_{H_1} \cap h_{H_2}$.

Proof.

1) For any two EHFEs, H_1 and H_2, by the operation of HFEs and Eq. (6), we can get:

$$h_{H_1} \cup h_{H_2} = \cup_{\gamma_1 \in h_{H_1}, \gamma_2 \in h_{H_2}} \{\gamma_1, \gamma_2 \mid \gamma_1, \gamma_2 \geq \max(\gamma_{H_1}^-, \gamma_{H_2}^-)\}. \tag{13}$$

By Eq. (6), it can be shown that:

$$h_{(H_1 \cup H_2)} = \cup_{\gamma \in (h_{H_1} \cup h_{H_2})} \{\gamma\}. \tag{14}$$

Since:

$$H_1 \cup H_2 = \cup_{u \in (H_1 \cup H_2)} \{u \mid u^- \geq \max(\gamma_{H_1}^-, \gamma_{H_2}^-)\}; \tag{15}$$

then:

$$\begin{aligned} & h_{(H_1 \cup H_2)} = \cup_{\gamma \in u} \{\gamma \mid u \in (H_1 \cup H_2), u^- \geq \max(\gamma_{H_1}^-, \gamma_{H_2}^-)\} = \\ & \cup_{\gamma_1 \in h_{H_1}, \gamma_2 \in h_{H_2}} \{\gamma_1, \gamma_2 \mid \gamma_1, \gamma_2 \in u, u \in (H_1 \cup H_2), u^- \geq \max(\gamma_{H_1}^-, \gamma_{H_2}^-)\} = \\ & \cup_{\gamma_1 \in h_{H_1}, \gamma_2 \in h_{H_1}} \{\gamma_1, \gamma_2 \mid \gamma_1, \gamma_2 \geq \max(\gamma_{H_1}^-, \gamma_{H_2}^-)\} = \\ & h_{H_1} \cup h_{H_2}, \end{aligned} \tag{16}$$

which completes the proof.

The proof of the intersection of EHFEs is similar to that of the proof of union above, which is not listed here.

HFEs and IFNs have a close relationship that HFEs are deemed IFNs when HFEs are nonempty closed intervals. Given an IFN, (μ, ν), we can get a corresponding HFE, *h*, denoted by an interval $h = [\mu, 1-\nu]$ if $\mu \neq 1-\nu$; given a HFE, *h*, the envelope of h is an IFN, i.e., $A_{env}(h) = (h^-, 1-h^+)$. The envelope of EHFEs also has close connections with HFEs and IFNs. We now give a definition of the envelope of EHFEs.

Definition 10. Given an EHFE $H = \cup_{u \in H} \{u\}$, the envelope of *H* can be defined as $A_{env}(H) = \cup_{\mu \in u^-, \nu \in u^+} \{(\mu, \nu) \mid u \in H\}$, where u^- and u^+ are the minimum and maximum memberships of u, respectively.

It's clear that the envelope of an EHFE may include several IFNs. In addition, in the particular case that an EHFE, *H*, is equivalent to a HFE, *h* (proposed in Proposition 1), the envelope of H is equivalent to the envelope of h, i.e., $A_{env}(H) = A_{env}(h)$. Thus, $A_{env}(h)$ is also a particular case of $A_{env}(H)$, which is stated as below.

Proposition 5. $A_{env}(h)$ is a particular case of $A_{env}(H)$, when H is equivalent to h.

We can further propose a proposition of $A_{env}(H)$ below.

Proposition 6. For an EHFEs H and its envelope $A_{env}(H)$, we have $A_{env}(H^c) = (A_{env}(H))^c$.

Proof. For an EHFE H, and its envelope $A_{env}(H)$, since:

$$A_{env}(H^c) = A_{env}(\cup_{u \in H}\{u\}) = \cup_{u^+,u^- \in u}\{(1-u^+, 1-1+u^-) \mid u \in H\} = \cup_{u^+,u^- \in u}\{(1-u^+, u^-) \mid u \in H\} \tag{17}$$

and

$$(A_{env}(H))^c = \cup_{u^+,u^- \in u}\{(u^-, 1-u^+)^c \mid u \in H\} = \cup_{u^+,u^- \in u}\{(1-u^+, u^-) \mid u \in H\}, \tag{18}$$

then $A_{env}(H^c) = (A_{env}(H))^c$, which completes the proof.

We now develop some operations of EHFEs further.

Definition 11. Given three EHFEs, $H = \cup_{u \in H}\{u\}$, $H_1 = \cup_{u_1 \in H_1}\{u_1\}$, $H_2 = \cup_{u_2 \in H_2}\{u_2\}$, $\lambda > 0$, since the MUs u, u_1 and u_2 can be considered as three HFEs, we have the following operations:

1) $H^\lambda = \cup_{u \in H}\{u^\lambda\}$;
2) $\lambda H = \cup_{u \in H}\{\lambda u\}$;
3) $H_1 \oplus H_2 = \cup_{u_1 \in H_1, u_2 \in H_2}\{u_1 \oplus u_2\}$;
4) $H_1 \otimes H_2 = \cup_{u_1 \in H_1, u_2 \in H_2}\{u_1 \otimes u_2\}$.

Based on Definition 11, we can prove the following proposition.

Proposition 7. For two EHFEs, H_1 and H_2, $\lambda > 0$, we have:

1) $H_1 \oplus H_2 = H_2 \oplus H_1$;
2) $H_1 \otimes H_2 = H_2 \otimes H_1$;
3) $\lambda(H_1 \oplus H_2) = \lambda H_1 \oplus \lambda H_2$;
4) $(H_1 \otimes H_2)^\lambda = H_1^\lambda \otimes H_2^\lambda$.

Proof. For any two EHFEs, H_1 and H_2, $\lambda > 0$, based on the operations and relationships of HFEs, we can get:

1) $H_1 \oplus H_2 = \cup_{u_1 \in H_1, u_2 \in H_2}\{u_1 \oplus u_2\} = \cup_{u_1 \in H_1, u_2 \in H_2}\{u_2 \oplus u_1\} = H_2 \oplus H_1$;

2) $H_1 \otimes H_2 = \cup_{u_1 \in H_1, u_2 \in H_2}\{u_1 \otimes u_2\} = \cup_{u_1 \in H_1, u_2 \in H_2}\{u_2 \otimes u_1\} = H_2 \otimes H_1$;

3) $\lambda(H_1 \oplus H_2) = \cup_{u_1 \in H_1, u_2 \in H_2}\{\lambda(u_1 \oplus u_2)\} = \cup_{u_1 \in H_1, u_2 \in H_2}\{\lambda u_1 \oplus \lambda u_2\} = \cup_{u_1 \in H_1}\{\lambda u_1\} \oplus \cup_{u_2 \in H_2}\{\lambda u_2\} = \lambda H_1 \oplus \lambda H_2$;

4) $\left(H_1 \otimes H_2\right)^\lambda = \cup_{u_1 \in H_1, u_2 \in H_2}\{(u_1 \otimes u_2)^\lambda\} = \cup_{u_1 \in H_1, u_2 \in H_2}\{u_1^\lambda \otimes u_2^\lambda\} = H_1^\lambda \otimes H_2^\lambda$.

Proposition 8. For any three EHFEs H, H_1 and H_2, and their reduced EHFEs h_H, h_{H_1} and h_{H_2}, $\lambda > 0$, the following are valid:

1) $h_{H^\lambda} = (h_H)^\lambda$;

2) $h_{\lambda H} = \lambda(h_H)$;

3) $h_{(H_1 \oplus H_2)} = h_{H_1} \oplus h_{H_2}$;

4) $h_{(H_1 \otimes H_2)} = h_{H_1} \otimes h_{H_2}$.

Proof. For any three EHFEs H, H_1 and H_2, and their reduced EHFEs h_H, h_{H_1} and h_{H_2}, $\lambda > 0$, we have:

1) $h_{H^\lambda} = \bigcup_{\gamma \in u}\{\gamma^\lambda \mid u \in H\} = \bigcup_{\gamma \in h_H}\{\gamma^\lambda\} = (h_H)^\lambda$;

2) $h_{\lambda H} = \bigcup_{\gamma \in u}\{1-(1-\gamma)^\lambda \mid u \in H\} = \bigcup_{\gamma \in h_H}\{1-(1-\gamma)^\lambda\} = \lambda(h_H)$;

3) $h_{(H_1 \oplus H_2)} = \bigcup_{\gamma_1 \in u_1, \gamma_2 \in u_2}\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2 \mid u_1 \in H_1, u_2 \in H_2\} =$

$\bigcup_{\gamma_1 \in h_{H_1}, \gamma_2 \in h_{H_2}}\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\} = h_{H_1} \oplus h_{H_2}$;

4) $h_{(H_1 \otimes H_2)} = \bigcup_{\gamma_1 \in u_1, \gamma_2 \in u_2}\{\gamma_1\gamma_2 \mid u_1 \in H_1, u_2 \in H_2\} = \bigcup_{\gamma_1 \in h_{H_1}, \gamma_2 \in h_{H_2}}\{\gamma_1\gamma_2\} = h_{H_1} \otimes h_{H_2}$.

Proposition 9. For any three EHFEs H, H_1 and H_2, $\lambda > 0$, we have:

1) $A_{env}(H^\lambda) = (A_{env}(H))^\lambda$;

2) $A_{env}(\lambda H) = \lambda(A_{env}(H))$;

3) $A_{env}(H_1 \oplus H_2) = A_{env}(H_1) \oplus A_{env}(H_2)$;

4) $A_{env}(H_1 \otimes H_2) = A_{env}(H_1) \otimes A_{env}(H_2)$.

Proof. For any three EHFEs H, H_1 and H_2, $\lambda > 0$, we have:

1) $A_{env}(H^\lambda) = A_{env}(\bigcup_{u \in H}\{u^\lambda\}) =$

$\bigcup_{u^-, u^+ \in u}\{((u^-)^\lambda, 1-(u^+)^\lambda) \mid u \in H\} =$

$\bigcup_{u^-, u^+ \in u}\{((u^-)^\lambda, 1-(1-(1-u^+))^\lambda) \mid u \in H\} =$

$\bigcup_{u^-, u^+ \in u}\{((u^-, 1-u^+)^\lambda) \mid u \in H\} = (A_{env}(H))^\lambda =$;

2) $A_{env}(\lambda H) = A_{env}(\bigcup_{u \in H}\{\lambda u\}) =$

$\bigcup_{u^-, u^+ \in u}\{(1-(1-u^-)^\lambda, 1-(1-(1-u^+)^\lambda)) \mid u \in H\} =$

$\bigcup_{u^-, u^+ \in u}\{(1-(1-u^-)^\lambda, (1-u^+)^\lambda)) \mid u \in H\} =$

$\lambda(\bigcup_{u^-, u^+ \in u}\{(u^-, 1-u^+) \mid u \in H\}) =$

$\lambda(A_{env}(H))$;

3) $A_{env}(H_1 \oplus H_2) = A_{env}(\bigcup_{u_1 \in H_1, u_2 \in H_2} \{u_1 \oplus u_2) =$

$\bigcup_{u_1^-, u_1^+ \in u_1, u_2^-, u_2^+ \in u_2} \left\{(u_1^- + u_2^- - u_1^- u_2^-, 1-(u_1^+ + u_2^+ - u_1^+ u_2^+)) \,|\, u_1 \in H_1, u_2 \in H_2\right\} =$

$\bigcup_{u_1^-, u_1^+ \in u_1, u_2^-, u_2^+ \in u_2} \left\{(u_1^- + u_2^- - u_1^- u_2^-, (1-u_1^+)(1-u_2^+)) \,|\, u_1 \in H_1, u_2 \in H_2\right\} =$

$(\bigcup_{u_1^-, u_1^+ \in u_1} \{(u_1^-, 1-u_1^+) \,|\, u_1 \in H_1\}) \oplus (\bigcup_{u_2^-, u_2^+ \in u_2} \{(u_2^-, 1-u_2^+) \,|\, u_2 \in H_2\}) =$

$A_{env}(H_1) \oplus A_{env}(H_2)$;

4) $A_{env}(H_1 \otimes H_2) = A_{env}(H_1 \otimes H_2) = A_{env}(\bigcup_{u_1 \in H_1, u_2 \in H_2} \{u_1 \otimes u_2 \,|\, u_1 \in H_1, u_2 \in H_2\}) =$

$\bigcup_{u_1^-, u_1^+ \in u_1, u_2^-, u_2^+ \in u_2} \{(u_1^- u_2^-, 1-u_1^+ u_2^+) \,|\, u_1 \in H_1, u_2 \in H_2\} =$

$\bigcup_{u_1^-, u_1^+ \in u_1, u_2^-, u_2^+ \in u_2} \left\{((u_1^- u_2^-, (1-u_1^+) + (1-u_2^+) - (1-u_1^+)(1-u_2^+))) \,|\, u_1 \in H_1, u_2 \in H_2\right\} =$

$\bigcup_{u_1^-, u_1^+ \in u_1} \{(u_1^-, 1-u_1^+) \,|\, u_1 \in H_1\} \oplus \bigcup_{u_2^-, u_2^+ \in u_2} \{(u_2^-, 1-u_2^+) \,|\, u_2 \in H_2\} =$

$A_{env}(H_1) \otimes A_{env}(H_2)$.

Proposition 10. For any three EHFEs H, H_1 and H_2, $\lambda > 0$, we have:

1) $H_1^c \cup H_2^c = (H_1 \cap H_2)^c$;
2) $H_1^c \cap H_2^c = (H_1 \cup H_2)^c$;
3) $\lambda(H^c) = (H^\lambda)^c$;
4) $(H^c)^\lambda = (\lambda H)^c$;
5) $H_1^c \oplus H_2^c = (H_1 \otimes H_2)^c$;
6) $H_1^c \otimes H_2^c = (H_1 \oplus H_2)^c$.

Proof. For any three EHFEs H, H_1 and H_2, $\lambda > 0$, we have:

1) $H_1^c \cup H_2^c = \bigcup_{u_1 \in H_1, u_2 \in H_2} \{u_1^c, u_2^c \,|\, (u_1^c)^-, (u_2^c)^- \geq \max((1-\gamma_{H_1}^+), (1-\gamma_{H_2}^+)\} =$

$\bigcup_{u_1 \in H_1, u_2 \in H_2} \{u_1^c, u_2^c \,|\, (u_1^c)^-, (u_2^c)^- \geq 1 - \min(\gamma_{H_1}^+, \gamma_{H_2}^+)\} =$

$(\bigcup_{u_1 \in H_1, u_2 \in H_2} \{u_1, u_2 \,|\, u_1^+, u_2^+ \leq \min(\gamma_{H_1}^+, \gamma_{H_2}^+)\})^c =$

$(H_1 \cap H_2)^c$;

2) $H_1^c \cap H_2^c = \bigcup_{u_1 \in H_1, u_2 \in H_2} \{u_1^c, u_2^c \,|\, (u_1^c)^+, (u_2^c)^+ \leq \min((1-\gamma_{H_1}^-), (1-\gamma_{H_2}^-))\} =$

$\bigcup_{u_1 \in H_1, u_2 \in H_2} \{u_1^c, u_2^c \,|\, (u_1^c)^+, (u_2^c)^+ \leq 1 - \max(\gamma_{H_1}^-, \gamma_{H_2}^-)\} =$

$(\bigcup_{u_1 \in H_1, u_2 \in H_2} \{u_1, u_2 \,|\, u_1^-, u_2^- \geq \max(\gamma_{H_1}^-, \gamma_{H_2}^-)\})^c =$

$(H_1 \cup H_2)^c$;

3) $\lambda(H^c) = \bigcup_{u \in H} \{\lambda(u^c)\} = \bigcup_{(\gamma_1, \ldots, \gamma_m) \in H} \left\{(1-(1-(1-\gamma_1))^\lambda, \ldots, 1-(1-(1-\gamma_m))^\lambda)\right\} =$

$\bigcup_{(\gamma_1, \ldots, \gamma_m) \in H} \{((1-\gamma_1)^\lambda, \ldots, (1-\gamma_m)^\lambda)\} = \bigcup_{u \in H} \{(u^\lambda)^c\} = (H^\lambda)^c$;

4) $(H^c)^\lambda = \bigcup_{u\in H}\{(u^c)^\lambda\} = \bigcup_{(\gamma_1,\ldots,\gamma_m)\in H}\{((1-\gamma_1)^\lambda,\ldots,(1-\gamma_m)^\lambda)\} =$

$\bigcup_{(\gamma_1,\ldots,\gamma_m)\in H}\left\{((1-(1-\gamma_1)^\lambda)^c,\ldots,(1-(1-\gamma_m)^\lambda)^c)\right\} =$

$\bigcup_{u\in H}\{(\lambda u)^c\} = (\lambda H)^c$;

5) Since $u_1^c \oplus u_2^c = \bigcup_{\gamma_1\in u_1,\gamma_2\in u_2}\{(1-\gamma_1)+(1-\gamma_2)-(1-\gamma_1)(1-\gamma_2)\} =$

$\bigcup_{\gamma_1\in u_1,\gamma_2\in u_2}\{1-\gamma_1\gamma_2\} = (u_1\otimes u_2)^c$,

we can get $H_1^c \oplus H_2^c = \bigcup_{u_1\in H_1,u_2\in H_2}\{u_1^c\oplus u_2^c\} = \bigcup_{u_1\in H_1,u_2\in H_2}\{(u_1\otimes u_2)^c\} = (H_1\otimes H_2)^c$;

6) Since $u_1^c \otimes u_2^c = \bigcup_{\gamma_1\in u_1,\gamma_2\in u_2}\{(1-\gamma_1)(1-\gamma_2)\} = \bigcup_{\gamma_1\in u_1,\gamma_2\in u_2}\{(1-\gamma_1-\gamma_2+\gamma_1\gamma_2)\} =$

$\bigcup_{\gamma_1\in u_1,\gamma_2\in u_2}\{1-(\gamma_1+\gamma_2-\gamma_1\gamma_2)\} = (u_1\oplus u_2)^c$,

then $H_1^c \otimes H_2^c = \bigcup_{u_1\in H_1,u_2\in H_2}\{u^c\otimes u^c\} = \bigcup_{u_1\in H_1,u_2\in H_2}\{(u_1\oplus u_2)^c\} = (H_1\oplus H_2)^c$.

For a given EHFS *H* on *X*, we have *H(x)* for all *x* in *X*. Then, we can define the EHFS as a fuzzy multiset (FMS) as:

$$FMS_H = \oplus_{x\in X}\oplus_{\gamma\in u}\{(x,\gamma)\,|\,u\in H(x)\}. \tag{19}$$

Thus, we can give the relationship between EHFSs and FMSs below.

Proposition 11. EHFSs can be represented as FMSs.

Similar to HFSs, the operations for FMSs also do not apply correctly to the EHFSs. Given an EHFS *H* on *X*, for all *x* in *X*, we can also define the EHFS as the following type-2 fuzzy set (T2FS):

$$\mu_{H(x)}(\gamma) = \begin{cases} 1, \gamma\in u, u\in H(x) \\ 0, \gamma\notin u, u\in H(x) \end{cases} \quad (x\in X).$$

Thus, we can derive the following result.

Proposition 12. EHFSs can be represented as T2FSs.

3. Extended hesitant distance measures

In the following, we put forward the axioms of extended hesitant distance measures.

Definition 12. Let H_1 and H_2 be two EHFEs, then the extended hesitant distance measure is denoted by $d(H_1,H_2)$, which satisfies the following properties:

1) $0\le d(H_1,H_2)\le 1$;

2) $d(H_1,H_2)=0$ if and only if $H_1 = H_2$;

3) $d(H_1,H_2)=d(H_2,H_1)$.

For any finite universe set $X=\{x_1,\ldots,x_n\}$, Bustince and Burillo (1995) defined some distance measures between two IFSs $A_1(x)$ and $A_2(x)$ on X. For two IFNs, A_1 and A_2, we have:

1) The normalized Hamming distance

$$d_1(A_1,A_2) = \frac{1}{2}(|\mu_{A_1}-\mu_{A_2}|+|\nu_{A_1}-\nu_{A_2}|);$$

2) The normalized Euclidean distance

$$d_2(A_1,A_2)=\sqrt{\frac{1}{2}((\mu_{A_1}-\mu_{A_2})^2+(\nu_{A_1}-\nu_{A_2})^2)}\ .$$

With respect to HFSs, Xu and Xia (2011) further defined the hesitant distance measures. For two HFEs h_1 and h_2, the hesitant distance measures can be stated as follows:

1) The hesitant normalized Hamming distance

$$d_1(h_1,h_2)=\frac{1}{\#h}\left(S_s\left(\bigcup\nolimits_{\gamma_1^{\sigma(i)}\in h_1,\gamma_2^{\sigma(i)}\in h_2}\{|\gamma_1^{\sigma(i)}-\gamma_2^{\sigma(i)}|\}\right)\right);$$

2) The hesitant normalized Euclidean distance

$$d_2(h_1,h_2)=\left(\frac{1}{\#h}\left(S_s\left(\bigcup\nolimits_{\gamma_1^{\sigma(i)}\in h_1,\gamma_2^{\sigma(i)}\in h_2}\{(\gamma_1^{\sigma(i)}-\gamma_2^{\sigma(i)})^2\}\right)\right)\right)^{1/2},$$

where $\#h=\max(\#h_1,\#h_2)$, S_s is a function that indicates a summation of all values in a set, $\gamma_1^{\sigma(i)}$ and $\gamma_2^{\sigma(i)}$ are the *i*th largest values in h_1 and h_2 respectively.

To use the hesitant distance measures above, we need to make sure that there is the same number of memberships between two HFEs. Zhu and Xu (2013) developed an optimized parameter ς $(0\le\varsigma\le1)$ to add linguistic terms in hesitant fuzzy linguistic term sets. Motivated by the optimized parameter, we give the following definition.

Definition 13. For a MU, $u=(\gamma_1,\ldots,\gamma_m)$, let $u^-=\min\{\gamma\,|\,\gamma\in u\}$ and $u^+=\max\{\gamma\,|\,\gamma\in u\}$ be the minimum and maximum memberships in u respectively, and ς $(0\le\varsigma\le1)$ be the optimized parameter, then we call $\bar{\gamma}=\varsigma u^+ +\left(1-\varsigma\right)u^-$ an added membership.

For two EHFEs with different number of MUs, we further utilize the optimized parameter to obtain a MU.

Definition 14. Given an EHFE, $H_{h_D}=\bigcup_{\gamma_1\in h_1,\ldots,\gamma_m\in h_m}\{(\gamma_1,\ldots,\gamma_m)\}$ $(D=1,\cdots,m)$, let h_D^- and h_D^+ be the minimum and maximum memberships in h_D respectively, and ς $(0\le\varsigma\le1)$ be the optimized parameter, then an added MU is defined as $\bar{u}=(\bar{\gamma}_1,\ldots,\bar{\gamma}_m)$, where $\bar{\gamma}_D=\varsigma(h_D^+)+\ (1-\varsigma)(h_D^-)$ $(D=1,\cdots,m)$.

To compare two MUs, the comparison law can be stated below.

Definition 15. For a MU, $u=(\gamma_1,\ldots,\gamma_m)$, then we call $s(u)=(1/\#u)\sum_{\gamma\in u}\gamma$ the score function of *u*, where $\#u$ is the number of memberships in *u*. For any two MUs, u_1 and u_2, if $s(u_1)>s(u_2)$, then $u_1\succ u_2$; if $s(u_1)=s(u_2)$, then $u_1\sim u_2$, where "$\succ$" denotes "be superior to", and "$\sim$" means "be indifferent to".

Furthermore, we can also consider the deviation degree to compare MUs. A small deviation degree of all elements with respect to the average value in a MU reflects how these elements agree with each other, that is, they have a higher consistency.

Definition 16. For a MU, $u=(\gamma_1,\ldots,\gamma_m)$, let *s(u)* be the score function of *u*, then we call $\rho(u)=\left[(1/\#u)\sum_{\gamma\in u}(\gamma-s(u))^2\right]^{\frac{1}{2}}$ the deviation function of HFSs, where $\#u$ is the number of memberships in *u*.

The deviation function reflects the deviation degree between all possible memberships in a MU and their average value. Based on the score function and the deviation function, we develop the following comparison laws.

Definition 17. Let u_1 and u_2 be two MUs, $s(u_1)$ and $s(u_2)$ the scores of u_1 and u_2 respectively, and $\rho(u_1)$ and $\rho(u_2)$ the deviation degrees of u₁ and u₂ respectively, then:

1) if $s(u_1) < s(u_2)$, then $u_1 \prec u_2$;
2) if $s(u_1) = s(u_2)$, then
   (1) if $\rho(u_1) = \rho(u_2)$, then $u_1 \sim u_2$;
   (2) if $\rho(u_1) < \rho(u_2)$, then $u_1 \succ u_2$;
   (3) if $\rho(u_1) > \rho(u_2)$, then $u_1 \prec u_2$.

For any two EHFEs, $H_1 = \bigcup_{u_1 \in H_1} \{u_1\}$ and $H_2 = \bigcup_{u_2 \in H_2} \{u_2\}$, and ς $(0 \le \varsigma \le 1)$, we make them have the same number of MUs and the same number of memberships in each MU by Definitions 13 and 14, respectively. According to Definitions 15–17 to rank MUs, and combining the normalized Hamming distance and the normalized Euclidean distance, we define the following distance measures:

1) Extended hesitant Normalized Hamming distance

$$d_h(H_1,H_2) = \frac{1}{(\#H)(\#u)} S_s\left(\sum_{i=1}^{\#H}\sum_{j=1}^{\#u}\left(\bigcup_{(\gamma_1)^{\sigma(j)} \in u_1^{\sigma(i)},(\gamma_2)^{\sigma(j)} \in u_2^{\sigma(i)}} \{|(\gamma_1)^{\sigma(j)} - (\gamma_2)^{\sigma(j)}|\} \,\Big|\, u_1^{\sigma(i)} \in H_1, u_2^{\sigma(i)} \in H_2\}\right)\right); \quad (20)$$

2) Extended hesitant Normalized Euclidean distance

$$d_e(H_1,H_2) = \left(\frac{1}{(\#H)(\#u)} S_s\left(\sum_{i=1}^{\#H}\sum_{j=1}^{\#u}\left(\bigcup_{(\gamma_1)^{\sigma(j)} \in u_1^{\sigma(i)},(\gamma_2)^{\sigma(j)} \in u_2^{\sigma(i)}} \{((\gamma_1)^{\sigma(j)} - (\gamma_2)^{\sigma(j)})^2 \,\Big|\, u_1^{\sigma(i)} \in H_1, u_2^{\sigma(i)} \in H_2\}\right)\right)\right)^{\frac{1}{2}}; \quad (21)$$

3) Extended hesitant Normalized generalized distance

$$d_g(H_1,H_2) = \left(\frac{1}{(\#H)(\#u)} S_s\left(\sum_{i=1}^{\#H}\sum_{j=1}^{\#u}\left(\bigcup_{(\gamma_1)^{\sigma(j)} \in u_1^{\sigma(i)},(\gamma_2)^{\sigma(j)} \in u_2^{\sigma(i)}} \{((\gamma_1)^{\sigma(j)} - (\gamma_2)^{\sigma(j)})^\lambda \,\Big|\, u_1^{\sigma(i)} \in H_1, u_2^{\sigma(i)} \in H_2\}\right)\right)\right)^{\frac{1}{\lambda}} (\lambda > 0), \quad (22)$$

where $\#H = \#H_1 = \#H_2$ ($\#H_1$ and $\#H_2$ are the number of MUs in H_1 and H_2 respectively), $\#u = \#u_1 = \#u_2$, S_s is a function that indicates a summation of all values in a set, $(\gamma_1)^{\sigma(j)}$ and $(\gamma_2)^{\sigma(j)}$ are the jth largest memberships in u_1 and u_2 respectively, $u_1^{\sigma(i)}$ and $u_2^{\sigma(i)}$ are the ith largest MUs in H_1 and H_2 respectively.

It's clear that the extended hesitant generalized normalized distance can reduce to the extended hesitant normalized Hamming distance and the extended hesitant normalized Euclidean distance when $\lambda = 1$ and $\lambda = 2$ respectively.

Example 2. Let $H_1 = \{(0.2,0.3),(0.2,0.4),(0.3,0.3),(0.3,0.4)\}$ and $H_2 = \{(0.3,0.4),(0.1,0.6)\}$ be two EHFEs, and $\varsigma = 0.8$. Since $\#H_1 (= 4) > \#H_2 (= 2)$, then we should add two MUs to $\#H_2$. According to Definition 14, we can get the added MU, $\bar{u} = (\bar{\gamma}_1, \bar{\gamma}_2) = (0.26,0.56)$, where

$$\bar{\gamma}_1 = 0.8 \times 0.3 + 0.2 \times 0.1 = 0.26, \quad \bar{\gamma}_2 = 0.8 \times 0.6 + 0.2 \times 0.4 = 0.56.$$

Thus, an adjusted H_2 with the added MUs is:

$$H_2' = \bigcup_{u_2 \in H_2} \{u_2\} = \{(0.3,0.4),(0.1,0.6),(0.26,0.56),(0.26,0.56)\}.$$

By Eqs (20) and (21), we have:

$$d_h\left(H_1,H_2\right)=\frac{1}{2\times 4}S_s\left(\sum_{i=1}^{4}\sum_{j=1}^{2}\left(\begin{array}{l}\bigcup_{(\gamma_1)^{\sigma(j)}\in u_1^{\sigma(i)},(\gamma_2')^{\sigma(j)}\in u_2'^{\sigma(i)}}\{\left|(\gamma_1)^{\sigma(j)}-(\gamma_2')^{\sigma(j)}\right|\\ \left|u_1^{\sigma(i)}\in H_1,u_2'^{\sigma(i)}\in H_2'\right.\}\end{array}\right)\right)=0.1250,$$

$$d_e\left(H_1,H_2\right)=\left(\frac{1}{2\times 4}S_s\left(\sum_{i=1}^{4}\sum_{j=1}^{2}\left(\begin{array}{l}\bigcup_{(\gamma_1)^{\sigma(j)}\in u_1^{\sigma(i)},(\gamma_2')^{\sigma(j)}\in u_2'^{\sigma(i)}}\{((\gamma_1)^{\sigma(j)}-(\gamma_2')^{\sigma(j)})^2\\ \left|u_1^{\sigma(i)}\in H_1,u_2'^{\sigma(i)}\in H_2'\right.\}\end{array}\right)\right)\right)^{\frac{1}{2}}=0.1442.$$

In Example 2, the distances vary with two parameters, the optimized parameter ς, and the parameter λ. More details can be found in Figure 1.

From Figure 1, we can partly conclude that the distances between H_1 and H_2 increase with the increase of ς. However, as λ increases, the distances increase approaching the path of the sine function.

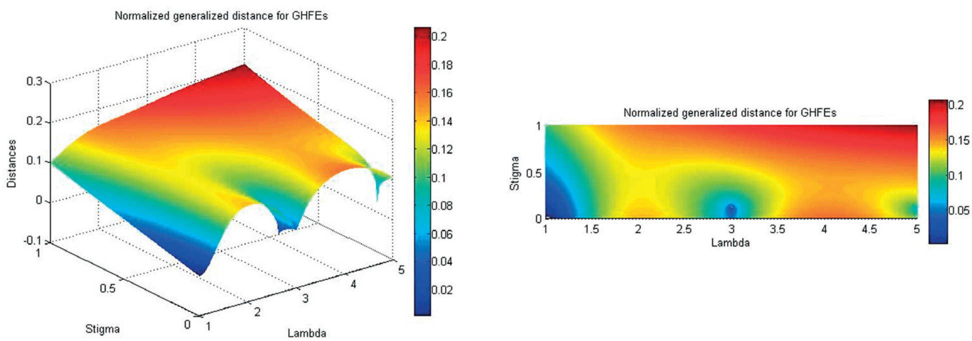

Fig. 1. Extended hesitant Normalized generalized distance $(\lambda\in[1,5],\varsigma\in[0,1])$

4. Weighted extended hesitant distance measures

The hesitant distance measures proposed by Xu and Xia (2011) cannot consider the weights of the DMs. To save all the information provided by the DMs, distinct them from each other, and consider their different importance in decision making, we now propose the weighted extended hesitant distance measures in this section.

Assume a group decision making problem with m DMs. For two EHFEs, $H_k=\bigcup_{u_k\in H_k}\{u_k\}=\bigcup_{(\gamma_1^{(k)},\ldots,\gamma_m^{(k)})\in H_k}\{(\gamma_1^{(k)},\ldots,\gamma_m^{(k)})\}$ $(k=1,2)$, the weights of DMs are ω_D $(D=1,\ldots,m)$ with $\omega_D\in[0,1]$ and $\sum_{D=1}^{m}\omega_D=1$. Let $\gamma_{\omega_D}^{(k)}=\omega_D\gamma_D^{(k)}$ $(D=1,\ldots,m)$ be memberships associated with the DMs' weights. According to the extended hesitant generalized distance, we now develop a weighted extended hesitant generalized distance as:

$$d_{wg}\left(H_1,H_2\right)=\left(\frac{1}{\#H}S_s\left(\sum_{i=1}^{\#H}\sum_{j=1}^{\#u}\left(\begin{array}{l}\bigcup_{\gamma^{(1)}\in u_1^{\sigma(i)},\gamma^{(2)}\in u_2^{\sigma(i)}}\{((\gamma_{\omega_D}^{(1)})^{\sigma(j)}-(\gamma_{\omega_D}^{(2)})^{\sigma(j)})^{\lambda}\\ \left|u_1^{\sigma(i)}\in H_1,u_2^{\sigma(i)}\in H_2\right.\}\end{array}\right)\right)\right)^{\frac{1}{\lambda}}\quad(\lambda>0),\tag{23}$$

where $(\gamma_{\omega_D}^{(1)})^{\sigma(j)}$ and $(\gamma_{\omega_D}^{(2)})^{\sigma(j)}$ are the jth largest memberships associated with the DMs' weights in u_1 and u_2 respectively.

In particular, if $\lambda = 1$ and $\lambda = 2$, then the weighted extended hesitant generalized distance reduces to a weighted extended hesitant weighted Hamming distance and a weighted extended hesitant Euclidean distance respectively.

To emphasize the importance of DMs in group decision making further, we now combine the weighted extended hesitant distance measures with the Dempster-Shafer belief structure (Dempster 1967; Shafer 1976). The Dempster-Shafer belief structure has proven to be a useful tool for representing uncertainty, which has been used in an astonishingly wide range of applications (Yager 1992; Yager *et al.* 1994).

Definition 18 (Dempster 1967; Shafer 1976). A Dempster-Shafer belief structure consists of a collection of r non-null subsets B_k $(k = 1,2,\cdots,r)$ of X defined on a space $X = \{x_1, x_2, ..., x_n\}$, called focal elements, and a mapping p called the probability assignment, defined as $p: 2^X \to [0,1]$ such that:

1) $p(B_j) \in [0,1]$;
2) $\sum_{k=1}^{r} p(B_k) = 1$;
3) $p(C) = 0$, $\forall\ C \neq B_k$.

Assume a group decision making problem associated with m DMs with a collection of weights w_D $(D = 1,\ldots,m)$, and a set of the alternatives, $\{A_1, \ldots, A_q\}$, with the set of states of nature $\{S_1, \ldots, S_n\}$. The DMs provide all possible preferences over all the alternatives A_i $(i = 1,\ldots,q)$ with respect to the states of nature (criteria) S_j $(j = 1,2,\ldots,n)$, then we can get $q \times n$ EHFEs H_{ij} $(i = 1,\ldots,q; j = 1,\ldots,n)$, which indicates the group preferences over the alternative A_i of the criterion S_j.

Let C_{ij} be a payoff to the alternative A_i and the state of nature is S_j, $C = (C_{ij})_{q\times n}$ a payoff matrix, and ς $(0 \le \varsigma \le 1)$ the optimized parameter. The DMs' knowledge of the states of nature is captured in terms of a belief structure p with the focal elements $B_1, B_2, \ldots, B_r$, each of which is associated with a weight $p(B_k)$, where $\sum_{k=1}^{r} p(B_k) = 1$. We now develop the following approach to deal with group decision making.

Step 1. Construct the extended hesitant fuzzy decision matrix $H = (H_{ik})_{q\times n}$ by EHFEs, H_{ij} $(i = 1,\ldots,q; j = 1,\ldots,n)$.

Step 2. Assume a standard EHFE H*, and the optimized parameter, then calculate the distance between H^* and H_{ij} by the extended hesitant distance measures or the weighted extended hesitant distance measure. Let C_{ij} equal to the distance, and construct the payoff matrix $C = (C_{ij})_{q\times n}$.

Step 3. Calculate the belief function p about the states of nature.

Step 4. Utilize the optimized parameter to calculate the collection of weights (O'Hagan 1988; Yager 1993) which are used in the OWA aggregation for each cardinality of focal elements.

Step 5. Determine the payoff collection, $M_{ik} = \{C_{ij} \mid S_j \in B_k\}$, which is a set of payoffs that are possible if we select the alternative A_i and the focal element B_k occurs, and calculate the aggregated payoff, $V_{ik} = OWA(M_{ik})$.

Step 6. Calculate $C_i = \sum_{k=1}^{r} V_{ik} p(B_k)$, and select the alternative which has the best generalized expected value as the optimal alternative.

Example 3 (Kahraman, Kaya 2010)**.** Energy is an indispensable factor for the social-economic development of societies. Thus the correct energy policy affects economic development and environment, the most appropriate energy policy selection is very important. Suppose that there are five alternatives (energy projects) A_i $(i = 1,2,3,4,5)$ to be invested, and four criteria to be considered: S_1 – technological; S_2 – environmental; S_3– socio-political; S_4– economic. Five DMs are invited to evaluate the performances of the five alternatives.

Xu and Xia (2011) used HFSs to collect the DMs' preferences and utilized the hesitant distance measures to deal with this problem. The precondition of the method introduced by Xu and Xia (2011) is that the DMs should give their preferences anonymously so as to ignore the repeated preferences. To deal with this energy policy problem without information loss, consider weights of the DMs in group decision making, and compare Xu and Xia (2011)'s resolution, we now give the following approach to deal with this problem.

Step 1. The DMs D_k $(k = 1,2,3,4,5)$ provide their preferences over all the alternatives A_i A_i $(i = 1,2,\ldots,5)$ with respect to the criteria S_j $(j = 1,2,3,4)$ based on HFSs, then we can construct EHFSs, and get an extended hesitant fuzzy matrix $H = (H_{ij})_{5\times 4}$, which indicates the group preferences over the alternative A_i of the criterion S_j. Assume that the matrix is shown in Table 1.

Table 1. Extended hesitant fuzzy decision matrix

	S_1	S_2
A_1	$\{(0.3,0.4,0.3,0.4,0.5)\}$	$\{(0.7,0.8,0.3,0.8,0.6),(0.7,0.8,0.4,0.8,0.6)\}$
A_2	$\{(0.3,0.4,0.5,0.2,0.5),(0.3,0.4,0.5,0.3,0.5)\}$	$\{(0.5,0.6,0.5,0.6,0.6)\}$
A_3	$\{(0.4,0.5,0.5,0.5,0.6)\}$	$\left\{\begin{matrix}(0.5,0.6,0.7,0.6,0.5),(0.6,0.6,0.7,0.6,0.5),\\(0.5,0.6,0.8,0.6,0.5),(0.6,0.6,0.8,0.6,0.5)\end{matrix}\right\}$
A_4	$\{(0.3,0.2,0.2,0.3,0.1)\}$	$\{(0.6,0.5,0.7,0.5,0.5)\}$
A_5	$\{(0.3,0.4,0.6,0.2,0.2),(0.3,0.3,0.6,0.2,0.2)\}$	$\{(0.6,0.8,0.5,0.4,0.6),(0.6,0.8,0.5,0.5,0.6)\}$
	S_3	S_4
A_1	$\left\{\begin{matrix}(0.3,0.4,0.2,0.3,0.2),(0.4,0.4,0.2,0.3,0.2),\\(0.3,0.4,0.3,0.3,0.2),(0.4,0.4,0.3,0.3,0.2)\end{matrix}\right\}$	$\{(0.6,0.5,0.5,0.4,0.6)\}$
A_2	$\{(0.6,0.4,0.5,0.3,0.5),(0.6,0.4,0.4,0.3,0.5)\}$	$\{(0.3,0.4,0.5,0.2,0.2),(0.3,0.4,0.4,0.2,0.2)\}$
A_3	$\{(0.7,0.3,0.9,0.8,0.6),(0.7,0.3,0.8,0.8,0.6)\}$	$\{(0.7,0.8,0.7,0.8,0.8)\}$
A_4	$\{(0.4,0.3,0.2,0.3,0.5)\}$	$\{(0.3,0.2,0.7,0.2,0.1)\}$
A_5	$\{(0.7,0.5,0.6,0.8,0.6)\}$	$\{(0.6,0.4,0.5,0.4,0.6),(0.7,0.4,0.5,0.4,0.6)\}$

Step 2. Let $A^*=\{(1,1,1,1,1)\}$ be the ideal values of alternative seen as a standard EHFE H^*, $\varsigma = 0.75$ be the optimized parameter, $w=(0.3,0.1,0.3,0.2,0.1)$ be the weighting vector of the DMs, and $\lambda = 1$. By Eq. (23), we can calculate the distance between H^* and H_{ij}, i.e. $d_{wg}(H^*,H_{ij})$. Let $C_{ij}=d_{wg}(H^*,H_{ij})$, then construct the payoff matrix $C=(C_{ij})_{5\times 4}$, shown in Table 2.

Table 2. The payoff matrix

	S_1	S_2	S_3	S_4
A_1	0.6800	0.4000	0.7750	0.5000
A_2	0.6550	0.4900	0.5700	0.7600
A_3	0.5500	0.4550	0.4000	0.2700
A_4	0.8700	0.4800	0.7000	0.7800
A_5	0.7400	0.4350	0.4100	0.5050

Step 3. The DMs analyze the energy policy problem so as to obtain the probabilistic information about the states of nature. Assume that the DMs' knowledge of the states of nature consists of the following belief structure, shown in Table 3.

Table 3. Belief structure

Focal element	Weights
$B_1=\{S_1,S_3\}$	0.15
$B_2=\{S_2,S_4\}$	0.25
$B_3=\{S_1,S_3,S_4\}$	0.6

Step 4. We use the O'Hagan (1988) method to obtain weighting vectors associated with the OWA operators for various numbers of arguments. Since $\varsigma = 0.75$, then we can get the weighting vectors shown in Table 4.

Table 4. Weighting vectors for various numbers of arguments

Number of arguments	w_1	w_2	w_3
2	0.75	0.25	
3	0.62	0.27	0.11

Step 5. Since $M_{ik}=\{C_{ij}\mid S_j\in B_k\}$ and $V_{ik}=OWA(M_{ik})$, then we can get V_{ik} for all i and k $(i=1,2,3,4,5; k=1,2,3)$.

Step 6. Since $C_i=\sum_{k=1}^{3}V_{ik}p(B_k)$, and according to V_{ik} and the belief structure, we have $C_1=0.5884$, $C_2=0.5801$, $C_3=0.4135$, $C_4=0.6617$, $C_5=0.5390$. Thus, A_3 is the optimal alternative closest to the ideal values of alternative with the minimum generalized expected value, $C_3=0.4135$.

According to the approach above, we know that C_i $(i = 1,2,3,4,5)$ vary with the parameter λ. For different λ, we get Table 5 below.

Table 5. Generalized expected values for different λ

	C_1	C_2	C_3	C_4	C_5	Rankings
λ = 1	0.5884	0.5801	0.4135	0.6617	0.5390	$A_4 \succ A_1 \succ A_2 \succ A_5 \succ A_3$
λ = 2	0.3006	0.2873	0.2069	0.3221	0.2663	$A_4 \succ A_1 \succ A_2 \succ A_5 \succ A_3$
λ = 5	0.2258	0.2074	0.1531	0.2315	0.1922	$A_4 \succ A_1 \succ A_2 \succ A_5 \succ A_3$

It's clear that for the three different values of λ, the same ranking result can be obtained in Example 3. In practice, we let λ = 1 without loss of generality.

The DMs influence the final decision result due to their different importance, if we let $w' = (0.2,0.2,0.2,0.2,0.2)$ be the weighting vector of the DMs, which means that there is no difference among the DMs, we can obtain a different ranking result. By the developed approach, and all other conditions are still the same, we have $C_1 = 0.5673$, $C_2 = 0.5764$, $C_3 = 0.4179$, $C_4 = 0.6820$, $C_5 = 0.5463$, and the ranking $A_4 \succ A_2 \succ A_1 \succ A_5 \succ A_3$, where a change in ranking happens between A_1 and A_2.

The hesitant distance measures cannot consider the differences among the DMs, which is similar to the situation that the weighting vector of DMs is $w' = (0.2,0.2,0.2,0.2,0.2)$. For comparison, we apply the hesitant distance measures to Example 3. According to the definition of reduced EHFEs, we first transform Table 1 to a hesitant fuzzy decision matrix, shown in Table 6.

Table 6. Hesitant fuzzy decision matrix

	S_1	S_2	S_3	S_4
A_1	{0.3,0.4,0.5}	{0.7,0.8,0.3,0.6,0.4}	{0.3,0.4,0.2}	{0.6,0.5,0.4}
A_2	{0.3,0.4,0.5,0.2}	{0.5,0.6}	{0.6,0.4,0.5,0.3}	{0.3,0.4,0.5,0.2}
A_3	{0.4,0.5,0.6}	{0.5,0.6,0.7,0.8}	{0.7,0.3,0.9,0.8,0.6}	{0.7,0.8}
A_4	{0.3,0.2,0.1}	{0.6,0.5,0.7}	{0.4,0.3,0.2,0.5}	{0.3,0.2,0.7,0.1}
A_5	{0.3,0.4,0.6,0.2}	{0.6,0.8,0.5,0.4}	{0.7,0.5,0.6,0.8}	{0.6,0.4,0.5,0.7}

With other conditions still being the same, and according to the hesitant normalized Hamming distance, we have $C_1 = 0.7043$, $C_2 = 0.7153$, $C_3 = 0.5869$, $C_4 = 0.7851$, $C_5 = 0.6201$, and the ranking $A_4 \succ A_2 \succ A_1 \succ A_5 \succ A_3$, which is the same as the ranking result when the weighting vector of the DMs is $w' = (0.2,0.2,0.2,0.2,0.2)$. Thus, the existing hesitant distance measures can be considered as a particular case of the extended hesitant distance measures with certain conditions. And the extended hesitant distance measures appear to be more extensive and effective in practical applications.

Conclusions

We have developed the extended hesitant fuzzy sets (EHFSs) to resolve the information loss problem of hesitant fuzzy sets (HFSs) in this paper, and have shown that intuitionistic fuzzy sets (IFSs), HFSs and dual hesitant fuzzy sets (DHFSs) are particular cases of EHFSs with certain conditions. EHFSs can also be represented as fuzzy multisets (FMSs) or type-2 fuzzy sets (T2FSs). Given several hesitant fuzzy elements (HFEs), we can construct an extended hesitant fuzzy element (EHFE) by their Cartesian product. On the contrary, given an EHFE, we can get its reduced EHFE which is a HFE. As an extension of HFSs, EHFSs increase the richness of numerical representation based on the membership units (MUs), enhance the modeling abilities of HFSs, and can identify different DMs in group decision making. We have further developed some extended hesitant distance measures which take advantages of EHFSs without the information loss problem. A weighted extended hesitant distance measure has been developed, which can take the different importance of the DMs into account in group decision making comparing with the existing hesitant distance measures. Combining the proposed weighted extended hesitant distance measure with Dempster-Shafer belief structure, we have proposed an approach to deal with group decision making problems with an illustrative example. In the future, EHFSs are likely to play an importance role in group decision making with more studies on the theory and applications.

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